Talk 7 - The surgery step

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Blockseminar on surgery - Kastell Windsor

August 3, 2012

Abstract: We show how to make a normal map highly connected within a normal cobordism class. The main source is [5, Sec. 3.4].

1 Motivation

Recall that, by definition, given a C^0 real vector bundle $\xi \longrightarrow X$ over a finite n-dimensional connected Poincaré complex and a closed smooth n-dimensional manifold M, a *normal map w.r.t. the tangent bundle of* M is a fibrewise bijective vector bundle homomorphism $(\overline{f}, f) : TM \oplus \underline{\mathbb{R}}^a \longrightarrow \xi$ for some $a \in \mathbb{N}$. As usual, $\underline{\mathbb{R}}^a \longrightarrow X$ denotes the trivial a-ranked real vector bundle over X. A Poincaré complex is a CW-complex satisfying a kind of Poincaré duality, see [5, Def. 3.6].

We are interested in the following

Problem [5, Problem 3.54]: *Given a normal map $(\overline{f}, f)$ from a closed smooth manifold M to a finite n-dimensional connected Poincaré complex X, can one change M and $(\overline{f}, f)$ - but neither ξ nor X - to a normal map $(\overline{f'}, f')$ from a new closed smooth manifold M' to X such that $f' : M' \to X$ is a homotopy equivalence?*

Although we shall not give a full (and firmly positive) answer to that question, we shall show how to make the map f highly connected, that is, "near to" a homotopy equivalence in some sense. This is first done in a naive way by

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attaching a cell to M (Section 2), which is a purely topological construction killing non-trivial elements of the "relative" homotopy groups $\pi_k(f)$ introduced below (see Definition 2.1). Since this construction does not preserve the structure of manifolds, we shall consider the well-known way of attaching cells (then called handles) to manifolds so as to obtain a new manifold: surgery (Section 3). This new construction still kills non-trivial elements of $\pi_k(f)$ but does not take care of the bundles and normal maps we had at the beginning. To succeed, we have to make use of Hirsch and Smale deep results on regular homotopy classes of immersions (Theorem 3.6), see Carolina's part in Section 4.

2 Attaching cells

First we want to indicate how to partially solve the problem on the topological level and without taking care of the bundles or even of the manifold structure. More precisely, the problem we want to solve in this section is the following:

Problem: *Let $f : Y \longrightarrow X$ be a continuous map between CW-complexes. Can one find a new CW-complex Y' and a continuous map $f' : Y' \longrightarrow X$ which is a homotopy equivalence?*

The answer will eventually be "Yes, well, at least f can be made highly connected". First we have to explain what "highly connected" for f means.

2.1 A generalization of relative homotopy groups

To that extent, we introduce the following homotopy groups.

Definition 2.1 *Given a continuous map $f : Y \to X$ between CW-complexes and $k \in \mathbb{N} \setminus \{0\}$, we denote by $\pi_k(f)$[1] the set of all homotopy classes of commutative diagrams of the form*

$$\begin{array}{ccc} \mathbb{S}^{k-1} & \xrightarrow{q} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^k & \xrightarrow{Q} & X, \end{array}$$

where $j : \mathbb{S}^{k-1} = \partial D^k \to D^k$ is the inclusion and q, Q are continuous maps.

[1]not to be confused with the map $\pi_k(Y) \to \pi_k(X)$ induced by f!

With "homotopy classes of commutative diagrams" of the form above, we mean that two such diagrams with maps (q_0, Q_0) and (q_1, Q_1) respectively are homotopic if and only if there exist continuous maps $h : \mathbb{S}^{k-1} \times [0,1] \to Y$ and $H : D^k \times [0,1] \to X$ with $H(\cdot, t) \circ j = f \circ h$ on $\mathbb{S}^{k-1} \times [0,1]$ as well as $h(i, \cdot) = q_i$ and $H(\cdot, i) = Q_i$ for both $i = 0, 1$.

The sets $\pi_k(f)$ can be seen as generalizing the relative homotopy groups when replacing the inclusion map $Y \subset X$ by an arbitrary map $Y \xrightarrow{f} X$. Just as the relative homotopy groups, the set $\pi_k(f)$ has a natural group structure for $k \geq 2$ which is abelian as soon as $k \geq 3$[2]. There is also a long exact sequence available:

$$\ldots \to \pi_k(Y) \to \pi_k(X) \to \pi_k(f) \to \pi_{k-1}(Y) \to \ldots \to \pi_0(X).$$

Important as well is to mention is that there is, at least when Y is path-connected, a natural $\pi_1(Y)$-action on $\pi_k(f)$ for each $k \geq 2$.

Definition 2.2 *A continuous map $f : Y \to X$ between CW-complexes is called* k-connected *for some $k \in \mathbb{N}$ if and only if $\pi_j(f) = 0$ for all $1 \leq j \leq k$ and $\pi_0(Y) \to \pi_0(X)$ is surjective.*

Equivalently, f is k-connected if and only if the induced map $\pi_j(Y) \to \pi_j(X)$ is an isomorphism for all $1 \leq j \leq k-1$ and is surjective for $j = 0, k$.

By Whitehead's theorem (see e.g. [2, Sec. 4.1]), a map $f : Y \to X$ between connected CW-complexes is a homotopy equivalence if and only if it induces an isomorphism $\pi_j(Y) \to \pi_j(X)$ for all $j \geq 1$, that is, if and only if $\pi_j(f) = 0$ for all $j \geq 1$ and $\pi_0(Y) \to \pi_0(X)$ is bijective. The approach to solve our problem is hence to "kill" all homotopy groups $\pi_j(f)$ inductively, the decisive step consisting of annihilating a non-trivial element $\omega \in \pi_k(f)$ by a suitable operation on Y and f. The basic and central concept needed here is that of (topological) *pushout*.

2.2 Pushouts

Definition 2.3 *Given three topological spaces X, Y, Z and two continuous maps $f : X \to Y$ and $g : X \to Z$, the* pushout *of f and g is defined to be the quotient space ${Y \bigsqcup Z}/{f(x) \sim g(x)}$ endowed with the quotient topology.*

[2] and $\pi_0(f)$ is not defined, at least not canonically!

By construction, there exist continuous maps $\iota_Y : Y \to {}^{Y \bigsqcup Z}/_{\sim}$ and $\iota_Z : Z \to {}^{Y \bigsqcup Z}/_{\sim}$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\ Z & \xrightarrow{\iota_Z} & {}^{Y \bigsqcup Z}/_{\sim}. \end{array}$$

Moreover, the pushout satisfies the following universal property: given any topological space W and any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j_Y} \\ Z & \xrightarrow{j_Z} & W \end{array}$$

with continuous maps j_Y, j_Z, there exists a unique continuous map $u : {}^{Y \bigsqcup Z}/_{\sim} \to W$ with $u \circ \iota_Y = j_Y$ and $u \circ \iota_Z = j_Z$. Mind however that the maps ι_Y and ι_Z are not necessarily injective; this is the case if and only if *both* maps f and g are injective.

Example 2.4 (Mapping cylinder) The pushout of a map $f : Y \to X$ and the inclusion $i_0 : Y \to Y \times [0,1]$, $y \mapsto (y, 0)$, is called the *mapping cylinder* of f and is denoted by $\mathrm{cyl}(f)$. It is easy to check that the map $\iota_X : X \to \mathrm{cyl}(f)$ is a homotopy equivalence: there is a well-defined map $p : \mathrm{cyl}(f) \to X$ with $p \circ \iota_X = \mathrm{id}_X$ and such that $\iota_X \circ p$ is homotopic to $\mathrm{id}_{\mathrm{cyl}(f)}$.

2.3 Killing homotopy classes via pushouts

The main result of Section 2 is based on the following

Lemma 2.5 *For $k \geq 1$ let $f \colon Y \to X$ be a k-connected continuous map between CW-complexes and $\omega \in \pi_{k+1}(f)$. Represent ω by a commutative diagram*

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{q} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow{Q} & X \end{array}$$

with continuous maps q, Q. Let Y' be the pushout of q and j and $f' : Y' \to X$ be the unique continuous map with $f' \circ \iota_Y = f$ and $f' \circ \iota_{D^{k+1}} = Q$. Then f' is k-connected and the natural map $\pi_{k+1}(f) \to \pi_{k+1}(f')$ is a surjective group homomorphism containing ω in its kernel.

We say that f' is obtained from f by attaching a $(k+1)$-cell.

Sketch of proof: For every $l \geq 1$, there is a well-defined natural map $\pi_l(f) \to \pi_l(f')$ induced by

$$\begin{array}{ccc} \mathbb{S}^{l-1} & \xrightarrow{\hat{q}} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^l & \xrightarrow{\hat{Q}} & X \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} \mathbb{S}^{l-1} & \xrightarrow{\iota_Y \circ \hat{q}} & Y' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f'} \\ D^l & \xrightarrow{\hat{Q}} & X \end{array} .$$

One can show that this map is actually a group homomorphism which is surjective, at least for $l \leq k+1$. The image of ω is zero: this is due to the fact that the map $D^{k+1} \xrightarrow{Q} X$ lifts through f' to a continuous map $D^{k+1} \xrightarrow{\iota_{D^{k+1}}} Y'$ by construction of the pushout; now $\iota_{D^{k+1}}$ is null-homotopic since D^{k+1} is contractible, hence projecting the homotopy down using f' gives a homotopy of the diagram with the "trivial" one (where the horizontal arrows are constant maps). □

One can actually show that, if $k \geq 2$, then the kernel of the above map $\pi_{k+1}(f) \to \pi_{k+1}(f')$ is exactly the $\mathbb{Z}\pi_1(Y)$-module generated by ω, see [6].

We come to the main result of this section [5, Lemma 3.55].

Proposition 2.6 *Let $f\colon Y \to X$ be a $(k-1)$-connected continuous map between finite CW-complexes for some $k \in \mathbb{N} \setminus \{0\}$.*

1. *If X is connected, $k \geq 2$ and $\pi_1(Y) \to \pi_1(X)$ is bijective, then $\pi_k(f)$ is a finitely generated $\mathbb{Z}\pi_1(Y)$-module.*
2. *The map f can be made k-connected by attaching finitely many cells.*

Sketch of proof:
1. Note that, by assumption, $\pi_0(Y) \to \pi_0(X)$ has to be bijective, therefore Y is also connected. Denote by $\widetilde{Y} \xrightarrow{p_Y} Y$ and $\widetilde{X} \xrightarrow{p_X} X$ the universal coverings of X and Y. Let $\widetilde{f}\colon \widetilde{Y} \to \widetilde{X}$ be a lift of f. Then, for each $l \in \mathbb{N} \setminus \{0\}$, there is a well-defined natural map $\pi_l(\widetilde{f}) \xrightarrow{\psi_l} \pi_l(f)$ induced by

$$\begin{array}{ccc} \mathbb{S}^{l-1} & \xrightarrow{\hat{q}} & \widetilde{Y} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ D^l & \xrightarrow{\hat{Q}} & \widetilde{X} \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} \mathbb{S}^{l-1} & \xrightarrow{p_Y \circ \hat{q}} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^l & \xrightarrow{p_X \circ \hat{Q}} & X \end{array} .$$

For $l \geq 2$, the map ψ_l can be shown to be bijective (and to be a group homomorphism). Namely, using the lifting property of maps and homotopies through coverings, one can construct a map

$$\begin{array}{ccc} \mathbb{S}^{l-1} & \xrightarrow{q} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^l & \xrightarrow{Q} & X \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} \mathbb{S}^{l-1} & \xrightarrow{\widetilde{q}} & \widetilde{Y} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ D^l & \xrightarrow{\widetilde{Q}} & \widetilde{X} \end{array},$$

where $\widetilde{q}$ and $\widetilde{Q}$ are lifts of q and Q through p_Y and p_X respectively. Here one has to pay attention to two things: first, we only consider pointed maps so as to make the lifts unique; second, this lifting property still holds in the case $l = 2$ (where *a priori* not every loop lifts through p_Y) because $\pi_1(X) \to \pi_1(Y)$ is bijective by assumption. One checks that this map induces a group homomorphism $\chi_l \colon \pi_l(f) \to \pi_l(\widetilde{f})$ satisfying $\psi_l \circ \chi_l = \mathrm{id}$ and $\chi_l \circ \psi_l = \mathrm{id}$, in particular ψ_l is a group isomorphism.

Now recall the following version of the Hurewicz theorem, see e.g. [2] for the relative homotopy groups: *If $f : Y \to X$ is a $(k-1)$-connected map with $k \geq 2$ and Y is (non-empty and) 1-connected, then $\pi_l(f)$ is canonically isomorphic to $H_l(f)$ for all $1 \leq l \leq k$.* Here we can assume that (integral) homology groups can be constructed generalizing the usual (integral) relative homology groups just as above for homotopy groups; those are what we denote by $H_l(f)$. In our situation, $\widetilde{Y}$ and $\widetilde{X}$ are 1-connected and $\pi_l(\widetilde{f}) \cong \pi_l(f) = 0$ for all $1 \leq l \leq k-1$, so that Hurewicz theorem applies and provides in particular an isomorphism $\pi_k(\widetilde{f}) \cong H_k(\widetilde{f})$. This shows by the way that, even if $k = 2$, the group $\pi_k(\widetilde{f})$ - and hence $\pi_k(f)$ - has to be abelian. It remains to show that $H_k(\widetilde{f})$ is a finitely generated $\mathbb{Z}\pi_1(Y)$-module, see [6].

2. Making a map 0-connected simply means adding points to Y - one in each path-connected component of X not meeting Y - and extending f by the inclusion of those points in X. This can be done by adding finitely many points since by assumption X is a finite CW-complex.

Assuming f to be 0-connected, it will be 1-connected as soon as the induced map $\pi_0(Y) \to \pi_0(X)$ will be injective and $\pi_1(Y) \to \pi_1(X)$ will be surjective. The injectivity of $\pi_0(Y) \to \pi_0(X)$ can be attained by linking all path-connected components of Y lying in the same path-connected component of X by continuous curves; this can be achieved in a finite number of steps since Y is a finite CW-complex. The surjectivity of $\pi_1(Y) \to \pi_1(X)$ can be attained by attaching at one point loops representing generators of $\pi_1(X)$ to Y. This shows the case $k = 1$.

For $k = 2$, one has to notice that the kernel of $\pi_1(Y) \to \pi_1(X)$ is finitely generated since $\pi_1(Y)$ is finitely generated and $\pi_1(X)$ is finitely presented.

Each element in this kernel can be killed by attaching a 2-dimensional cell, so that we obtain after finitely many such attachments an isomorphism $\pi_1(Y) \to \pi_1(X)$ and can apply part 1 of the proposition. The case $k \geq 3$ follows directly from part 1. □

On the whole, the answer to our problem at the beginning of Section 2 is positive, at least for finite CW-complexes. The major difficulty in our context is that we want to obtain a manifold after attaching cells and the way we did it (using pushouts) obviously destroys the manifold structure in general. Therefore, we have to be more careful and use another way to attach cells. This can be achieved by doing *surgery* on the manifold.

3 Performing surgery

The new problem we are interested in is the following:

Problem: *Let $f : M \longrightarrow X$ be a continuous map from a smooth closed n-dimensional manifold M to a CW-complex X. Can one find a new smooth closed n-dimensional manifold M' and a continuous map $f' : M' \longrightarrow X$ which is a homotopy equivalence?*

The answer will be "Yes, at least if X is a finite CW-complex". The main tool we use is surgery.

3.1 Attaching handles

Definition 3.1 *Let M and M' be two smooth n-dimensional manifolds without boundary and $k \in \{0, \ldots, n-1\}$. We say that M' is obtained from M by* attaching a k-handle *- or by k-dimensional surgery - if and only if there exists a smooth embedding $\mathbb{S}^k \times D^{n-k} \xrightarrow{q} M$ such that*

$$M' \overset{\text{diff.}}{\cong} \left(M \setminus \overset{\circ}{\operatorname{im}}(q)\right) \bigsqcup_q \left(D^{k+1} \times \mathbb{S}^{n-k-1}\right),$$

where $\overset{\circ}{\operatorname{im}}(q)$ denotes the interior of the range of q in M and the symbol "$\bigsqcup_q$" means we glue the topological space $D^{k+1} \times \mathbb{S}^{n-k-1}$ to $M \setminus \overset{\circ}{\operatorname{im}}(q)$ along $q|_{\mathbb{S}^k \times \mathbb{S}^{n-k-1}}$.

The glueing process is made possible thanks to the identities $\partial(\mathbb{S}^k \times D^{n-k}) = \mathbb{S}^k \times \mathbb{S}^{n-k-1} = \partial(D^{k+1} \times \mathbb{S}^{n-k-1})$. It can be easily shown that, if a k-handle is attached to M, then the new space M' has a unique structure of topological manifold such that both inclusions from $D^{k+1} \times \mathbb{S}^{n-k-1}$ and $M \setminus \overset{\circ}{\mathrm{im}(q)}$ into M' are continuous embeddings. With a bit of work ("straightening the angles" is the name of the method, meaning we smooth out along $\partial \mathrm{im}(q)$) one can even prove the existence of a unique smooth structure on M' such that both embeddings above are smooth. Moreover, if M is closed (resp. oriented), then so is M'.

Note that, by construction, if M' is obtained from M by attaching a k-handle, then conversely M is obtained from M' by attaching a $(n-k-1)$-handle. A less trivial but still central property of attaching handles is that it preserves the cobordism class. Namely, if M' is obtained from a closed smooth manifold M by attaching a k-handle, then the space

$$W := \big(M \times [0,1]\big) \bigsqcup_q \big(D^{k+1} \times D^{n-k}\big)$$

obtained by attaching the "full" handle $D^{k+1} \times D^{n-k}$ to $M \times \{1\}$ along q, is a compact manifold with boundary $M \bigsqcup M'$. In case M is oriented, the manifolds M' and W can be given an orientation such that $\partial W = -M \bigsqcup M'$, where $-M$ denotes the manifold M with the opposite orientation.

What is the effect of attaching handles on the homotopy groups?

Lemma 3.2 *For $k \in \{0, \ldots, [\frac{n-2}{2}]\}$ let $f\colon M \to X$ be a k-connected continuous map from a closed smooth n-dimensional manifold M to a CW-complex X. Let $\omega \in \pi_{k+1}(f)$ be represented by a commutative diagram*

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{\;q\;} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow{\;Q\;} & X \end{array}$$

with continuous maps q, Q. Assume that q can be extended to a smooth embedding $\mathbb{S}^k \times D^{n-k} \xrightarrow{\overline{q}} M$ and let M' be the closed smooth n-dimensional manifold obtained by attaching a k-handle along $\overline{q}$. Then f induces a k-connected map $f'\colon M' \to X$ which is bordant to f and such that the induced map $\pi_{k+1}(f) \to \pi_{k+1}(f')$ is surjective and contains ω in its kernel.

Proof: First, there exists a continuous map $\overline{Q}\colon D^{k+1} \times D^{n-k} \to X$ making the following diagram

$$\begin{array}{ccc} \mathbb{S}^k \times D^{n-k} & \xrightarrow{\overline{q}} & M \\ {\scriptstyle j\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} \times D^{n-k} & \xrightarrow{\overline{Q}} & X \end{array}$$

commute and such that $\overline{Q}_{|_{D^{k+1}\times\{0\}}} = Q$: indeed there exists a continuous map $D^{k+1}\times D^{n-k} \xrightarrow{\varphi} D^{k+1}\times D^{n-k}$ with $\varphi_{|_{D^{k+1}\times\{0\}}} = \mathrm{id}$ and $\varphi_{|_{\mathbb{S}^k\times D^{n-k}}} = \mathrm{id}$ ("compress" the interior of the cylinder $D^{k+1}\times D^{n-k}$ onto the "walls" $\mathbb{S}^k\times D^{n-k}$ and the "bottom" $D^{k+1}\times\{0\}$); now just consider the map obtained by composing $\overline{q}$ and Q by ϕ (which is then defined and continuous on $D^{k+1}\times D^{n-k}$).[3] Define $f'\colon M' \to X$ by $f'_{|_{M\setminus \mathrm{im}(\overset{\circ}{\overline{q}})}} := f$ and $f'_{|_{D^{k+1}\times\mathbb{S}^{n-k-1}}} := \overline{Q}_{|_{D^{k+1}\times\mathbb{S}^{n-k-1}}}$. Observe that f' is well-defined (and smooth) since by construction $f\circ\overline{q} = \overline{Q}\circ(j\times\mathrm{id})$. Moreover, if W is the cobordism between M and M' described above, then the assignments $F_{|_{M\times[0,1[}} := f\circ p_1$, $F_{|_{(M\setminus\mathrm{im}(\overset{\circ}{\overline{q}}))\times\{1\}}} := f$ and $F_{|_{D^{k+1}\times D^{n-k}}} := \overline{Q}$ define a (continuous) map $F\colon W\to X$ with $F_{|_{M\times\{0\}}} = f$ and $F_{|_{M'}} = f'$. Hence f and f' are cobordant. Next notice that the inclusion map $M\setminus \mathrm{im}(\overset{\circ}{q}) \subset M$ is $(n-k-1)$-connected. This follows from an excision argument in homology, from the inclusion map $\mathbb{S}^k\times\mathbb{S}^{n-k-1}\subset\mathbb{S}^k\times D^{n-k}$ being $(n-k-1)$-connected (which itself follows from $D^{n-k}/\mathbb{S}^{n-k-1} = \mathbb{S}^{n-k}$ being $(n-k-1)$-connected) and from the Hurewicz theorem. Using a long exact homotopy sequence for both maps $M\setminus \mathrm{im}(\overset{\circ}{q}) \xrightarrow{f} X$ and $M \xrightarrow{f} X$ and the five lemma (still applicable in the case where $j = n-k-1$), this implies that, for all $j\in\{1,\dots,n-k-1\}$, one has $\pi_j(f)\cong\pi_j(f_{|_{M\setminus\mathrm{im}(\overset{\circ}{q})}})$. Similarly, since M is obtained from M' by attaching a $(n-k-1)$-handle, we have $\pi_j(f')\cong\pi_j(f'_{|_{M\setminus\mathrm{im}(\overset{\circ}{q})}})$ for all $1\le j\le n-(n-k-1)-1 = k$; but since by construction $f'_{|_{M\setminus\mathrm{im}(\overset{\circ}{q})}} = f_{|_{M\setminus\mathrm{im}(\overset{\circ}{q})}}$, we deduce that $\pi_j(f)\cong\pi_j(f')$ for all $1\le j\le \min(k, n-k-1)$. Moreover, using an argument analogous as the one we used for the pushout, there is a natural (and surjective) group homomorphism $\pi_{k+1}(f)\to\pi_{k+1}(f')$ and ω lies in its kernel. All in all, we see that, if $k\le\frac{n-1}{2}$[4], then the above glueing does not modify $\pi_j(f)$ for all $1\le j\le k$ while it kills ω. □

The first question Lemma 3.2 raises is whether $q\colon\mathbb{S}^k\to M$ can be extended to an embedding $\overline{q}\colon\mathbb{S}^k\times D^{n-k}\to M$, or if, at least, there exists such a q. Note that q has to be itself an embedding then. Now for a smooth embedding

[3] Thanks to Matthias Blank for explaining this to me.

[4] problem with the bound, it should be $k\le\frac{n}{2}-1$.

$q\colon \mathbb{S}^k \to M$, it is easy to show that q extends to a smooth embedding $\overline{q}\colon \mathbb{S}^k \times D^{n-k} \to M$ if and only if its normal bundle $\nu(q) \to \mathbb{S}^k$ is trivial (where $\nu(q)$ is the quotient vector bundle ${q^*TM}/{T\mathbb{S}^k}$): the reason is that the total space $\nu(q)$ is diffeomorphic (by a diffeomorphism preserving fibres) to a fibre bundle with fibre D^{n-k}, the latter bundle being diffeomorphic to a tubular neighbourhood of $q(\mathbb{S}^k)$ in M.
In our situation, any map $q\colon \mathbb{S}^k \to M$ can be approximated (in the C^0-topology) by an embedding as soon as $2k < n$ in virtue of the Whitney embedding theorem (see e.g. [1]):

Theorem 3.3 (H. Whitney) *Let M^m and N^n be smooth manifolds and $f\colon M \to N$ be a continuous map.*

i) If $2m \leq n$, then every C^0-neighbourhood of f contains an immersion .

ii) If $2m < n$, then every C^0-neighbourhood of f contains an embedding.

The main remark now is that, if $q\colon \mathbb{S}^k \to M$ is an embedding with $2k < n$, then q extends to an embedding $\overline{q}\colon \mathbb{S}^k \times D^{n-k} \to M$ as soon as it comes from a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{q} & M \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow{Q} & X \end{array}$$

where f is the base map of a *normal map* $(\overline{f}, f)$: for then we have

$$T\mathbb{S}^k \oplus \nu(q) \oplus \underline{\mathbb{R}}^a \cong q^*TM \oplus \underline{\mathbb{R}}^a \cong q^*(f^*\xi) = (f \circ q)^*\xi \cong j^*(Q^*\xi)$$

for some $a \in \mathbb{N}$. But $Q^*\xi \to D^{k+1}$ is trivial since D^{k+1} is contractible, so that $j^*(Q^*\xi)$ is trivial, that is, $T\mathbb{S}^k \oplus \nu(q) \oplus \underline{\mathbb{R}}^a$ is trivial. Since $TS^k \oplus \underline{\mathbb{R}}$ is trivial (the normal bundle of the canonical embedding $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ is obviously trivial), we deduce that $\nu(q) \oplus \underline{\mathbb{R}}^{a+k+1} \to \mathbb{S}^k$ is trivial. The following lemma [4, Lemma 3.5] together with $n-k > k$ implies that $\nu(q) \to \mathbb{S}^k$ itself is trivial - and that therefore q extends to an embedding $\overline{q}\colon \mathbb{S}^k \times D^{n-k} \to M$.

Lemma 3.4 *Let $E \to X$ be a real n-ranked vector bundle over an k-dimensional CW-complex X. Assume $n > k$. Then $E \to X$ is trivial if and only if $E \oplus \underline{\mathbb{R}}^r \to X$ is trivial for some $r \in \mathbb{N}$.*

Proof: Assume $E \oplus \underline{\mathbb{R}}^r \to X$ to be trivial. Note that showing the case $r = 1$ suffices (then prove the result by induction over r). First, if $E \oplus \underline{\mathbb{R}}^1 \to X$ is trivial, then the first Stiefel-Whitney class $w_1(E) = w_1(E \oplus \underline{\mathbb{R}}^1) = 0$, so that $E \to X$ is orientable. Fix an orientation on $E \to X$ and an orientation-preserving isomorphism $\phi : E \oplus \underline{\mathbb{R}}^1 \to \underline{\mathbb{R}}^{n+1} = X \times \mathbb{R}^{n+1}$. Denoting by $\widetilde{\gamma}^n(\mathbb{R}^{n+1}) \to \widetilde{G}_n(\mathbb{R}^{n+1})$ the oriented universal bundle over the Grassmannian of oriented n-planes in $\mathbb{R}^{n+1}$, the map $\psi\colon X \to \widetilde{G}_n(\mathbb{R}^{n+1})$, $x \mapsto \phi(E_x)$, is well-defined, continuous and obviously pulls $\widetilde{\gamma}^n(\mathbb{R}^{n+1}) \to \widetilde{G}_n(\mathbb{R}^{n+1})$ back to $E \to X$. Now $\widetilde{G}_n(\mathbb{R}^{n+1})$ is diffeomorphic to $\widetilde{G}_1(\mathbb{R}^{n+1})$, which itself is diffeomorphic to $\mathbb{S}^n$. Because of $n > k$, the map $\psi\colon X \to \mathbb{S}^n$ is necessarily null-homotopic, therefore $E = \psi^*\widetilde{\gamma}^n(\mathbb{R}^{n+1}) \to X$ is isomorphic to the trivial vector bundle $\underline{\mathbb{R}}^n \to X$. □

The next question is whether the map f' obtained after adding a k-handle is still a normal map. This is not the case in general, however it can be made possible after having a closer look to the connection between immersions and their differential. This is explained below.

3.2 Regular homotopy classes of immersions

Recall first the following

Definition 3.5 *Let M and N be smooth manifolds.*

i) Two immersions $f_i\colon M \to N$, $i = 0, 1$, are called regularly homotopic *if and only if there exists a smooth map $h\colon M \times [0,1] \to N$ with $h(\cdot, i) = f_i$ for $i = 0, 1$ and each $h(\cdot, t)\colon M \to N$ is an immersion.*

ii) Let $E \to M$ and $F \to N$ be smooth vector bundles and $(\overline{f}_i, f_i)\colon E \to F$ be smooth vector-bundle-monomorphisms (i.e., fibrewise injective vector-bundle-homomorphisms). Then $(\overline{f}_0, f_0)$ and $(\overline{f}_1, f_1)$ are called bundle homotopic *if and only if there exists a pair $(\overline{h}, h)$ of smooth maps making the following diagram commute*

$$\begin{array}{ccc} E \times [0,1] & \xrightarrow{\overline{h}} & F \\ {\scriptstyle \pi_E \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M \times [0,1] & \xrightarrow{h} & N \end{array}$$

with $\overline{h}(\cdot, i) = \overline{f}_i$, $i = 0, 1$ and $\overline{h}(\cdot, t)\colon E \to F$ is a vector-bundle-monomorphism for every $t \in [0,1]$.[5]

[5]In particular, h is a (smooth) homotopy between f_0 and f_1.

Regular homotopies and bundle homotopies define equivalence relations. We denote by $\pi_0(\mathrm{Imm}(M,N))$ the set of regular homotopy classes of immersions from M into N and by $\pi_0(\mathrm{Mono}(E,F))$ that of bundle homotopy classes of monomorphisms of E to F[6].

It is a striking fact due to Hirsch and Smale that the sets $\pi_0(\mathrm{Imm}(M,N))$ and $\pi_0(\mathrm{Mono}(TM,TN))$ correspond (see [5, Sec. 3.4.2] for references):

Theorem 3.6 (Hirsch and Smale) *Let M^m be an m-dimensional closed smooth manifold and N^n be an n-dimensional smooth manifold.*

1. *If $1 \leq m < n$, then taking the differential yields a bijection*

$$T\colon \pi_0(\mathrm{Imm}(M,N)) \longrightarrow \pi_0(\mathrm{Mono}(TM,TN)).$$

2. *If $1 \leq m \leq n$ and M has a handlebody decomposition consisting solely of k-handles with $k \leq n-2$. Then taking the differential yields a bijection*

$$T\colon \pi_0(\mathrm{Imm}(M,N)) \longrightarrow \varinjlim_{a\to\infty} \pi_0(\mathrm{Mono}(TM \oplus \underline{\mathbb{R}}^a, TN \oplus \underline{\mathbb{R}}^a)).$$

Before returning to our original problem, we give two applications of Theorem 3.6.

Examples 3.7

1. We first claim that $\pi_0(\mathrm{Imm}(\mathbb{S}^2,\mathbb{R}^3)) = 0$. Note first that this is an amazing fact since it implies - among others - that a so-called *sphere eversion* exists: one can turn $\mathbb{S}^2$ "inside out" using a regular homotopy! Namely, if ι_0 is the canonical inclusion $\mathbb{S}^2 \subset \mathbb{R}^3$ and $\iota_1 := \iota \circ I$, where $I := -\mathrm{id}_{\mathbb{S}^2}$ is the point symmetry about 0, then there must exist a regular homotopy from ι_0 to ι_1 - though ι_1 is orientation-reversing!
   To prove the claim, we show that any two bundle-monomorphisms $(\overline{f}_i, f_i)$, $i = 0, 1$, from $T\mathbb{S}^2$ to $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ are bundle-homotopic; Theorem 3.6 will imply the result. Given any two such bundle-monomorphisms, we first notice that, since $\mathbb{S}^2$ is orientable and $\overline{f}_i$ is fibrewise injective, the bundle $\overline{f}_i(T\mathbb{S}^2)^\perp \to \mathbb{S}^2$ is trivial and $\overline{f}_i$ can be extended to orientation-preserving bundle-monomorphisms $\overline{g}_i\colon T\mathbb{S}^2 \oplus$

[6] The notation is chosen so as to coincide with that of path-connected components of the spaces $\mathrm{Imm}(M,N)$ and $\mathrm{Mono}(E,F)$ respectively - provided those are endowed with the right topology!

$\underline{\mathbb{R}} \to \mathbb{R}^3 \times \mathbb{R}^3$. On the other hand, since $\mathbb{R}^3$ is contractible, both f_i are homotopic (though not regularly homotopic) to the constant map $c_0\colon \mathbb{S}^2 \to \mathbb{R}^3$, $x \mapsto 0 \in \mathbb{R}^3$, in particular $\overline{g}_i$ is bundle-homotopic to a bundle-monomorphism $T\mathbb{S}^2 \oplus \underline{\mathbb{R}} \xrightarrow{G_i} \{0\} \times \mathbb{R}^3$. Consider now the map $u : \mathbb{S}^2 \to \mathrm{GL}(3, \mathbb{R})$ defined by $u(x) := G_1(x) \circ G_0(x)^{-1}$. Since both $\overline{g}_i$ and hence G_i are orientation-preserving, the map u actually takes its values in $\mathrm{GL}^+(3, \mathbb{R})$, which is known to be homotopy-equivalent to $\mathrm{SO}(3, \mathbb{R})$. But $\pi_2(\mathrm{SO}(3, \mathbb{R})) \cong \pi_2(\widetilde{\mathrm{SO}(3, \mathbb{R})}) \cong \pi_2(\mathbb{S}^3) = 0$, so that u has to be homotopic to a constant map. Therefore, G_0 and G_1 are bundle-homotopic and hence $\overline{g}_0$ and $\overline{g}_1$ are bundle-homotopic; by restriction onto $T\mathbb{S}^2$, the vector-bundle-monomorphisms $\overline{f}_0$ and $\overline{f}_1$ are hence bundle-homotopic.

2. We claim that[7], given $n \geq 1$,

$$\pi_0(\mathrm{Imm}(\mathbb{S}^n, \mathbb{S}^{2n})) \cong \pi_0(\mathrm{Imm}(\mathbb{S}^n, \mathbb{R}^{2n})) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \text{ is even.} \end{cases}$$

The first isomorphism is obvious since any immersion $\mathbb{S}^n \to \mathbb{S}^{2n}$ cannot be surjective and hence can be regularly homotoped to an immersion $\mathbb{S}^n \to \mathbb{S}^{2n} \setminus \{e_{n+1}\}$, the latter space being diffeomorphic to $\mathbb{R}^{2n}$. To see how the second isomorphism is contructed, consider again two vector-bundle-monomorphisms $(\overline{f}_i, f_i)$ from $T\mathbb{S}^n$ to $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Fixing a vector-bundle-isomorphism $T\mathbb{S}^n \oplus \mathbb{R} \xrightarrow{\phi} \underline{\mathbb{R}}^{n+1}$, we obtain two vector-bundle-monomorphisms

$$\underline{\mathbb{R}}^{n+1} \xrightarrow{\phi^{-1}} T\mathbb{S}^n \oplus \underline{\mathbb{R}} \xrightarrow{\overline{f}_i \oplus (f_i \times \mathrm{id}_{\mathbb{R}})} \mathbb{R}^{2n} \times \mathbb{R}^{2n+1}$$

and hence smooth maps $\mathbb{S}^n \to V_{2n+1,n+1} := V_{n+1}(\mathbb{R}^{2n+1})$, where $V_k(\mathbb{R}^l)$ denotes the Stiefel manifold of all k-frames in $\mathbb{R}^l$. Now it can be shown that $(\overline{f}_0, f_0)$ and $(\overline{f}_1, f_1)$ are bundle-homotopic if and only if the corresponding maps $\mathbb{S}^n \xrightarrow{u_i} V_{2n+1,n+1}$ are homotopic.[8] Thus we are reduced to finding all homotopy classes of maps $\mathbb{S}^n \to V_{2n+1,n+1}$ and hence to computing the homotopy groups $\pi_n(V_{2n+1,n+1})$. The first step is to prove that $\pi_n(V_{2n+1,n+1}) \cong \pi_n(V_{2n,n})$: this follows by considering the long exact homotopy sequence associated to the fibration

$$V_{2n,n} = {}^{\mathrm{O}_{2n}}\!/_{\mathrm{O}_n} \hookrightarrow {}^{\mathrm{O}_{2n+1}}\!/_{\mathrm{O}_n} = V_{2n+1,n+1} \twoheadrightarrow {}^{\mathrm{O}_{2n+1}}\!/_{\mathrm{O}_{2n}} = \mathbb{S}^{2n},$$

[7] Thanks to Diarmuid Crowley for explaining part of this result to me.
[8] Why?

for $\pi_k(\mathbb{S}^{2n}) = 0$ for $k \leq n+1$ as soon as $n > 1$. The next step is to consider the long exact homotopy sequence associated to the fibration

$$V_{n,n} = \mathrm{O}_n \hookrightarrow \mathrm{O}_{2n} = V_{2n,2n} \twoheadrightarrow {}^{\mathrm{O}_{2n}}\!/_{\mathrm{O}_n} = V_{2n,n}.$$

See [6] for the rest of the proof.

4 Performing surgery and carrying bundles

References

[1] T. Bröcker, K. Jänich, *Einführung in die Differentialtopologie*, Heidelberger Taschenbücher **143**, Springer-Verlag, Berlin-New York, 1973.

[2] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[3] M.W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276.

[4] M. Kervaire, J. Milnor, *Groups of homotopy spheres I*, Ann. of Math. (2) **77** (1963), 504–537.

[5] W. Lück, *A basic introduction to surgery theory*, Topology of high-dimensional manifolds, no. 1, 2 (Trieste, 2001), 1–224, ICTP Lect. Notes **9**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, available at `http://131.220.77.52/lueck/data/ictp.pdf`.

[6] *The Manifold Atlas Project, Category: Exercises*, available at `http://www.map.him.uni-bonn.de/Category:Exercises`.