The parametrized Seiberg-Witten moduli space

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January 7, 2013

Abstract: We study a slightly perturbed system of equations for the Seiberg-Witten equations and prove compactness and generic smoothness of the corresponding moduli space of solutions. The presentation follows [1, Sec. 6.1–6.4] except for the functional analytical aspects, which can be found in [2, Sec. 1.5.2].

1 Functional analytical background

In this section, we give a theorem, generalizing that of Sard-Brown to the infinite-dimensional setting, and which can be used to solve the following problem: given a smooth map $F : \Lambda \times M \longrightarrow N$ between manifolds and a point $y \in N$, where Λ is thought of as a set of parameters, when can be said that $\{x \in M\,,\ F(\lambda, x) = y\}$ is a smooth submanifold of M for "most" parameters λ?

First we fix the framework. Let Λ, M, N be (infinite-dimensional) Hilbert manifolds and $F : \Lambda\times M \longrightarrow N$ be a smooth map. We fix some $y \in N$ and set $S := F^{-1}(\{y\}) \subset \Lambda \times M$. For each $\lambda \in \Lambda$, we let $F_\lambda := F(\lambda, \cdot) : M \to N$ and $S_\lambda := F_\lambda^{-1}(\{y\}) \subset S$. The first projection $\Lambda \times M \to \Lambda$ induces by restriction to S a map $\pi : S \to \Lambda$ with $\pi^{-1}(\{\lambda\}) = \{\lambda\} \times S_\lambda$ for all $\lambda \in \Lambda$. Even if we assume y to be a regular value for F – that is, that $d_{\lambda,x}F : T_\lambda\Lambda \oplus T_xM \to T_yN$ is a bounded surjective linear map for every $(\lambda, x) \in S$ –, which, with the help of the inverse function theorem, implies that S is a smooth submanifold of $\Lambda \times X$, there is no reason *a priori* for each S_λ to be a smooth submanifold of M. Note however that, using Lemma 1.5 below, λ is a regular value of π iff

y is a regular value F_λ; in the finite-dimensional setting, the theorem of Sard-Brown would state that the set of singular values of π has zero measure in Λ, in particular "almost all" values of π are regular. In the infinite-dimensional case, the theorem of Sard-Brown no longer holds true, however it can be generalized to a category of maps F which are called *Fredholm maps.*

Definition 1.1 *A map $F : M \longrightarrow N$ between Hilbert manifolds is called* Fredholm *iff it is differentiable*[1] *and, for each $x \in M$, the differential $d_xF : T_xM \to T_{F(x)}N$ is Fredholm, that is, it has (closed range and) finite-dimensional kernel and cokernel.*

Given a C^1 Fredholm map $F : M \to N$ (in the strong sense), one can define the index $\operatorname{ind}(d_xF) := \dim(\ker(d_xF)) - \dim(\operatorname{coker}(d_xF)) \in \mathbb{Z}$ of the Fredholm linear map d_xF at each $x \in M$; since d_xF depends continuously on x, this index is locally constant in $x \in M$. In particular, if M is connected, one may define the *index* of F to be

$$\operatorname{ind}(F) := \operatorname{ind}(d_xF)$$

for some (and hence all) $x \in M$. Before we formulate the main result of this section, we make the concept of generic subset precise.

Definition 1.2 *A subset in a topological space is called* generic *iff it contains the intersection of countably many open dense subsets of the space.*

For instance, any generic subset of a complete metric space is dense by Baire's theorem (even if it is not necessarily open). In the following, we say that a property is satisfied by "most $x \in X$" iff the subset of X consisting of all x for which that property is fulfilled is a generic subset.

Theorem 1.3 (Sard-Smale [3]) *Let $F : M \longrightarrow N$ be a smooth* Fredholm *map between paracompact*[2] *Hilbert manifolds, where M is connected. Then most $y \in N$ are regular values of F and for those the subset $F^{-1}(\{y\})$ is a* (possibly empty) *smooth* $\operatorname{ind}(F)$*-dimensional submanifold of M.*

We mean in particular that, if $\operatorname{ind}(F) < 0$, then most $y \in N$ do not belong to the range of F, that is, $F^{-1}(\{y\}) = \varnothing$ for most $y \in N$.

Now we can answer our first question in the context of Fredholm maps.

[1] This means in particular that $d_xF : T_xM \to T_{F(x)}N$ is a *bounded* linear map for each $x \in M$.

[2] Might be already contained in the definition.

Theorem 1.4 *Let Λ, M, N be paracompact smooth Hilbert manifolds with M connected and $F : \Lambda \times M \longrightarrow N$ be a smooth map. Assume that $y \in N$ is a* regular value *of F and that $F_\lambda : M \to N$, $x \mapsto F(\lambda, x)$, is* Fredholm *for every $\lambda \in \Lambda$.*
Then the restriction $\pi : F^{-1}(\{y\}) \to \Lambda$ of the first projection $\Lambda \times M \to \Lambda$ to $F^{-1}(\{y\})$ is Fredholm. In particular for most $\lambda \in \Lambda$, the subset $F_\lambda^{-1}(\{y\})$ is a (possibly empty) *smooth* $\mathrm{ind}(F_\lambda)$*-dimensional submanifold of M.*

The proof of Theorem 1.4 relies on Theorem 1.3 and on the following elementary remarks.

Lemma 1.5 *Let E, F, G be vector spaces and $E \xrightarrow{f} F$, $E \xrightarrow{g} G$ be* surjective *linear maps. Then the subspaces $\ker(g_{|\ker(f)}) = \ker(f) \cap \ker(g) = \ker(f_{|\ker(g)})$ coincide and*

$$\mathrm{coker}(g_{|\ker(f)}) \cong {}^{E}\!/\!{\ker(f) + \ker(g)} \cong \mathrm{coker}(f_{|\ker(g)}).$$

In particular, the map $g_{|\ker(f)}$ is surjective iff $f_{|\ker(g)}$ is surjective. In the case of bounded linear maps f, g between Hilbert spaces, the map $g_{|\ker(f)}$ is Fredholm iff $f_{|\ker(g)}$ is Fredholm, and then $\mathrm{ind}(f_{|\ker(g)}) = \mathrm{ind}(g_{|\ker(f)})$.

Proof: The first statement is trivial. For the identification of the cokernels, note that, if $U \subset E$ is a vector subspace, then the composition $E \xrightarrow{f} F \xrightarrow{\text{proj.}} {}^{F}\!/\!{f(U)}$ is surjective with kernel $f^{-1}(f(U)) = U + \ker(f)$. Remember also that the closedness condition for the range automatically follows from the boundedness of the map and the cokernel be finite-dimensional. $\square$

Theorem 1.4 has a "transverse" version, which will be the main one we use:

Theorem 1.6 *Let Λ, M, N be paracompact smooth Hilbert manifolds with M connected and $F : \Lambda \times M \longrightarrow N$ be a smooth map. Let $L \subset N$ be a smooth Hilbert submanifold of N such that F is* transverse *to L and, for all $(\lambda, x) \in F^{-1}(L)$, the map $[d_x F_\lambda] : T_x M \to {}^{T_{F(\lambda,x)}N}\!/\!{T_{F(\lambda,x)}L}$, is* Fredholm.
Then, for most $\lambda \in \Lambda$, the map $F_\lambda : M \to N$ is transverse to L, in particular $F_\lambda^{-1}(L)$ is a (possibly empty) smooth $\mathrm{ind}([d_x F_\lambda])$*-dimensional*[3] *submanifold of M.*

Proof: Note that, since by assumption $F \pitchfork L$, the subset $S := F^{-1}(L)$ is a smooth Hilbert submanifold of $\Lambda \times M$ with $T_{(\lambda,x)}S = (d_{(\lambda,x)}F)^{-1}(T_{F(x)}L)$ for

[3]Since M is connected, the index $\mathrm{ind}([d_x F_\lambda]) \in \mathbb{Z}$ is independent of $x \in M$.

all $(\lambda, x) \in S$. Let $\pi := p_{|_S} : S \to \Lambda$ be the restriction to S of the first projection $p : \Lambda \times M \to \Lambda$. Since $\ker(d_{(\lambda,x)}p) = T_xM$ and $T_{(\lambda,x)}S = \ker([d_{(\lambda,x)}F])$, where $[d_{(\lambda,x)}F] = \text{proj.} \circ d_{(\lambda,x)}F : T_\lambda\Lambda \oplus T_xM \to {T_{F(\lambda,x)}N}/{T_{F(\lambda,x)}L}$, Lemma 1.5 implies that, for all $(\lambda, x) \in S$,

$$\begin{aligned}
\ker(d_{(\lambda,x)}\pi) &= \ker(d_{(\lambda,x)}p_{|_{T_{(\lambda,x)}S}}) \\
&= \ker(d_{(\lambda,x)}p_{|_{\ker([d_{(\lambda,x)}F])}}) \\
&= \ker(d_{(\lambda,x)}p) \cap \ker([d_{(\lambda,x)}F]) \\
&= T_xM \cap \ker([d_{(\lambda,x)}F]) \\
&= \ker([d_{(\lambda,x)}F]_{|_{T_xM}}) \\
&= \ker([d_xF_\lambda])
\end{aligned}$$

and analogously $\text{coker}(d_{(\lambda,x)}\pi) \cong \text{coker}([d_xF_\lambda])$. As a consequence, the map $d_{(\lambda,x)}\pi$ is Fredholm iff $[d_xF_\lambda]$ is, and in that case they have the same index. Now Theorem 1.3 implies that most $\lambda \in \Lambda$ are regular values of π. For those λ's, the map $[d_xF_\lambda]$ becomes surjective (for all $x \in M$ with $(\lambda, x) \in S$), that is, the map $F_\lambda : M \to N$ becomes transverse to L and therefore $F_\lambda^{-1}(L)$ becomes a smooth submanifold of M of dimension $\dim(\ker([d_xF_\lambda])) = \text{ind}([d_xF_\lambda])$. □

Corollary 1.7 *Let $E \longrightarrow \Lambda \times M$ be a smooth Hilbert space bundle over the product of two paracompact Hilbert manifolds and $F : \Lambda \times M \longrightarrow E$ be a smooth section of E. Assume that F is* transverse *to the zero-section of E and that $d_xF_\lambda : T_xM \to T_{0_{(\lambda,x)}}E$, is* Fredholm *for every $(\lambda, x) \in F^{-1}(\{0\})$. Then for most $\lambda \in \Lambda$, the subset $F_\lambda^{-1}(\{0\})$ is a* (possibly empty) *smooth* $\text{ind}(d_xF_\lambda)$*-dimensional submanifold of M.*

Proof: Just notice that $T_{0_{(\lambda,x)}}E \cong T_xM \oplus E_{(\lambda,x)}$ and that d_xF_λ is Fredholm iff its vertical projection $\text{pr}_{E_{(\lambda,x)}} \circ d_xF_\lambda$ is (since its horizontal projection defines an isomorphism $T_xM \to T_xM$). Now $\text{pr}_{E_{(\lambda,x)}} \circ d_xF_\lambda$ can be identified with $[d_xF_\lambda]$ via $E_{(\lambda,x)} \cong {T_{0_{(\lambda,x)}}E}/{T_xM}$. Apply Theorem 1.6. □

2 The parametrized moduli space

In this section, we introduce a new system of equations obtained by adding a parameter to the Seiberg-Witten equations. Let M be a 4-dimensional closed oriented Riemannian manifold. Fix a spin^c structure[4] $\widetilde{P} = P_{\text{Spin}_4^c}TM \to M$ on

[4] By a theorem due to Hirzebruch-Hopf and W. Wu, every such manifold is spin^c, see first talk of the seminar.

M, with associated $\mathbb{U}_1$-bundle (called *determinant bundle*) $P = P_{\mathbb{U}_1} \to M$. Denote by $\Sigma M \to M$ (resp. $\Sigma^+ M \to M$, $\Sigma^- M \to M$) the spinor (resp. positive, negative spinor) bundle of M associated to $\widetilde{P}$. Recall that the *configuration space* is defined by

$$\mathcal{C}(\widetilde{P}) := \mathcal{A}^{4,2}(P) \oplus H^{4,2}(\Sigma^+ M),$$

where $\mathcal{A}^{4,2}(P)$ is the space of $H^{4,2}$-connection 1-forms on P and $H^{4,2}(\Sigma^+ M)$ that of $H^{4,2}$-sections of $\Sigma^+ M \to M$. Recall that $\mathcal{A}^{4,2}(P)$ is an *affine* space with associated vector space $H^{4,2}(T^*M \otimes i\mathbb{R})$.
The group of $H^{5,2}$-gauge transformations of $\widetilde{P}$ which project to the identity on the frame bundle $P_{\mathrm{SO}_4} TM \to M$ is denoted by $\mathcal{G}(\widetilde{P})$; since the center of Spin_4^c is $\mathbb{U}_1$, we have in fact $\mathcal{G}(\widetilde{P}) \cong H^{5,2}(M, \mathbb{U}_1)$, the $H^{5,2}$-Sobolev space of $\mathbb{U}_1$-valued functions on M (the action on $\widetilde{P}$ is just given by pointwise right multiplication with such a function). Define the map

$$\begin{array}{rcl} F : \mathcal{C}(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M) & \longrightarrow & H^{3,2}\left((\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus \Sigma^- M\right) \\ (A, \psi, h) & \longmapsto & (F_A^+ - q_\psi - ih, D^A \psi), \end{array}$$

where $\Lambda^2_+ T^*M \to M$ is the (3-ranked real) vector bundle of self-dual 2-forms on M, F_A^+ is the self-dual component of the curvature form $F_A \in H^{3,2}(\Lambda^2 T^*M \otimes i\mathbb{R})$ of A, $q_\psi(\varphi) := \langle \varphi, \psi \rangle \psi - \frac{|\psi|^2}{2} \varphi$ and $D^A := \sum_{j=1}^4 e_j \cdot \nabla^A_{e_j}$ is the Dirac-operator associated to A and $\widetilde{P}$. Here and as usual, we identify imaginary-valued self-dual two forms with traceless Hermitian endomorphisms of $\Sigma^+ M$ via Clifford multiplication, see e.g. [1, Lemma 2.3.4].

By definition, the map F almost coincides with that defining the Seiberg-Witten equations, the difference consisting in the supplementary term $-ih$ in the form component. Given $h \in H^{3,2}(\Lambda^2_+ T^*M)$, we call (SW_h) the following system of equations provided by $\{(A, \psi) \, \text{s.t.} \, F(A, \psi, h) = 0\}$, that is:

$$(\mathrm{SW}_h) \left\{ \begin{array}{ll} F_A^+ & = q_\psi + ih \\ D^A \psi & = 0 \end{array} \right.$$

and we ask the same questions as for the (unperturbed) Seiberg-Witten equations (SW) (obtained by putting $h = 0$ in (SW_h)): do the solutions form a smooth submanifold, is there any gauge invariance, and if there is, what can be said about the corresponding moduli space (smooth, compact etc.)?

The first surprising fact when comparing to (SW) is that any (A, ψ, h) solving (SW_h) with $\psi \neq 0$ is a *regular* point for the map F:

Lemma 2.1 *Let $(A,\psi,h) \in \mathcal{C}(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ with $F(A,\psi,h)=0$. If $\psi \neq 0$, then*

$$d_{(A,\psi,h)}F : H^{4,2}\big((T^*M \otimes i\mathbb{R}) \oplus \Sigma^+ M\big) \oplus H^{3,2}(\Lambda^2_+ T^*M) \to H^{3,2}\big((\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus \Sigma^- M\big)$$

is surjective.

Proof: An elementary computation leads to

$$d_{(A,\psi,h)}F = \begin{pmatrix} p_+ \circ d & -\eta_\psi & -i \\ \frac{1}{2}\cdot\psi & D^A & 0 \end{pmatrix},$$

where $p_+ := \frac{1}{2}(\mathrm{Id} + *) : \Lambda^2 T^*M \to \Lambda^2_+ T^*M$ is the (pointwise) orthogonal projection onto the space of self-dual 2-forms, the endomorphism $\eta_\psi(\varphi)$ of $\Sigma^+ M$ is defined by $\eta_\psi(\varphi) := \langle \cdot, \psi\rangle \otimes \varphi + \langle \cdot, \varphi\rangle \otimes \psi - \Re e(\langle \psi, \varphi\rangle)\mathrm{Id}$ and $\frac{1}{2}\cdot\psi : T^*M \otimes i\mathbb{R} \to \Sigma^- M$ is defined by $B \mapsto \frac{1}{2} B\cdot\psi$. The surjectivity of $d_{(A,\psi,h)}F$ onto the first factor $H^{3,2}\big(\Lambda^2_+ T^*M \otimes i\mathbb{R}\big)$ is clear (given $\overline{h} \in H^{3,2}\big(\Lambda^2_+ T^*M \otimes i\mathbb{R}\big)$, one has $d_{(A,\psi,h)}F(0,0,i\overline{h}) = \overline{h}$). The surjectivity of $d_{(A,\psi,h)}F$ onto the second factor is equivalent to that of the map $H^{4,2}\left((T^*M \otimes i\mathbb{R}) \oplus \Sigma^+ M\right) \xrightarrow{G} H^{3,2}(\Sigma^- M)$, $(B,\varphi) \mapsto D^A\varphi + \frac{1}{2} B\cdot\psi$. To show that G is surjective, pick any $\phi \in H^{3,2}(\Sigma^- M)$ which is L^2-orthogonal to the range $\mathrm{Im}(G)$ of G. Then in particular, $\phi \in \big(\mathrm{Im}(D^A_+)\big)^{\perp, L^2} = \ker(D^A_-)$, where, as usual $D^A_\pm := (D^A)_{|_{L^2(\Sigma^\pm M)}} : L^2(\Sigma^\pm M) \to L^2(\Sigma^\mp M)$ (so that $D^A = D^A_+ \oplus D^A_-$). If we assume that $\varphi \neq 0$, then by the unique continuation property[5] for eigenvectors of elliptic self-adjoint operators, ϕ cannot vanish on *any* open subset of M. Analogously, since by assumption $\psi \neq 0$, the section ψ cannot vanish on any open subset of M. Therefore there exists an open subset U of M where $\varphi(x) \neq 0$ and $\psi(x) \neq 0$ for almost all $x \in U$.
Claim: *For any $\sigma_+ \in \Sigma_4^+ \setminus \{0\}$, the linear map $\mathbb{R}^4 \longrightarrow \Sigma_4^-$, $v \mapsto \delta_4(v) = v\cdot$, is an isomorphism.*
Proof of the claim: The injectivity clearly follows from $\delta_4(v)^2 = -|v|^2\mathrm{Id}$. $\surd$
Note in particular that, for any $(\sigma_+, \sigma_-) \in \Sigma_4^+ \times \Sigma_4^-$ with $\sigma_\pm \neq 0$, there exists a unique $v \in i\mathbb{R}^4$ with $\delta_4(v)(\sigma_+) = \sigma_-$. If we fix $x_0 \in U$, then the claim implies the existence of an $a \in T^*_{x_0}M \otimes i\mathbb{R}$ with $\langle a\cdot\psi(x_0), \varphi(x_0)\rangle > 0$. Extend a onto some open neighbourhood U' of x_0 in U such that $\langle a(x)\cdot\psi(x), \varphi(x)\rangle > 0$ and $a(x) \in T^*_x M \otimes i\mathbb{R}$ for all $x \in U'$. Using a cut-off function, we can extend a to an imaginary-valued 1-form on M such that $\langle a(x)\cdot\psi(x), \varphi(x)\rangle \geq 0$ for all $x \in M$. Since $\langle a(x)\cdot\psi(x), \varphi(x)\rangle > 0$ for all $x \in U'$, we obtain

$$\int_M \langle a(x)\cdot\psi(x), \varphi(x)\rangle dv_g(x) > 0.$$

[5] Is there any version of the UCP available for elliptic operators with *non-smooth* coefficients?

But this means precisely that φ is *not* L^2-orthogonal to $G(2a,0)$ and therefore not L^2-orthogonal to $\mathrm{Im}(G)$, which is a contradiction. We conclude that $\mathrm{Im}(G)^{\perp,L^2} \cap H^{3,2}(\Sigma^- M) = 0$, which shows $\mathrm{Im}(G) = H^{3,2}(\Sigma^- M)$ and the proposition. $\square$

As a consequence, if we let $\mathcal{C}^*(\widetilde{P}) := \mathcal{A}^{4,2}(P) \times H^{4,2}(\Sigma^+ M) \setminus \{0\}$ (which is a Hilbert manifold by the first talk), the set

$$\Big\{(A,\psi,h) \in \mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)\,,\ F(A,\psi,h) = 0\Big\}$$

is a smooth Hilbert submanifold of $\mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$.

Next we look at the gauge invariance of F. As in the Seiberg-Witten setting, the group $\mathcal{G}(\widetilde{P})$ can be made acting from the right on $\mathcal{C}(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ through

$$(A,\psi,h)\cdot\sigma := (R^*_{\sigma^2}A, \sigma^{-1}\psi, h)$$

for all $(A,\psi,h) \in \mathcal{C}(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ and $\sigma \in \mathcal{G}(\widetilde{P})$. Note that $R^*_{\sigma^2}A \neq A$ unless σ is *locally constant* on M, however $F_{R^*_{\sigma^2}A} = F_A$. Obviously $q_{\sigma^{-1}\psi} = q_\psi$. If we define

$$\begin{array}{rcl} H^{3,2}\big((\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus \Sigma^- M\big) \times \mathcal{G}(\widetilde{P}) & \longrightarrow & H^{3,2}\big((\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus \Sigma^- M\big) \\ ((B,\varphi),\sigma) & \longmapsto & (B,\varphi)\cdot\sigma := (B, \sigma^{-1}\varphi), \end{array}$$

then we see, as for the Seiberg-Witten equations, that $F((A,\psi,h)\cdot\sigma) = F(A,\psi,h)\cdot\sigma$ for all $(A,\psi,h) \in \mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ and $\sigma \in \mathcal{G}(\widetilde{P})$. In particular, the space of solutions to (SW_h) is preserved by the action of $\mathcal{G}(\widetilde{P})$.

As in the preceding talks, we call an element $(A,\psi,h) \in \mathcal{C}(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ *irreducible* iff its stabilizer is trivial, i.e., iff

$$\mathrm{Stab}_{(A,\psi,h)} := \Big\{\sigma \in \mathcal{G}(\widetilde{P})\,,\ (A,\psi,h)\cdot\sigma = (A,\psi,h)\Big\} = 1,$$

and *reducible* otherwise. If M is connected, the condition $R^*_{\sigma^2}A = A$ is equivalent to σ being constant by the remark above; therefore, $\sigma^{-1}\psi = \psi$ iff $\sigma = 1$ when $\psi \neq 0$, whereas no supplementary condition comes in if $\psi = 0$. Therefore, $\mathrm{Stab}_{(A,\psi,h)} = 1$ if $\psi \neq 0$ and $\mathrm{Stab}_{(A,\psi,h)} = \mathbb{U}_1$ if $\psi = 0$. In particular, the space of irreducible elements of configurations is $\mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$.

Notations 2.2

1. We denote by $\widetilde{\mathcal{PM}}(\widetilde{P}) := F^{-1}(\{0\}) \subset \mathcal{C}(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ the space of all solutions (A, ψ, h) of $F(A, \psi, h) = 0$.

2. We denote by $\widetilde{\mathcal{PM}}^*(\widetilde{P}) := F^{-1}(\{0\}) \cap \Big(\mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)\Big)$ the subspace of all irreducible solutions (A, ψ, h) of $F(A, \psi, h) = 0$.

3. For $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the space of all solutions (A, ψ) of (SW_h) is denoted by $\widetilde{\mathcal{M}}(\widetilde{P}, h) := \Big\{(A, \psi) \in \mathcal{C}(\widetilde{P})\,,\ F(A, \psi, h) = 0\Big\}$.

4. Similarly, we denote by $\widetilde{\mathcal{M}}^*(\widetilde{P}, h) := \widetilde{\mathcal{M}}(\widetilde{P}, h) \cap \mathcal{C}^*(\widetilde{P})$ the space of all irreducible solutions of (SW_h).

By the remark above, $\widetilde{\mathcal{PM}}^*(\widetilde{P})$ is a smooth Hilbert submanifold of $\mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$.

Definition 2.3

1. *The quotient space* $\mathcal{PM}(\widetilde{P}) := \widetilde{\mathcal{PM}}(\widetilde{P}) / \mathcal{G}(\widetilde{P})$ *is called the* parametrized moduli space *of all solutions* (A, ψ, h) *of* $F(A, \psi, h) = 0$. *The subset* $\mathcal{PM}^*(\widetilde{P}) := \widetilde{\mathcal{PM}}^*(\widetilde{P}) / \mathcal{G}(\widetilde{P})$ *is called the parametrized moduli space of all* irreducible *solutions* (A, ψ, h) *of* $F(A, \psi, h) = 0$.

2. *For* $h \in H^{3,2}(\Lambda^2_+ T^*M)$, *we call* $\mathcal{M}(\widetilde{P}, h) := \widetilde{\mathcal{M}}(\widetilde{P}, h) / \mathcal{G}(\widetilde{P})$ *the moduli space of all solutions* (A, ψ) *of* (SW_h). *Its elements are called* monopoles. *Similarly, we call* $\mathcal{M}^*(\widetilde{P}, h) := \widetilde{\mathcal{M}}^*(\widetilde{P}, h) / \mathcal{G}(\widetilde{P})$ *the moduli space of all irreducible solutions of* (SW_h).

Recall that, for any $(A, \psi) \in \mathcal{C}(\widetilde{P})$, we have an associated elliptic complex

$$0 \xrightarrow{d_0} H^{5,2}(M; i\mathbb{R}) \xrightarrow{d_1} H^{4,2}(T^*M \otimes i\mathbb{R} \oplus \Sigma^+ M) \xrightarrow{d_2} H^{3,2}\left((\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus \Sigma^- M\right) \xrightarrow{d_3} 0$$

where $d_1 f := (2df, -f\psi)$ provides the differential of the group action and $d_2(B, \varphi) := \left((dB)^+ - \eta_\psi(\varphi), \frac{1}{2} B \cdot \psi + D^A \varphi\right)$ provides that of $F(\cdot, \cdot, h)$ (for an arbitrary h). The associated (finite-dimensional) "cohomology groups" are defined by $\mathcal{H}^i_{[A,\psi]} := \ker(d_{i+1}) / \mathrm{im}(d_i)$, $i = 0, 1, 2$. If $\psi \neq 0$ (e.g. if (A, ψ) is irreducible), then $\mathcal{H}^0_{[A,\psi]} = 0$. The space $\mathcal{H}^1_{[A,\psi]}$ is the formal (or Zariski) tangent space to the moduli space $\mathcal{M}(\widetilde{P})$ (or $\mathcal{M}(\widetilde{P}, h)$). The space $\mathcal{H}^2_{[A,\psi]}$ is the so-called *obstruction space* at $[(A, \psi)]$. In case $\psi = 0$, the Euler characteristic of the complex is given by $-\frac{c_1(P)^2 - 2\chi(M) - 3\mathrm{sign}(M)}{4}$: indeed we have $\mathcal{H}^0_{[A,0]} = \mathbb{R}$,

$\mathcal{H}^1_{[A,0]} = {\ker(d) \cap \Omega^1(M)}/{\mathrm{im}(d) \cap \Omega^1(M)} \oplus \ker(D^A_+) = H^1(M; i\mathbb{R}) \oplus \ker(D^A_+)$ and $\mathcal{H}^2_{[A,0]} = \mathrm{coker}(d_2) = \mathrm{coker}(p_+ \circ d) \oplus \mathrm{coker}(D^A_+) = \mathscr{H}^2_+ \oplus \mathrm{coker}(D^A_+)$, where we have used Lemma 2.6 below. As a consequence, using the Atiyah-Singer index theorem and Poincaré duality,

$$\begin{aligned}
\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{R}}(\mathcal{H}^i_{[A,0]}) &= 1 - b_1(M) - \dim_{\mathbb{R}}(\ker(D^A_+)) \\
&\quad + b_2^+(M) + \dim_{\mathbb{R}}(\mathrm{coker}(D^A_+)) \\
&= 1 - b_1(M) + b_2^+(M) - 2\mathrm{ind}_{\mathbb{C}}(D^A_+) \\
&= 1 - b_1(M) + b_2^+(M) - \frac{1}{4}(c_1(P)^2[M] - \mathrm{sign}(M)) \\
&= 1 - b_1(M) + b_2^+(M) \\
&\quad - \frac{1}{4} c_1(P)^2[M] + \frac{1}{4}(b_2^+(M) - b_2^-(M)) \\
&= -\frac{1}{4} c_1(P)^2[M] + \frac{3}{4}(b_2^+(M) - b_2^-(M)) \\
&\quad + 1 - b_1(M) + \frac{1}{2}(b_2^+(M) + b_2^-(M)) \\
&= -\frac{1}{4}(c_1(P)^2[M] - 3\mathrm{sign}(M)) + \frac{1}{2}(2 - 2b_1(M) + b_2(M)) \\
&= -\frac{1}{4}\left(c_1(P)^2[M] - 3\mathrm{sign}(M) - 2\chi(M)\right).
\end{aligned}$$

Proposition 2.4

1. *The set $\mathcal{PM}^*(\widetilde{P})$ is a smooth Hilbert submanifold of the Hilbert manifold ${\mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)}/{\mathcal{G}(\widetilde{P})} = {\mathcal{C}^*(\widetilde{P})}/{\mathcal{G}(\widetilde{P})} \times H^{3,2}(\Lambda^2_+ T^*M)$.*

2. *If $\pi : \mathcal{PM}^*(\widetilde{P}) \longrightarrow H^{3,2}(\Lambda^2_+ T^*M)$ is the projection onto the second factor, then π is smooth with $\ker(d_{([A,\psi],h)}\pi) = \mathcal{H}^1_{[A,\psi]}$, the formal (or Zariski) tangent space to $\mathcal{M}^*(\widetilde{P}, h)$ at $[A, \psi]$).*

3. *The cokernel of $d_{([A,\psi],h)}\pi$ is isomorphic to the obstruction space $\mathcal{H}^2_{[A,\psi]}$ for $\mathcal{M}^*(\widetilde{P}, h)$ at $[A, \psi]$. In particular, $d_{([A,\psi],h)}\pi$ is Fredholm with index $d = d(P) := \frac{c_1(P)^2[M] - 2\chi(M) - 3\mathrm{sign}(M)}{4}$.*

Proof: Since $\widetilde{\mathcal{PM}}^*(\widetilde{P})$ is a smooth Hilbert submanifold of the Hilbert manifold $\mathcal{C}^*(\widetilde{P}) \times H^{3,2}(\Lambda^2_+ T^*M)$ and the right $\mathcal{G}(\widetilde{P})$-action is free (and properly discontinuous), the quotient space $\mathcal{PM}^*(\widetilde{P})$ is a smooth Hilbert submanifold of the quotient Hilbert manifold ${\mathcal{C}^*(\widetilde{P})}/{\mathcal{G}(\widetilde{P})} \times H^{3,2}(\Lambda^2_+ T^*M)$. This shows 1. Statement 2. follows from Lemma 1.5, where $\ker(d_{[(A,\psi)]}[F](\cdot,\cdot,h)) = \ker(d_{(A,\psi)}F(\cdot,\cdot,h))/_{\mathrm{im}(d_1)} = \ker(d_2)/_{\mathrm{im}(d_1)} = \mathcal{H}^1_{[A,\psi]}$ and $[F]$ denotes the *section* of the *vector bundle* $E := H^{3,2}(\Lambda^2_+T^*M \otimes i\mathbb{R} \oplus \Sigma^- M) \times \mathcal{C}^*(\widetilde{P})/_{\mathcal{G}(\widetilde{P})} \to H^{3,2}(\Lambda^2_+T^*M) \times \mathcal{C}^*(\widetilde{P})/_{\mathcal{G}(\widetilde{P})}$ induced by F. Similarly, statement 3. follows from $\mathrm{coker}(d_{[(A,\psi)]}[F](\cdot,\cdot,h)) = \mathrm{coker}(d_2) = \mathcal{H}^2_{[A,\psi]}$. □

Corollary 2.5 *For most $h \in H^{3,2}(\Lambda^2_+T^*M)$, the parametrized moduli space $\mathcal{M}^*(\widetilde{P},h)$ of all irreducible solutions (A,ψ) of* (SW_h) *is a (possibly empty) smooth $d(P)$-dimensional submanifold of $\mathcal{C}^*(\widetilde{P})/_{\mathcal{G}(\widetilde{P})}$.*

Proof: By Lemma 2.1, the map $d_{(A,\psi,h)}F$ is surjective for every $(A,\psi,h) \in \widetilde{\mathcal{PM}}^*(\widetilde{P})$, that is, the section $[F]$ of the vector bundle E introduced above is *transverse* to its zero-section[6]. Since the map $d_{[(A,\psi)]}[F](\cdot,\cdot,h)$ is Fredholm for every $([A,\psi],h) \in \mathcal{PM}^*(\widetilde{P})$, Corollary 1.7 applies and provides the result. □

We have made (and shall again make) use of Lemma 2.7 from Andreas' second talk, in which the following useful lemma from Bernd's first talk plays a central role:

Lemma 2.6 *Let (M^4,g) be any closed oriented smooth Riemannian manifold.*

i) Let $\mathscr{H}^2_+(M)$ be the space of self-dual harmonic 2-forms on M. Then the L^2-orthogonal decomposition $\Omega^2_+(M) = \mathscr{H}^2_+(M) \oplus p_+ \circ d(\Omega^1(M))$ holds. In particular, for any $h \in \Omega^2_+(M)$, there exists (at least one) $\eta \in \Omega^2(M) \cap \ker(d)$ with $\eta_+ = h$.

ii) Given $\alpha \in \Omega^1(M)$, we have $(d\alpha)_+ = 0$ iff $d\alpha = 0$.

Proof: The splitting in *i)* follows from $*$ exchanging $d\Omega^1(M)$ with $\delta\Omega^3(M)$, so that $d\Omega^1(M) \oplus \delta\Omega^3(M) = p_+(d\Omega^1(M)) \oplus p_-(d\Omega^1(M))$. As a consequence of the splitting $\Omega^2_+(M) = \mathscr{H}^2_-(M) \oplus p_+ \circ d(\Omega^1(M))$, one can write $h = [h] + (d\alpha)_+ = (\underbrace{[h] + d\alpha}_{=:\eta})_+$, where $d\eta = d[h] + d^2\alpha = 0$. Statement *ii)* follows from $\int_M d\alpha \wedge d\alpha = \int_M d(\alpha \wedge d\alpha) = 0$. □

[6] would need a few details.

Lemma 2.7 *Given any* $h \in H^{3,2}(\Lambda^2_+ T^*M)$*, the space* $\mathcal{M}(\widetilde{P}, h)$ *contains at least one reducible monopole iff* $[h] = 2\pi[c_1(P)]_+$*. In that case, the set of reducible monopoles in* $\mathcal{M}(\widetilde{P}, h)$ *is an affine space modelled on* $\ker(d) \cap \Omega^1(M)$*.*

Proof: A reducible monopole is of the form $[(A, 0)]$, where $F_A^+ = ih$. If there exists such a monopole, then taking the harmonic components we obtain $[F_A^+] = [F_A]_+ = i[h]_+ = i[h]$, where $[F_A] = 2i\pi[c_1(P)]$ (property of the first Chern class), so that $[h] = 2\pi[c_1(P)]_+$. Conversely, if $[h] = 2\pi[c_1(P)]_+$, then choosing an arbitrary connection 1-form $\widehat{A}$ on $P \to M$, we have $[F^+_{\widehat{A}}] = i[h]$, so that there exists a real-valued $\alpha \in \Omega^1(M)$ with $F^+_{\widehat{A}} + i(d\alpha)_+ = ih$ by Lemma 2.6. In particular, the connection 1-form $A_0 := \widehat{A} + i\alpha$ on $P \to M$ satisfies $F^+_{A_0} = ih$, i.e., $[(A_0, 0)] \in \mathcal{M}(\widetilde{P}, h)$. Moreover, for any other connection 1-form A on $P \to M$, we can write $A = A_0 + i\beta$ with $\beta \in \Omega^1(M)$ and then $F_A^+ = ih$ iff $(d\beta)_+ = 0$, that is, using Lemma 2.6, iff $d\beta = 0$. This concludes the proof. □

3 Reducible solutions

As in the last talks, we denote by $b_2^+(M)$ the real dimension of the space of self-dual harmonic 2-forms on M.

Proposition 3.1 *Assume that* $b_2^+(M) > 0$ *and that* $c_1(P) \notin \mathrm{Tor}(H^2(M, \mathbb{Z}))$*. Then we have the following.*

1. *For most metrics and spin^c structures on M with P as determinant bundle, there are no reducible solutions to* (SW)*.*

2. *If, for a given metric and spin^c structure, there are no reducible solutions to* (SW)*, then for sufficiently small* $h \in H^{3,2}(\Lambda^2_+ T^*M)$ *there are no reducible solutions to* (SW_h)*. Moreover, for any metric and most* $h \in H^{3,2}(\Lambda^2_+ T^*M)$ *there are no reducible solutions to* (SW_h)*.*

Proof: By Lemma 2.7, there exists a reducible solution to (SW) iff the self-dual harmonic part $[c_1(P)]_+$ of $c_1(P)$ vanishes. But by a result by C. Taubes [4], if $c_1(P) \notin \mathrm{Tor}(H^2(M, \mathbb{Z}))$, i.e., $c_1(P) \neq 0 \in H^2(M; \mathbb{R})$, then the set of Riemannian metrics satisfying that condition is contained in a $b_2^+(M)$-codimensional submanifold of the space of all Riemannian metrics.[7] In particular, the existence of reducible solutions is generically not fulfilled. This

[7]Could this be a reason: the "section" $g \mapsto [c_1(P)]_{+,g}$ is transverse to the zero section of the bundle $\mathscr{H}^2_{+,\cdot} \to \mathrm{Riem}(M)$? Is that true? Check!

proves 1.
Again, Lemma 2.7 states that, if there is a reducible solution to (SW_h), then $[h] = 2\pi[c_1(P)]_+$. But, if $[c_1(P)]_+ \neq 0$ for some metric g on M, then there is a small neighbourhood V of the (finite-dimensional) space $\mathscr{H}_+^2$ such that $[h] - 2\pi[c_1(P)]_+ \neq 0$ for all $[h] \in V$, therefore there is no reducible solution to (SW_h) for every sufficiently small $h \in H^{3,2}(\Lambda_+^2 T^*M)$. For the last statement, given any metric on M, the set of $h \in H^{3,2}(\Lambda_+^2 T^*M)$ with $[h] - 2\pi[c_1(P)]_+ = 0$ is a closed $b_2^+(M)$-codimensional affine subspace of $H^{3,2}(\Lambda_+^2 T^*M)$, therefore its complement is open and dense in $H^{3,2}(\Lambda_+^2 T^*M)$.[8] $\square$

Corollary 3.2 *Let M be a 4-dimensional closed Riemannian manifold with $b_2^+(M) > 0$ and a spin^c structure $\widetilde{P} \to M$ with associated determinant bundle $P \to M$. Then for most $h \in H^{3,2}(\Lambda_+^2 T^*M)$, the parametrized moduli space $\mathcal{M}(\widetilde{P}, h)$ of all solutions (A, ψ) of (SW_h) is a (possibly empty) smooth $d(P)$-dimensional submanifold of $\mathcal{C}^*(\widetilde{P})/\mathcal{G}(\widetilde{P})$.*

Proof: Direct consequence of Corollary 2.5 and Proposition 3.1. $\square$

Proposition 3.3 *Let M be a 4-dimensional closed Riemannian manifold with $b_2^+(M) = 0$ and a spin^c structure $\widetilde{P} \to M$ with associated determinant bundle $P \to M$. Then there exists*[9] *an $h \in H^{3,2}(\Lambda_+^2 T^*M)$ and an $A \in \mathcal{A}^{4,2}(P)$ such that, the obstruction space $\mathcal{H}^2_{[A,0]} = \mathrm{coker}(D_+^A)$ associated to the reducible monopole $[(A, 0)]$ to (SW_h) vanishes.*

Proof: Note first that, since $b_2^+(M) = 0$, Lemma 2.7 implies that, whatever $h \in H^{3,2}(\Lambda_+^2 T^*M)$ is, there exists at least one reducible solution to (SW_h) (namely the condition $[h] = 2\pi[c_1(P)]_+$ is void). The map

$$\begin{array}{rcl} \mathcal{A}^{4,2}(P) \times H^{3,2}(\Lambda_+^2 T^*M) \times H^{4,2}(\Sigma^+ M) & \stackrel{f}{\longrightarrow} & H^{3,2}(\Lambda_+^2 T^*M \otimes i\mathbb{R} \oplus \Sigma^- M) \\ (A, h, \psi) & \longmapsto & (F_A^+ - ih, D_+^A \psi), \end{array}$$

is smooth with differential $d_{(A,h,\psi)} f(B, \overline{h}, \varphi) = ((dB)^+ - i\overline{h}, D_+^A \varphi + \frac{1}{2} B \cdot \psi)$, see proof of Lemma 2.1. Let now $\widehat{\psi} \in H^{3,2}(\Sigma^- M)$ be such that $\widehat{\psi}$ vanishes on no open subset of M; for instance, take any (non-zero) eigenspinor ϕ associated to any eigenvalue λ of D^A (for some A) and let $\widehat{\psi} := \phi_-$

[8] It may not be the full answer, since one may want a set of h which works for *all* Riemannian metrics. Check.

[9] Is this true for most h and all A?

(the UCP implies that ϕ cannot vanish on any open subset of M; since $D^A\phi_\pm = \lambda\phi_\mp$, this implies that both $\phi_\pm$ cannot vanish on any open subset of M). Then any solution ψ to $D^A\psi = \widehat{\psi}$ cannot vanish on any open subset of M either, whatever A is. Therefore and as in the proof of Lemma 2.1, the element $(0, \widehat{\psi}) \in H^{3,2}(\Lambda^2_+ T^*M \otimes i\mathbb{R} \oplus \Sigma^- M)$ is a regular value of f. If now $(A, h, \psi) \in f^{-1}(\{(0, \widehat{\psi})\})$ is arbitrary, then the differential of the map $f_A : (h, \psi) \mapsto f(A, h, \psi)$ is given by $d_{(h,\psi)} f_A(\overline{h}, \varphi) = (-i\overline{h}, D^A_+\varphi)$. In particular, $\ker(d_{(h,\psi)} f_A) = \ker(D^A_+)$ and $\mathrm{coker}(d_{(h,\psi)} f_A) = \mathrm{coker}(D^A_+)$ so that $d_{(h,\psi)} f_A$ is Fredholm. Theorem 1.4 states that, for most $A \in \mathcal{A}^{4,2}(P)$, the element $(0, \widehat{\psi}) \in H^{3,2}(\Lambda^2_+ T^*M \otimes i\mathbb{R} \oplus \Sigma^- M)$ is a regular value of f_A, that is, $\mathrm{coker}(D^A_+) = 0$ for any (h, ψ) with $f(A, h, \psi) = 0$. Since by assumption $\mathscr{H}^2_+(M) = 0$, we obtain $\mathcal{H}^2_{[A,0]} = \mathrm{coker}(D^A_+) = 0$ for any such triple (A, h, ψ). Picking such an A and setting $h := -iF^+_A$ concludes the proof[10]. $\square$

4 Compactness of the parametrized moduli space

Proposition 4.1 *Let M be a 4-dimensional closed Riemannian manifold with spin^c structure $\widetilde{P} \to M$. Then for any $h \in H^{3,2}(\Lambda^2_+ T^*M)$, the parametrized moduli space $\mathcal{M}(\widetilde{P}, h)$ is compact.*

Proof: By the Schrödinger-Lichnerowicz formula

$$(D^A)^2 = (\nabla^A)^*\nabla^A + \frac{S}{4}\mathrm{Id} + \frac{1}{2}F_A\cdot, \tag{1}$$

[10]The operator $D^A_+ + \frac{i\alpha}{2}\cdot$ need not be surjective when D^A_+ is, so that it is unclear whether any other reducible solution $(\widehat{A} = A + i\alpha, 0)$ to (SW_h) also satisfies $\mathcal{H}^2_{[\widehat{A},0]} = 0$.

where S is the scalar curvature of (M,g), we obtain, for any solution (A,ψ) of (SW_h),

$$\begin{aligned} 0 &= (D^A)^2\psi \\ &= (\nabla^A)^*\nabla^A\psi + \frac{S}{4}\psi + \frac{1}{2}F_A\cdot\psi \\ &= (\nabla^A)^*\nabla^A\psi + \frac{S}{4}\psi + \frac{1}{2}\left(F_A^+\cdot\psi + \underbrace{F_A^-\cdot\psi}_{0}\right) \\ &= (\nabla^A)^*\nabla^A\psi + \frac{S}{4}\psi + \frac{1}{2}\left(q_\psi\cdot\psi + ih\cdot\psi\right) \\ &= (\nabla^A)^*\nabla^A\psi + \frac{S}{4}\psi + \frac{|\psi|^2}{4}\psi + \frac{ih}{2}\cdot\psi. \end{aligned}$$

Taking the Hermitian inner product with ψ at a point x_0 where $|\psi(x_0)| = \max_{x\in M}(|\psi(x)|)$, we obtain

$$0 = \Re e(\langle(\nabla^A)^*\nabla^A\psi,\psi\rangle_{x_0}) + \frac{1}{4}(S(x_0)|\psi(x_0)|^2 + |\psi(x_0)|^4) + \frac{1}{2}\Re e(i\langle h(x_0)\cdot\psi(x_0),\psi(x_0)\rangle).$$

But, since $\Re e(\langle(\nabla^A)^*\nabla^A\psi,\psi\rangle) = |\nabla^A\psi|^2 + \frac{1}{2}\Delta(|\psi|^2)$ and by the assumption on x_0, we have $\Delta(|\psi|^2)(x_0) \geq 0$, we obtain

$$S(x_0)|\psi(x_0)|^2 + |\psi(x_0)|^4 + 2\Re e(i\langle h(x_0)\cdot\psi(x_0),\psi(x_0)\rangle) \leq 0.$$

Cauchy-Schwarz inequality yields $0 \geq |\psi(x_0)|^2\left(S(x_0) + |\psi(x_0)|^2 - 2|h(x_0)|\right)$. If $\psi(x_0) \neq 0$, then we deduce that $|\psi(x_0)|^2 \leq 2|h(x_0)| - S(x_0)$. On the whole, we obtain the pointwise upper bound[11]

$$\|\psi\|_\infty^2 \leq \max\left(\max_M(2|h(x)| - |S(x)|), 0\right).$$

An L^2 upper bound can also be deduced via (1) as follows:

$$\begin{aligned} \|\nabla^A\psi\|_2^2 &= \left((\nabla^A)^*\nabla^A\psi,\psi\right)_{L^2} \\ &= -\frac{1}{4}\left((S+|\psi|^2)\psi,\psi\right)_{L^2} - \frac{i}{2}\left(h\cdot\psi,\psi\right)_{L^2} \\ &= \frac{1}{4}\left(S\psi,\psi\right)_{L^2} - \frac{1}{4}\|\psi\|_4^4 - \frac{i}{2}\left(h\cdot\psi,\psi\right)_{L^2} \\ &\leq \left(\frac{1}{4}\|S\|_\infty + \|h\|_\infty\right)\|\psi\|_\infty^2\mathrm{Vol}(M,g). \end{aligned}$$

[11] Note that $h \in C^1(M)$ by the Sobolev embedding theorem.

Since $\|\psi\|_\infty^2$ can be bounded from above independently of A, ψ, we conclude, using a bootstrap argument as in the last talk, that $\mathcal{M}(\widetilde{P}, h)$ is compact.[12] $\square$

Corollary 4.2 *Let M be a 4-dimensional closed Riemannian manifold with $b_2^+(M) > 0$ and spin^c structure $\widetilde{P} \to M$. Then there exists an open dense subset $U_{\widetilde{P}}$ of $H^{3,2}(\Lambda_+^2 T^*M)$ such that, for any $h \in U_{\widetilde{P}}$, the parametrized moduli space $\mathcal{M}(\widetilde{P}, h)$ only consists of irreducible solutions and is a (possibly empty) compact smooth $d(P)$-dimensional submanifold of $\mathcal{C}^*(\widetilde{P})/\mathcal{G}(\widetilde{P})$.*

Proof: The idea is to use the compactness of the fibres $\mathcal{M}(\widetilde{P}, h)$ of the projection map $\pi : \mathcal{PM}^*(\widetilde{P}) \longrightarrow H^{3,2}(\Lambda_+^2 T^*M)$ (consequence of Proposition 4.1) to show that the dense subset $U_{\widetilde{P}}$ of those $h \in H^{3,2}(\Lambda_+^2 T^*M)$ for which $\mathcal{M}(\widetilde{P}, h)$ only consists of irreducible solutions (see Corollary 3.2) can be chosen to be *open* in $H^{3,2}(\Lambda_+^2 T^*M)$.[13] $\square$

5 Conclusion

Theorem 5.1 *Let M be a 4-dimensional closed Riemannian manifold with $b_2^+(M) > 0$. Then for most $h \in H^{3,2}(\Lambda_+^2 T^*M)$ the following holds: for any spin^c structure $\widetilde{P} \to M$ with associated determinant bundle $P \to M$, the parametrized moduli space $\mathcal{M}(\widetilde{P}, h)$ only consists of irreducible solutions and is a (possibly empty) compact smooth $d(P)$-dimensional submanifold of $\mathcal{C}^*(\widetilde{P})/\mathcal{G}(\widetilde{P})$, where $d(P) = \frac{c_1(P)^2 - 2\chi(M) - 3\mathrm{sign}(M)}{4}$.*

Proof: Let U be the intersection of all $U_{\widetilde{P}}$ from Corollary 4.2, where $\widetilde{P} \to M$ runs over the set of spin^c structures on M. Since there are countably many spin^c structures on M, the subset U of $H^{3,2}(\Lambda_+^2 T^*M)$ is generic. By definition of U, for any $h \in U$ and for any spin^c structure $\widetilde{P} \to M$ on M, the parametrized moduli space $\mathcal{M}(\widetilde{P}, h)$ only consists of irreducible solutions and is a (possibly empty) compact smooth $d(P)$-dimensional submanifold of $\mathcal{C}^*(\widetilde{P})/\mathcal{G}(\widetilde{P})$. $\square$

[12] The way to prove compactness is probably the following: show boundedness of the space of solutions in all $H^{k,2}$ topologies, then use a diagonal argument to conclude that the moduli space is sequentially compact; then hope for the quotient topology to be metrizable.

[13] Explain.

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