# The Dirac spectrum

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*Dedicated to my dear and loving mother*

# Preface

This overview is based on the talk [105] given at the mini-workshop 0648c *"Dirac operators in differential and non-commutative geometry"*, Mathematisches Forschungsinstitut Oberwolfach. Intended for non-specialists, it draws up a panorama about the spectrum of the fundamental Dirac operator on Riemannian spin manifolds, including recent research and open problems. No spin geometrical background is required, nevertheless the reader is assumed to be familiar with basic notions of differential geometry (manifolds, Lie groups, vector and principal bundles, coverings, connections, differential forms). The starting point was the surveys [45, 138], which already provide a very good insight for closed manifolds. We hope the content of this book reflects the wide range of results and sometimes amazing applications of the spin side of spectral theory and attracts a new audience to the topic.

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# Contents

# Introduction

*"Find a first order linear differential operator on* $\mathbb{R}^n$ *whose square coincides with the Laplace operator* $-\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$*."* Give this as exercise to a group of undergraduates. If they can solve it for $n = 1$ then they have heard of complex numbers. If they can do it for $n \geq 2$ then either they believe to have solved it, or they claim to be students, or they know about Dirac.

For this simple-minded question and its rather involved answer lie at the origin of the whole theory of Dirac operators. It was P. Dirac who introduced [84] the operator now bearing his name when looking for an equation describing the probability amplitude of spin-$\frac{1}{2}$-particles (fermions, e.g. electrons) and that would fit into the framework of both special relativity and quantum mechanics. Mathematically formulated, his problem consisted in finding a square root of the Klein-Gordon operator (d'Alembert plus potential) on the 4-dimensional Minkowski spacetime. It already came as a breakthrough when Dirac showed that the problem could be solved not for the scalar operator but for the $\mathbb{C}^2$-valued one using the so-called Pauli matrices as coefficients.

Like many objects invented by physicists, the Dirac operator was soon called upon to develop an own mathematical life. It was indeed later discovered that the setup of Clifford algebras allowed it to be defined in a general geometrical framework on "almost any" smooth semi-Riemannian manifold. Here "almost any" means that there exists a topological restriction on the manifold for the Dirac operator to be well-defined - the spin condition, see Chapter 1 - which is however satisfied on most "known" manifolds. This mathematical investigation gave birth to spin geometry. One of the first and probably most famous achievements of spin geometry was the discovery of a topological obstruction to positive scalar curvature as a relatively straightforward application of the Atiyah-Singer-Index theorem, see Chapter 3 and references therein.

It would be very modest to claim that spin geometry has remained lively since. Less than twenty years after Atiyah and Singer's breakthrough, the whole mathematical community could only gape at E. Witten's amazingly simple proof of the positive mass theorem based on the analysis of a Dirac-type operator of a bounding hypersurface [241]. At the same time, noncommutative geometry made the Dirac operator one of its keystones as it allows to reconstruct a given Riemannian spin manifold from its so-called canonical spectral triple [81, 115]. Independently, special eigenvectors of the Dirac operator called Killing spinors have become some of the physicists' main tools in the investigation of supersymmetric models for string theory in dimension 10, see e.g. [190]. In a more geometrical context, Dirac-type operators have been successfully applied in as varied situations as finding obstructions to minimal Lagrangian embeddings [145], rigidity issues in extrinsic geometry [142] or the Willmore conjecture [14, 19], just to cite a few of them.

Exploring the spin geometrical aspects of all the above-mentioned topics would require a small encyclopedia, therefore we focus on a particular one. Out of lack of up-to-date literature on the subject, we choose to deal in this book with the spectrum of the Dirac operator on complete (mainly compact) Riemannian spin manifolds with or without boundary. In particular we do not intend to give any kind of extensive introduction to spin geometry, see [66, 91] and the mother-reference [178] in this respect (the physics-oriented reader may prefer [233]). Since it was not possible to handle all facets of the Dirac spectrum in one volume, we had to leave some of them aside. To keep the book as self-contained as possible, we sketch those briefly in the last chapter.

We begin with introducing the Dirac operator and its geometrical background. Although the definition is rather involved, we try to remain as simple as possible so as not to drown the reader in technical considerations such as representation theory of Clifford algebras or the topological spin condition. In Chapter 1 we define the spin group, spin structures on manifolds, spinors (which are sections of a vector bundle canonically attached to manifolds carrying a spin structure) and the Dirac operator acting on spinors. We show that the Dirac operator is an elliptic, formally self-adjoint linear differential operator of first order and, if the underlying Riemannian manifold is furthermore complete, then it is essentially self-adjoint in $\mathrm{L}^2$. In particular, if the manifold is closed, then the spectrum of its Dirac operator is well-defined, real, discrete and unbounded. In case the boundary of the manifold is non-empty, elliptic boundary conditions have to be precised for the spectrum to be well-defined and discrete.
At this point we underline that only a so-called spin$^c$ structure is needed on the manifold in order for the Dirac operator to be well-defined. Spin$^c$ structures require weaker topological assumptions to exist than spin structures. Since however their treatment would bring us too far, we choose to ignore them in this book (see Section 8.4 for references).

The second chapter deals with examples of closed manifolds whose Dirac spectrum - or at least some eigenvalues - can be explicitly computed. They all belong to the class of homogeneous spaces, for which we recall the representation-theoretical method allowing one to describe the Dirac operator as a family of matrices, see Theorem 2.2.1.

Since it would be illusory to aim at the explicit knowledge of the Dirac spectrum in general, one way for studying it consists in estimating the eigenvalues. In Chapter 3 we consider an arbitrary closed Riemannian spin manifold and describe the main lower bounds that have been proved for its Dirac spectrum. Almost all of them rely on the Schrödinger-Lichnerowicz formula (1.15) and thus involve the scalar curvature of the manifold. Starting from the most general estimate - Friedrich's inequality (3.1) - we show how it can be improved in some particular cases. The equality-case of most of those inequalities is characterized by the existence of special sections (e.g. Killing spinors) which give rise to interesting geometrical features. We shift the treatment of some of them to Appendix A since they are of independent interest.

In the situation where the manifold has a non-empty boundary, we consider four different boundary conditions, two of which generalize those originally introduced by Atiyah, Patodi and Singer [30]. We describe in Chapter 4 the corresponding lower bounds *à la Friedrich* that have been obtained in this context.

The techniques involved for proving lower bounds drastically differ from those used in getting upper eigenvalue bounds. In the latter case - and if the manifold is closed - there exist two methods available for the Dirac operator, the first one based on index theory and the second one on the min-max principle. Chapter 5 collects the different geometrical upper bounds that have been proved with the help of those, separating the intrinsic - depending on the intrinsic geometry only - from the extrinsic ones, i.e., depending on some map from the manifold into another one.

In Chapter 6, we turn to the closely related issues of isospectrality and prescription of eigenvalues. In a first part, we discuss isospectrality results obtained on spaceforms of non-negative curvature and on circle bundles. Turning to the eigenvalue 0, we detail in Section 6.2 existence as well as non-existence results for harmonic spinors, i.e., sections lying in the kernel of the Dirac operator. Here there is a remarkable difference between dimensions 2 and greater than 2. We end this chapter with a brief account on how the lower part of the spectrum can always be prescribed provided it does not contain 0.

On non-compact Riemannian spin manifolds another part of the spectrum beside the eigenvalues must be taken into account, the so-called continuous spectrum. For the Dirac operator it is well-defined as soon as the underlying Riemannian manifold is complete, however the square of the Dirac operator always has a spectrum (see Section 7.1). Only few examples are known where the whole Dirac spectrum can be computed. In Chapter 7 we mainly discuss the interactions between the geometry or topology of the manifold with the Dirac spectrum, in particular we focus on whether it can be purely discrete or continuous.

# Chapter 1

# Basics of spin geometry

In this chapter we define spin structures, spinors, the Dirac operator and discuss the properties we need further on. Unless explicitly mentioned all objects (manifolds, bundles, sections) will be assumed smooth in the whole survey. For the thorough treatment of spin or spin$^c$ groups, spin or spin$^c$ structures on vector bundles, representation theory of Clifford algebras and Dirac operators on arbitrary semi-Riemannian Clifford modules we refer to [66, 91, 178].

## 1.1 Spin group and spin structure

**Definition 1.1.1** *Let $n$ be a positive integer. The* spin group *in dimension $n$, denoted by $\mathrm{Spin}_n$, is the non-trivial 2-fold covering of the special orthogonal group $\mathrm{SO}_n$.*

The spin group in dimension $n$ is a compact $\frac{n(n-1)}{2}$-dimensional Lie group, connected if $n \geq 2$ and simply-connected if $n \geq 3$. In fact, if $\mathrm{Spin}_n \xrightarrow{\xi} \mathrm{SO}_n$ denotes this non-trivial covering map, then $\xi(z) = z^2$ for any $z \in \mathrm{Spin}_2 \cong \mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$ and $\xi$ is the universal covering map if $n \geq 3$. In particular the spin group provides the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \xrightarrow{\xi} \mathrm{SO}_n \longrightarrow 1,$$

where we identify $\{\pm 1\}$ to $\mathbb{Z}_2$.
From now on we denote by $\mathrm{SO}(TM) \longrightarrow M$ the $\mathrm{SO}_n$-principal bundle of positively oriented orthonormal frames on the tangent bundle of an oriented Riemannian manifold $(M^n, g)$.

**Definition 1.1.2** *Let $n \in \mathbb{N} \setminus \{0\}$.*

*i) A* spin structure *on an oriented Riemannian manifold $(M^n, g)$ is a $\mathrm{Spin}_n$-principal bundle $\mathrm{Spin}(TM) \longrightarrow M$ together with a 2-fold covering map $\mathrm{Spin}(TM) \xrightarrow{\eta} \mathrm{SO}(TM)$ compatible with the respective group actions, i.e.,* *the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Spin}(TM)\times\mathrm{Spin}_n & \longrightarrow & \mathrm{Spin}(TM) & & \\ \Big\downarrow{\scriptstyle \eta\times\xi} & & \Big\downarrow{\scriptstyle \eta} & \searrow & \\ & & & & M \\ & & & \nearrow & \\ \mathrm{SO}(TM)\times\mathrm{SO}_n & \longrightarrow & \mathrm{SO}(TM) & & \end{array}$$

*ii) A* spin manifold *is an oriented Riemannian manifold admitting a spin structure.*

A spin structure is a reduction of the bundle of oriented orthonormal frames to the spin group. Not every oriented Riemannian manifold admits a spin structure, the condition for its existence being of topological nature.

**Proposition 1.1.3** *An oriented Riemannian manifold $(M^n, g)$ is spin if and only if the second Stiefel-Whitney class of its tangent bundle vanishes. If this condition is fulfilled, then the set of spin structures on $(M^n, g)$ stands in one-to-one correspondence with $H^1(M, \mathbb{Z}_2)$.*

In other words, a manifold is spin if and only both its first and second Stiefel-Whitney classes vanish (the vanishing of the first one being equivalent to the orientability of the manifold). This explains why the spin condition is sometimes presented as an orientability condition of second order.

*Proof of* Proposition 1.1.3: First recall that, for any Lie group $G$, there exists a bijection between the set of equivalence classes of $G$-principal bundles over some manifold $N$ and the set $H^1(N, G)$ (which, if $G$ is abelian, is the first Čech-cohomology group with coefficients in $G$), see e.g. [178, App. A]. In particular the set of two-fold coverings of - i.e., of $\mathbb{Z}_2$-bundles over - $N$ can be identified with $H^1(N, \mathbb{Z}_2)$.
For $n = 1$ the result is a trivial consequence of this observation, since in that case $H^2(M, \mathbb{Z}_2) = 0$ and a spin structure is a 2-fold covering of the manifold itself, hence $\mathbb{R}$ has exactly one and the circle 2 spin structures, see also Example 1.4.3.1 below for a more precise description.
Assume for the rest of the proof $n \geq 2$. First note that, from their definition, the spin structures on $(M^n, g)$ coincide with the 2-fold coverings of $\mathrm{SO}(TM)$ which are non-trivial on each fibre of the projection map $\mathrm{SO}(TM) \longrightarrow M$. This follows essentially from the standard lifting property of maps through coverings. From the remark above, the set of 2-fold coverings of $\mathrm{SO}(TM)$ can be identified with $H^1(\mathrm{SO}(TM), \mathbb{Z}_2)$. Now the second Stiefel-Whitney class of $TM$ can be defined as follows: the fibration $\mathrm{SO}_n \xrightarrow{\iota} \mathrm{SO}(TM) \xrightarrow{\pi} M$ induces the following short exact sequence of groups

$$0 \longrightarrow H^1(M, \mathbb{Z}_2) \xrightarrow{\pi^*} H^1(\mathrm{SO}(TM), \mathbb{Z}_2) \xrightarrow{\iota^*} H^1(\mathrm{SO}_n, \mathbb{Z}_2) \xrightarrow{w} H^2(M, \mathbb{Z}_2)$$

and the second Stiefel-Whitney class of $TM$ is the image under $w$ of the non-trivial element of $H^1(\mathrm{SO}_n, \mathbb{Z}_2) \cong \mathbb{Z}_2$. The spin structures on $(M^n, g)$ can therefore be reinterpreted as the elements in $H^1(\mathrm{SO}(TM), \mathbb{Z}_2)$ with non-zero image under $\iota^*$. In particular $(M^n, g)$ is spin if and only if such an element exists, that is, if and only if $\iota^*$ is surjective. From $w \circ \iota^* = 0$ this is equivalent to $w = 0$, i.e., to the vanishing of the second Stiefel-Whitney class of $TM$. This proves the first statement. If $(M^n, g)$ is spin then the set of its spin structures identifies through the above exact sequence with the non-zero coset in $H^1(\mathrm{SO}(TM), \mathbb{Z}_2)/_{\pi^*(H^1(M,\mathbb{Z}_2))}$, which has the same cardinality as $H^1(M, \mathbb{Z}_2)$ itself. This concludes the proof. □

**Notes 1.1.4**

1. In particular the existence of a spin structure does not depend on the metric or the orientation of a given manifold. Actually, if the manifold $M$ is oriented, spin structures can be defined independently of any metric (declare them to be non-trivial 2-fold coverings of the bundle of oriented frames of $TM$). In an equivalent way, a spin structure for a given metric canonically induces a spin structure for another one. For a detailed discussion of this point we refer to [178, Chap. 2], [10] and to [65].

2. Not every orientable manifold is spin. On surfaces the spin condition is equivalent to the vanishing of the mod 2 reduction of the Euler class, thus is fulfilled for orientable surfaces. In dimension 3, the second Stiefel-Whitney class is the square of the first one [178, p.86], hence any 3-dimensional orientable manifold is spin. The simplest counter-example comes up in dimension 4: the complex projective plane $\mathbb{C}\mathrm{P}^2$ is not spin (even if it canonically carries a so-called spin$^c$ structure as a Kähler manifold). Indeed a complex manifold is spin if and only if the mod 2 reduction of its first Chern class vanishes, see [178, App. D].

3. However any simply-connected manifold has a unique spin structure as soon as it is spin, since in that case $H^1(M, \mathbb{Z}_2) = 0$.

4. It was first noticed by J. Milnor [199] that different spin structures may provide equivalent principal $\mathrm{Spin}_n$-bundles. For instance, the 2-dimensional torus has 4 different spin structures (see Example 1.4.3.2), all of which are equivalent as principal $\mathrm{Spin}_2$-bundles. This is due to the fact that, for the torus, $H^1(M, \mathbb{Z}_2) = H^1(M, \mathbb{Z}) \otimes \mathbb{Z}_2$, see [178, p.84].

From now on, each time we assume that a manifold is spin we shall implicitly mean that a spin structure is fixed on it.

## 1.2 Spinor bundle and Clifford multiplication

In this section we define the natural algebraic and geometric objects on a Riemannian spin manifold, namely the Clifford multiplication and the induced compatible covariant derivative on its spinor bundle. We first recall the following central result in representation theory which can be found e.g. in [178, Prop. 5.15 p.36].

**Proposition 1.2.1** *Let* $n \in \mathbb{N} \setminus \{0\}$.

i) *If $n$ is odd then there exists up to equivalence exactly one fundamental irreducible complex representation* $\mathrm{Spin}_n \xrightarrow{\delta_n} \mathrm{Aut}(\Sigma_n)$ *of* $\mathrm{Spin}_n$ *that does not come from* $\mathrm{SO}_n$*. It is called the* spinor representation *in dimension $n$ and has dimension* $2^{\frac{n-1}{2}}$*.*

ii) *If $n$ is even then there exist up to equivalence exactly two fundamental irreducible complex representations* $\mathrm{Spin}_n \xrightarrow{\delta_n^{\pm}} \mathrm{Aut}(\Sigma_n^{\pm})$ *of* $\mathrm{Spin}_n$ *that do not come from* $\mathrm{SO}_n$*. The representations $\delta_n^+$ and $\delta_n^-$ are called the* positive *and* negative half spinor representation *in dimension $n$ respectively and each have dimension* $2^{\frac{n-2}{2}}$*.*

Recall that a fundamental representation of a compact Lie group $G$ is an irreducible complex representation whose highest weight is a fundamental weight (it belongs to a system of generators of all irreducible complex representations of $G$). That $\delta_n^{(\pm)}$ is a representation of $\mathrm{Spin}_n$ which does not come from $\mathrm{SO}_n$ means that there does not exist any representation $\rho$ of $\mathrm{SO}_n$ with $\rho \circ \xi = \delta_n^{(\pm)}$.

*Proof of* Proposition 1.2.1: The representation

$$\delta_n := \begin{cases} \delta_n^+ \oplus \delta_n^- & \text{for } n \text{ even} \\ \delta_n & \text{for } n \text{ odd} \end{cases}$$

is actually the restriction of the (or one of both if $n$ is odd) irreducible representation(s) of the corresponding finite-dimensional complex Clifford algebra. Remember that, for any given (real or complex) vector space $V$ endowed with a non-degenerate symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, the Clifford algebra of the pair $(V, \langle \cdot\,, \cdot\rangle)$ is the quotient of its tensor algebra through the two-sided ideal generated by the elements of the form $x \otimes y + y \otimes x + 2\langle x, y\rangle 1$ where $x, y \in V$. The product law induced on the quotient, which is usually denoted by "$\cdot$" and called the Clifford product or Clifford multiplication, satisfies the Clifford relations, namely

$$x \cdot y + y \cdot x = -2\langle x, y\rangle 1 \tag{1.1}$$

for all $x, y \in V$. Since we do not want to deal with Clifford algebras in detail we just recall their most important properties for our purpose (see e.g. [178, Chap. 1]):

1. The Clifford algebra is the smallest associative algebra with unit containing $V$ and satisfying the Clifford relations.

2. The Clifford algebra of the $n$-dimensional Euclidean space is linearly isomorphic to its exterior algebra $\Lambda^*\mathbb{R}^n$, the Clifford product being then given by
$$x \cdot \simeq x^\flat \wedge - x \lrcorner \tag{1.2}$$
for every $x \in \mathbb{R}^n$.

3. If $V$ is real then the complexification of the Clifford algebra of $(V, \langle \cdot\,, \cdot\rangle)$ coincides with the (complex) Clifford algebra of $(V \otimes \mathbb{C}, \langle \cdot\,, \cdot\rangle_{\mathbb{C}})$ (where $\langle \cdot\,, \cdot\rangle_{\mathbb{C}}$ is complex bilinear).

4. The Clifford algebra of $\mathbb{C}^{2n}$ with its canonical complex bilinear form is isomorphic to the algebra of all complex $2^n \times 2^n$ matrices, and that of $\mathbb{C}^{2n+1}$ to two copies of this algebra [178, Tab. I p.28].

Property 4 implies in particular the existence of exactly two and one irreducible non-trivial representations of the Clifford algebra of $\mathbb{C}^n$ for $n$ odd and even respectively. Now $\mathrm{Spin}_n$ can be identified with the set of even Clifford products of unit vectors of $\mathbb{R}^n$ (this can actually be used as definition of $\mathrm{Spin}_n$), in particular sits in the Clifford algebra of $\mathbb{C}^n$. After restriction onto $\mathrm{Spin}_n$ both Clifford-algebra-representations turn out to become equivalent for $n$ odd whereas the unique one splits into two inequivalent equally dimensional representations of $\mathrm{Spin}_n$ for $n$ even. The statement on their dimensions easily follows. □

The simplest way to distinguish $\delta_n^+$ from $\delta_n^-$ consists in looking at the action of the so-called *complex volume form* of $\mathbb{R}^n$ and which is defined for every $n$ by

$$\omega_n^{\mathbb{C}} := i^{[\frac{n+1}{2}]} e_1 \cdot \ldots \cdot e_n \tag{1.3}$$

for any positively-oriented orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$. The complex volume form does in general not lie in $\mathrm{Spin}_n$ but in the complex Clifford algebra of $\mathbb{C}^n$ and it can be shown that

$$\delta_n^{\pm}(\omega_n^{\mathbb{C}}) = \pm \mathrm{Id}_{\Sigma_n^{\pm}} \tag{1.4}$$

for $n$ even whereas $\delta_n(\omega_n^{\mathbb{C}}) = \mathrm{Id}_{\Sigma_n}$ or $-\mathrm{Id}_{\Sigma_n}$ for $n$ odd (in the latter case both possibilities can occur).

**Definition 1.2.2** *Let $(M^n, g)$ be a Riemannian spin manifold.*

*i) The* spinor bundle *of $M$ is the complex vector bundle associated to the principal bundle* $\mathrm{Spin}(TM)$ *via the spinor representation, i.e.,*

$$\Sigma M := \mathrm{Spin}(TM) \times_{\delta_n} \Sigma_n,$$

*where, for $n$ even, $\Sigma_n := \Sigma_n^+ \oplus \Sigma_n^-$ and $\delta_n := \delta_n^+ \oplus \delta_n^-$.*

*i') If $M$ is even-dimensional, the* positive *(resp.* negative*)* spinor bundle *of $M$ is the complex vector bundle associated to the principal bundle* $\mathrm{Spin}(TM)$ *via the positive (resp. negative) half-spinor representation, i.e.,*

$$\Sigma^+ M := \mathrm{Spin}(TM) \times_{\delta_n^+} \Sigma_n^+$$

*(resp. $\Sigma^- M := \mathrm{Spin}(TM) \times_{\delta_n^-} \Sigma_n^-$).*

*ii) A* Clifford multiplication *is a complex linear vector-bundle-homomorphism*

$$T^*M \otimes \Sigma M \xrightarrow{\mu} \Sigma M, \qquad X^\flat \otimes \varphi \longmapsto X \cdot \varphi$$

*such that*

$$X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X, Y)\varphi \tag{1.5}$$

*for all $X, Y \in TM$ and $\varphi \in \Sigma M$.*

*iii)* *A* compatible covariant derivative *on* $\Sigma M$ *is a covariant derivative* $\nabla^\Sigma$ *on* $\Sigma M$ *such that*

$$\nabla^\Sigma_X(Y\cdot\varphi) = (\nabla_X Y)\cdot\varphi + Y\cdot\nabla^\Sigma_X\varphi$$

*for all* $X, Y \in \Gamma(TM)$ *and* $\varphi \in \Gamma(\Sigma M)$*, where* $\nabla$ *denotes the Levi-Civita covariant derivative of* $(M^n, g)$*.*

In particular the spinor bundle of an even-dimensional Riemannian spin manifold always splits into $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. Note moreover that, in any dimension, $\mathrm{rk}_{\mathbb{C}}(\Sigma M) = 2^{[\frac{n}{2}]}$.

From its definition any Clifford multiplication can be canonically extended into an algebra-homomorphism from the so-called Clifford-algebra-bundle to $\mathrm{End}(\Sigma M)$. Since we essentially do not need the Clifford-algebra-bundle, we just indicate how $\mu$ extends from $T^*M \otimes \Sigma M$ to $\Lambda TM \otimes \Sigma M$: for any $p$-form $\omega$ and any spinor $\varphi$, the product $\omega \cdot \varphi$ is defined by

$$\omega\cdot\varphi := \sum_{1\le j_1<\ldots<j_p\le n} \omega_{j_1,\ldots,j_p} e_{j_1}\cdot(\ldots\cdot(e_{j_p}\cdot\varphi)),$$

where $\omega = \sum_{1\le j_1<\ldots<j_p\le n} \omega_{j_1,\ldots,j_p} e^*_{j_1}\wedge\ldots\wedge e^*_{j_p}$ in a local orthonormal basis $\{e_j\}_{1\le j\le n}$ of $TM$. Moreover the Clifford algebra being associative (see Property 1 in the proof of Proposition 1.2.1 above), we shall in the following forget about the parentheses and write $X\cdot Y\cdot\varphi$ instead of $X\cdot(Y\cdot\varphi)$.

The spinor bundle comes with a natural Hermitian inner product which together with Clifford multiplication exist and are unique in some sense.

**Proposition 1.2.3** *Let* $(M^n, g)$ *be a Riemannian spin manifold.*

a) *If* $n$ *is odd then there exist up to equivalence exactly two Clifford multiplications, which are opposite from each other.*

b) *If* $n$ *is even then there exists up to equivalence exactly one Clifford multiplication.*

c) *There exists a Hermitian inner product* $\langle\cdot\,,\cdot\rangle$ *on* $\Sigma M$*, pointwise unique up to scale, such that*

$$\langle X\cdot\varphi, \psi\rangle = -\langle\varphi, X\cdot\psi\rangle$$

*for all* $X \in TM$ *and* $\varphi, \psi \in \Sigma M$*, where "$\cdot$" denotes any fixed Clifford multiplication on* $\Sigma M$*.*

d) *There exists a metric compatible covariant derivative* $\nabla^\Sigma$ *on* $\Sigma M$*.*

*Proof*: The first two statements follow from Proposition 1.2.1 (see also its proof), since the spinor or half-spinor representations are exactly provided by representations of the complex Clifford algebra. The third statement is not a direct consequence of the existence of a Hermitian inner product on $\Sigma_n$ preserved by $\delta_n^{(\pm)}$ (which would simply follow from the compactness of $\mathrm{Spin}_n$), but from the following argument: choose any Hermitian inner product $\langle\cdot\,,\cdot\rangle'$ on $\Sigma_n$ and mean it over the Clifford action of the (finite) group generated by the canonical basis of $\mathbb{R}^n$ (which is a subgroup of the group of invertible elements in the Clifford

algebra). One obtains a new Hermitian inner product $\langle\cdot\,,\cdot\rangle_0$ on $\Sigma_n$ for which obviously every canonical basis vector acts unitarily and hence in a skew-adjoint way from (1.1). Therefore the Clifford action of every vector of $\mathbb{R}^n$ on $\Sigma_n$ becomes also skew-adjoint w.r.t. $\langle\cdot\,,\cdot\rangle_0$. Since $\mathrm{Spin}_n$ can be identified with the set of even Clifford products of unit vectors of $\mathbb{R}^n$, the inner product $\langle\cdot\,,\cdot\rangle_0$ remains invariant under the $\mathrm{Spin}_n$-action - which is precisely $\delta_n$. Hence $\langle\cdot\,,\cdot\rangle_0$ induces a Hermitian inner product $\langle\cdot\,,\cdot\rangle$ on $\Sigma M$ which obviously has the same properties. The pointwise uniqueness of $\langle\cdot\,,\cdot\rangle$ up to scale follows from the irreducibility of $\delta_n$ as a Clifford-algebra-representation.
A covariant derivative on $\Sigma M$ may be constructed by lifting locally the connection 1-form of the Levi-Civita covariant derivative of $(M^n, g)$, which is made possible by $\mathfrak{spin}_n \xrightarrow{d_1\xi} \mathfrak{so}_n$ being a Lie-algebra-isomorphism, see e.g. [138, (8) p.140] and the local formula (1.6) below. It is then straightforward to show that the covariant derivative defined in this way is metric and compatible in the sense of Definition 1.2.2.*iii*) (see e.g. [138, Prop. 4.4]). This explains d). □

From now on we choose a Hermitian inner product $\langle\cdot\,,\cdot\rangle$ on $\Sigma M$ coming from a Hermitian inner product on $\Sigma_n$ making Clifford multiplication of vectors of $\mathbb{R}^n$ skew-Hermitian, as in Proposition 1.2.3.
Note that, if $\nabla^\Sigma$ is a compatible covariant derivative on $\Sigma M$, then for any real 1-form $\theta$ on $M$ the covariant derivative $\nabla^\Sigma + i\theta \otimes \mathrm{Id}$ is again compatible, hence such a covariant derivative is not unique. We choose henceforth and denote by $\nabla$ the covariant derivative on $\Sigma M$ *naturally induced by the Levi-Civita covariant derivative* $\nabla$ *on* $(M^n, g)$ and which can be locally expressed as [138, Prop. 4.3]

$$\nabla\varphi_\alpha = \frac{1}{4}\sum_{j,k=1}^n g(\nabla e_j, e_k) e_j \cdot e_k \cdot \varphi_\alpha, \tag{1.6}$$

where $\{e_j\}_{1\leq j\leq n}$ denotes a local positively-oriented orthonormal basis of $TM$ and $\{\varphi_\alpha\}_{1\leq\alpha\leq 2^{[\frac{n}{2}]}}$ any corresponding local spinorial frame, that is: $\varphi_\alpha = [\widetilde{s}, \sigma_\alpha]$ with $\eta(\widetilde{s}) = (e_1, \ldots, e_n)$ and $\{\sigma_\alpha\}_{1\leq\alpha\leq 2^{[\frac{n}{2}]}}$ is a fixed orthonormal basis of $\Sigma_n$.

The curvature $R^\nabla$ of that covariant derivative can be explicitly expressed through that (denoted by $R$) of the Levi-Civita covariant derivative on $(M^n, g)$: for any local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $TM$,

$$R^\nabla_{X,Y}\varphi = \frac{1}{4}\sum_{j,k=1}^n g(R_{X,Y}e_j, e_k) e_j \cdot e_k \cdot \varphi, \tag{1.7}$$

for all $X, Y \in TM$ and $\varphi \in \Sigma M$ (see again [138, Prop. 4.3]). In dimension $n = 2$ this identity simplifies into

$$R^\nabla_{X,Y}\varphi = \frac{S}{8}(X \cdot Y - Y \cdot X) \cdot \varphi, \tag{1.8}$$

where $S$ is the scalar curvature of $(M^n, g)$.
The following very useful formula can be deduced from (1.7).

**Lemma 1.2.4** *Let* Ric *denote the Ricci-tensor of the Riemannian spin manifold* $(M^n, g)$*, then for all* $X \in TM$ *and* $\varphi \in \Sigma M$ *one has*

$$\sum_{j=1}^{n} e_j \cdot R^{\nabla}_{X,e_j}\varphi = \frac{1}{2}\mathrm{Ric}(X)\cdot\varphi, \tag{1.9}$$

*with the convention* $R^{\nabla}_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

*Proof*: The first Bianchi identity implies that

$$\begin{aligned}
\sum_{j=1}^{n} e_j \cdot R^{\nabla}_{X,e_j}\varphi &\stackrel{(1.7)}{=} \frac{1}{4}\sum_{j,k,l=1}^{n} g(R_{X,e_j}e_k, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi \\
&= -\frac{1}{4}\sum_{j,k,l=1}^{n} g(R_{e_j,e_k}X, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi \\
&\quad -\frac{1}{4}\sum_{j,k,l=1}^{n} g(R_{e_k,X}e_j, e_l)e_j\cdot e_k\cdot e_l\cdot\varphi \\
&= -\frac{1}{4}\sum_{j,k,l=1}^{n} g(R_{X,e_j}e_k, e_l)(e_k\cdot e_l\cdot e_j - e_k\cdot e_j\cdot e_l)\cdot\varphi,
\end{aligned}$$

with

$$\begin{aligned}
e_k\cdot e_l\cdot e_j - e_k\cdot e_j\cdot e_l &\stackrel{(1.5)}{=} -e_k\cdot e_j\cdot e_l - 2\delta_{jl}e_k + e_j\cdot e_k\cdot e_l + 2\delta_{jk}e_l \\
&\stackrel{(1.5)}{=} 2e_j\cdot e_k\cdot e_l + 4\delta_{jk}e_l - 2\delta_{jl}e_k.
\end{aligned}$$

We deduce that

$$\begin{aligned}
3\sum_{j=1}^{n} e_j \cdot R^{\nabla}_{X,e_j}\varphi &= -\sum_{j,l=1}^{n} g(R_{X,e_j}e_j, e_l)e_l\cdot\varphi + \frac{1}{2}\sum_{j,k=1}^{n} g(R_{X,e_j}e_k, e_j)e_k\cdot\varphi \\
&= \sum_{l=1}^{n} g(\mathrm{Ric}(X), e_l)e_l\cdot\varphi + \frac{1}{2}\sum_{k=1}^{n} g(\mathrm{Ric}(X), e_k)e_k\cdot\varphi \\
&= \frac{3}{2}\mathrm{Ric}(X)\cdot\varphi,
\end{aligned}$$

which is the result. □

## 1.3 The Dirac operator

We are now ready to define the central object of this survey.

**Definition 1.3.1** *The* fundamental Dirac operator *of a Riemannian spin manifold* $(M^n, g)$ *is the map* $D : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ *defined by*

$$D\varphi := \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\varphi$$

*for every* $\varphi \in \Gamma(\Sigma M)$*, where* $\{e_j\}_{1\le j\le n}$ *is any local orthonormal basis of* $TM$.

The Dirac operator is sometimes called Atiyah-Singer operator in the literature in honour to M. Atiyah and I. Singer who brought it to mathematical daylight through their famous index theorem [31]. To distinguish it from its twisted and/or generalized versions on Clifford bundles, it is also called the *fundamental* (or *spin*) Dirac operator. In this book, we only deal with $D$, see Chapter 8 for results related to twisted or generalized Dirac operators. When necessary we shall write $D_M$ or $D_{M,g}$, $D_g$ to precise the underlying manifold $M$ and/or the metric $g$.

The Dirac operator is obtained as the composition of the Clifford multiplication with the natural covariant derivative on $\Sigma M$. Alternatively one can check that the local expression defining the Dirac operator is independent of the local orthonormal basis chosen on $TM$. Beware here that, since $D$ depends on the choice of Clifford multiplication, it is only defined up to a sign if $n$ is odd. The usual convention in that case is to choose the Clifford multiplication such that the action of the complex volume form $\omega_n^{\mathbb{C}}$ (whose algebraic definition (1.3) makes sense on $M$ as an element of the so-called Clifford-algebra-bundle) is the identity, see e.g. [178, Prop. 5.9 p.34] for the real analog.
In case $n$ is even the Dirac operator can be split in a canonical way.

**Proposition 1.3.2** *Let $(M^n, g)$ be an even-dimensional Riemannian spin manifold, then its Dirac operator $D$ splits into*

$$D = D^+ \oplus D^-,$$

*where $D^\pm : \Gamma(\Sigma^\pm M) \longrightarrow \Gamma(\Sigma^\mp M)$.*

*Proof*: In even dimension the Clifford action of the complex volume form $\omega_n^{\mathbb{C}}$ is a non-trivial parallel involution of $\Sigma M$ anti-commuting with the Clifford multiplication with vectors (i.e., $X \cdot \omega_n^{\mathbb{C}} = -\omega_n^{\mathbb{C}} \cdot X$ for every $X \in TM$), so that

$$D(\omega_n^{\mathbb{C}} \cdot \varphi) = -\omega_n^{\mathbb{C}} \cdot D\varphi \tag{1.10}$$

for every $\varphi \in \Gamma(\Sigma M)$. From $\Sigma^\pm M = \{\varphi \in \Sigma M \,|\, \omega_n^{\mathbb{C}} \cdot \varphi = \pm\varphi\}$ we conclude. □

We come to the properties of the Dirac operator which are fundamental for the further study of its spectrum. First we need a formula computing the commutator of the Dirac operator with a function. For technical reasons we include into the next lemma the computation of commutators or anticommutators involving the Dirac operator and which we shall need in the following.

**Lemma 1.3.3** *Let $\varphi$ be a smooth spinor field, $f$ a smooth function and $\xi$ a smooth vector field on a Riemannian spin manifold $(M^n, g)$, then*

*i)*

$$D(f\varphi) = \mathrm{grad}(f) \cdot \varphi + fD\varphi, \tag{1.11}$$

*where* $\mathrm{grad}(f)$ *denotes the gradient vector field of $f$ on $(M^n, g)$.*

*ii)*

$$D(\xi \cdot \varphi) = -\xi \cdot D\varphi - 2\nabla_\xi \varphi + (d + \delta)\xi^\flat \cdot \varphi, \tag{1.12}$$

*where $d$ and $\delta$ denote the exterior differential and codifferential on $(M^n, g)$ respectively.*

*iii)*

$$D^2(f\varphi) = fD^2\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + (\Delta f)\varphi, \tag{1.13}$$

*where* $\Delta := \delta d = -\mathrm{tr}(\mathrm{Hess}_g(\cdot))$ *denotes the scalar Laplace operator on* $(M^n, g)$.

*Proof*: Fix a local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $TM$. We compute:

$$\begin{aligned} D(f\varphi) &= \sum_{j=1}^n e_j \cdot \nabla_{e_j}(f\varphi) \\ &= \sum_{j=1}^n e_j \cdot (e_j(f)\varphi + f\nabla_{e_j}\varphi) \\ &= \mathrm{grad}(f)\cdot\varphi + fD\varphi, \end{aligned}$$

which proves *i*). Moreover,

$$\begin{aligned} D(\xi\cdot\varphi) &= \sum_{j=1}^n e_j\cdot\nabla_{e_j}(\xi\cdot\varphi) \\ &= \sum_{j=1}^n e_j\cdot(\nabla_{e_j}\xi)\cdot\varphi + \sum_{j=1}^n e_j\cdot\xi\cdot\nabla_{e_j}\varphi \\ &\overset{(1.5)}{=} \sum_{j=1}^n e_j\cdot(\nabla_{e_j}\xi)\cdot\varphi - \xi\cdot\sum_{j=1}^n e_j\cdot\nabla_{e_j}\varphi - 2\sum_{j=1}^n g(\xi,e_j)\nabla_{e_j}\varphi \\ &\overset{(1.2)}{=} (\sum_{j=1}^n e_j\wedge\nabla_{e_j}\xi^\flat)\cdot\varphi - (\sum_{j=1}^n e_j\lrcorner\nabla_{e_j}\xi^\flat)\cdot\varphi \\ &\quad -\xi\cdot\sum_{j=1}^n e_j\cdot\nabla_{e_j}\varphi - 2\sum_{j=1}^n g(\xi,e_j)\nabla_{e_j}\varphi \\ &= -\xi\cdot D\varphi - 2\nabla_\xi\varphi + (d+\delta)\xi^\flat\cdot\varphi \end{aligned}$$

and

$$\begin{aligned} D^2(f\varphi) &\overset{(1.11)}{=} D(df\cdot\varphi + fD\varphi) \\ &\overset{(1.12)}{=} -df\cdot D\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + (d+\delta)df\cdot\varphi + df\cdot D\varphi + fD^2\varphi \\ &= fD^2\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + (\Delta f)\varphi, \end{aligned}$$

where we have identified 1-forms with vector fields through the metric $g$. This concludes the proof. □

**Proposition 1.3.4** *The Dirac operator of a Riemannian spin manifold is an elliptic and formally self-adjoint linear differential operator of first order.*

*Proof*: We deduce from Lemma 1.3.3 that $D$ is a linear differential operator of first order whose principal symbol is given by

$$\begin{aligned} T^*M &\longrightarrow \mathrm{End}(\Sigma M) \\ \xi &\longmapsto \xi^\sharp\cdot, \end{aligned}$$

where, as usual, $g(\xi^\sharp, X) := \xi(X)$ for every $X \in TM$. For any $\xi \in T^*M \setminus \{0\}$ the map $\xi^\sharp \cdot$ is an automorphism of $\Sigma M$ since it is injective (from (1.5) one has $\xi^\sharp \cdot \xi^\sharp \cdot \varphi = -g(\xi, \xi)\varphi$ and $g$ is Riemannian). Therefore $D$ is elliptic.
Let $\varphi, \psi \in \Gamma(\Sigma M)$, then using the mutual compatibility relations between $\langle \cdot\,, \cdot \rangle$, "$\cdot$" and $\nabla$ on $\Sigma M$ one has

$$\begin{aligned} \langle D\varphi, \psi \rangle &= \sum_{j=1}^{n} \langle e_j \cdot \nabla_{e_j}\varphi, \psi \rangle \\ &= -\sum_{j=1}^{n} \langle \nabla_{e_j}\varphi, e_j \cdot \psi \rangle \\ &= \sum_{j=1}^{n} -e_j(\langle \varphi, e_j \cdot \psi \rangle) + \langle \varphi, \nabla_{e_j} e_j \cdot \psi \rangle \\ &\quad + \sum_{j=1}^{n} \langle \varphi, e_j \cdot \nabla_{e_j}\psi \rangle \\ &= \mathrm{div}(V_{\varphi\psi}) + \langle \varphi, D\psi \rangle, \end{aligned} \tag{1.14}$$

where $V_{\varphi\psi} \in \Gamma(TM \otimes \mathbb{C})$ is defined by

$$(g \otimes \mathrm{Id}_{\mathbb{C}})(V_{\varphi\psi}, X) := \langle \varphi, X \cdot \psi \rangle$$

for all $X \in TM$. In particular, if $\varphi$ or $\psi$ has compact support on (and vanishes on the boundary of) $M$ then applying Green's formula we obtain

$$\int_M \langle D\varphi, \psi \rangle v_g = \int_M \langle \varphi, D\psi \rangle v_g,$$

which shows that $D$ is formally self-adjoint and concludes the proof. $\square$

In even dimension, Proposition 1.3.4 means that the formal adjoint of $D^\pm$ is $D^\mp$. Beware however that Proposition 1.3.4 does not prove the self-adjointness of $D$. It is indeed *a priori* not clear if $D$ and its adjoint $D^*$ - which is to be distinguished from the formal adjoint as a differential operator - have the same domain of definition and if they can be extended to the whole Hilbert space $\mathrm{L}^2(\Sigma M)$, which is defined as the completion of the space $\Gamma_c(\Sigma M) := \{\varphi \in \Gamma(\Sigma M) \,|\, \mathrm{supp}(\varphi) \text{ is compact}\}$ w.r.t. $(\cdot\,, \cdot) := \int_M \langle \cdot\,, \cdot \rangle v_g$. The concept needed here is that of *essential self-adjointness*: one has to show that the closure of the operator in $\mathrm{L}^2(\Sigma M)$ is self-adjoint. For $D$ this is possible as soon as the underlying Riemannian manifold is complete.

**Proposition 1.3.5** *Let $(M^n, g)$ be a complete Riemannian spin manifold, then its Dirac operator is essentially self-adjoint.*

*Proof*: The proof presented here is based on the talk [46]. Let $\mathrm{dom}(\cdot)$ denote the domain of definition of an operator. By construction the operator $D$ is densely defined in $\mathrm{L}^2(\Sigma M)$ and one can set $\mathrm{dom}(D) := \Gamma_c(\Sigma M)$. As a consequence it admits a unique closure $D^{**}$, whose graph is the closure of that of $D$. Proposition 1.3.4 implies that $D$ - thus $D^{**}$ - is symmetric in $\mathrm{L}^2(\Sigma M)$. Its adjoint $D^*$ is defined on $\{\psi \in \mathrm{L}^2(\Sigma M) \,|\, \varphi \mapsto (D\varphi, \psi) \text{ is bounded on } (\Gamma_c(\Sigma M), \|\cdot\|)\}$. Since the topological dual of $\mathrm{L}^2$ is $\mathrm{L}^2$ itself, $\mathrm{dom}(D^*) = \{\psi \in \mathrm{L}^2(\Sigma M) \,|\, \varphi \mapsto (D\varphi, \psi) \in \mathrm{L}^2(\Sigma M)\}$. Considering $D$ at the distributional level, one deduces from the formal self-adjointness of $D$ (Proposition 1.3.4) that

$$\mathrm{dom}(D^*) = \{\psi \in \mathrm{L}^2(\Sigma M) \,|\, D\psi \in \mathrm{L}^2(\Sigma M)\}.$$

It remains to show that $\mathrm{dom}(D^*) = \mathrm{dom}(D^{**})$, i.e., the inclusion "$\subset$". This is equivalent to proving that $\mathrm{Ker}(D^* - i\varepsilon\mathrm{Id}) = 0$ for both $\varepsilon \in \{\pm 1\}$, see e.g. [239, Cor. VII.2.9]. Let $\psi \in \mathrm{dom}(D^*)$ with $D^*\psi = i\varepsilon\psi$, then $(D^*\psi, \varphi) = i\varepsilon(\psi, \varphi)$ for every $\varphi \in \Gamma_c(\Sigma M)$, i.e., $(\psi, D\varphi) = i\varepsilon(\psi, \varphi)$. Since $D$ is formally self-adjoint this is equivalent to $D\psi = i\varepsilon\psi$ in the distributional sense. The operator $D - i\varepsilon\mathrm{Id}$ being elliptic, general elliptic theory (see e.g. [178, Thm. 4.5 p.190]) implies that $\psi$ is actually smooth. Let now $(\rho_k)_{k>0}$ be a sequence of smooth compactly supported $[0,1]$-valued functions converging pointwise to 1 and with $|\mathrm{grad}(\rho_k)| \le \frac{1}{k}$ for all $k$. Such a sequence exists because of the completeness assumption on $(M^n, g)$. The preceding identity together with Lemma 1.3.3 provide

$$\begin{aligned} i\varepsilon(\psi, \rho_k\psi) &= (D\psi, \rho_k\psi) \\ &= (\psi, D(\rho_k\psi)) \\ &\overset{(1.11)}{=} (\psi, \rho_k D\psi) + (\psi, \mathrm{grad}(\rho_k)\cdot\psi) \\ &= (\rho_k\psi, D\psi) + (\psi, \mathrm{grad}(\rho_k)\cdot\psi) \\ &= -i\varepsilon(\psi, \rho_k\psi) + (\psi, \mathrm{grad}(\rho_k)\cdot\psi), \end{aligned}$$

that is, $2i\varepsilon(\psi, \rho_k\psi) = (\psi, \mathrm{grad}(\rho_k)\cdot\psi)$, whose l.h.s. tends to $2i\varepsilon\|\psi\|^2$ and whose r.h.s. tends to 0 when $k$ goes to $\infty$. Therefore $\psi = 0$, QED. □

**Note 1.3.6** It seems at first glance that the completeness assumption on the metric enters the proof in a very weak manner and one could therefore think about getting rid of it. This is unfornutately hopeless. Consider for example $M := ]0, +\infty[$ with standard metric and canonical spin structure. Its Dirac operator is $D = i\frac{d}{dt}$ (the Clifford multiplication of $e_1 = 1 \in \mathbb{R}$ can be identified with that of $i$ on $\Sigma_1 = \mathbb{C}$). Set $\mathrm{dom}(D) := \Gamma_c(\Sigma M) = C_c^\infty(]0, +\infty[, \mathbb{C})$, then the Dirac operator remains symmetric in $\mathrm{L}^2(\Sigma M)$. However a simple computation shows that the kernel of $D^* - i\varepsilon\mathrm{Id}$ in the space of distributions is $\mathbb{R}e^{\varepsilon t}$. Since $e^t \notin \mathrm{L}^2(\Sigma M)$ one has $\mathrm{Ker}(D^* - i\mathrm{Id}) = 0$, however $e^{-t} \in \mathrm{L}^2(\Sigma M)$ so that $\mathrm{Ker}(D^* - i\mathrm{Id}) = \mathbb{R}e^{-t}$. Therefore $D$ is not essentially self-adjoint in $\mathrm{L}^2(\Sigma M)$. Actually the fact that $\mathrm{Ker}(D^* \mp i\mathrm{Id})$ do not have the same dimension imply the non-existence of self-adjoint extensions of $D$ in $\mathrm{L}^2(\Sigma M)$ (see [239, Thm. VII.2.10]).
Another example is $M := ]0, 1[$ with the same metric and spin structure: again $D = i\frac{d}{dt}$ is not essentially self-adjoint, but this time it has infinitely many self-adjoint extensions. Indeed the exponential $e^t$ belongs in that case to $\mathrm{L}^2(\Sigma M)$ so that $\mathrm{Ker}(D^* - i\varepsilon\mathrm{Id}) = \mathbb{R}e^{\varepsilon t}$ for both $\varepsilon \in \{\pm 1\}$. Moreover $\mathrm{Ker}(D^* \mp i\mathrm{Id})$ have the same dimension and it can be shown that there exists an $\mathbb{S}^1$-parametrized family of self-adjoint extensions of $D$, see [239, Ex. VII.2.a)].

We summarise the spectral properties of the Dirac operator on closed manifolds. Further basics on non-compact Riemannian spin manifolds are given in Section 7.1.

**Theorem 1.3.7** *Let $(M^n, g)$ be a closed Riemannian spin manifold and denote by* $\mathrm{Spec}(D)$ *the spectrum of its Dirac operator $D$, then the following holds:*

*i) The set* $\mathrm{Spec}(D)$ *is a closed subset of* $\mathbb{R}$ *consisting of an unbounded discrete sequence of eigenvalues.*

*ii) Each eigenspace of $D$ is finite-dimensional and consists of smooth sections.*

*iii) The eigenspaces of $D$ form a complete orthonormal decomposition of* $\mathrm{L}^2(\Sigma M)$*, i.e.,*

$$\mathrm{L}^2(\Sigma M) = \overline{\bigoplus_{\lambda \in \mathrm{Spec}(D)} \mathrm{Ker}(D - \lambda \mathrm{Id})}.$$

*iv) The set* $\mathrm{Spec}(D)$ *is unbounded on both sides of* $\mathbb{R}$ *and, if moreover $n \not\equiv 3\ (4)$, then it is symmetric about the origin.*

*Proof*: If $M$ is compact without boundary, then the statements $i) - iii)$ follow from the classical spectral theory of elliptic self-adjoint operators [178, Thm. 5.8 p.196], which is applicable here to the closure of $D$ as a consequence of Proposition 1.3.5.
As for the unboundedness of $\mathrm{Spec}(D)$ on both sides of $\mathbb{R}$, we give the proof presented in [11, Prop. 4.30] and which consists in the following. Assume it were wrong for $n \geq 3$ (for $n = 1, 2$ the spectrum of $D$ is symmetric about 0, see Note 2.1.2.1 and end of proof respectively). Since we already know that $\mathrm{Spec}(D)$ is unbounded, this would mean that either $\mathrm{Spec}(D) \subset [m, +\infty[$ or $\mathrm{Spec}(D) \subset ]-\infty, m]$ for some $m \in \mathbb{R}$. Up to changing $D$ into $-D$, one can assume that $\mathrm{Spec}(D) \subset [m, +\infty[$, i.e., that $(D\varphi, \varphi) \geq m\|\varphi\|^2$ for all $\varphi \in \Gamma(\Sigma M)$ (see e.g. Lemma 5.0.2 for a formulation of the min-max principle). Consider a finite open covering $(\Omega_k)_k$ of $M$ such that $TM_{|\Omega_k}$ is trivial for every $k$ and choose a partition of unity $(\chi_k)_k$ subordinated to that covering, that is, $\chi_k \in C^\infty(M, [0, 1])$ with $\mathrm{supp}(\chi_k) \subset \Omega_k$ and $\sum_k \chi_k = 1$. Let $(e_{j,k})_{1 \leq j \leq n}$ be an orthonormal frame trivializing $TM_{|\Omega_k}$ and set $\widetilde{e_{j,k}} := \sqrt{\chi_k} e_{j,k}$ for all $j, k$. Note that the $\widetilde{e_{j,k}}$ are globally defined smooth vector fields on $(M^n, g)$ satisfying

$$\begin{aligned} \sum_{j,k} \widetilde{e_{j,k}} \cdot \widetilde{e_{j,k}} \cdot &= \sum_{j,k} \chi_k e_{j,k} \cdot e_{j,k} \cdot \\ &= -\sum_{j,k} \chi_k \mathrm{Id} \\ &= -n \mathrm{Id} \end{aligned}$$

and analogously

$$\begin{aligned} \sum_{j,k} \widetilde{e_{j,k}} \cdot \nabla_{\widetilde{e_{j,k}}} &= \sum_{j,k} \chi_k e_{j,k} \cdot \nabla_{e_{j,k}} \\ &= \sum_k \chi_k D \\ &= D. \end{aligned}$$

Hence for any section $\varphi$ of $\Sigma M$ one has

$$\sum_{j,k} (D(\widetilde{e_{j,k}} \cdot \varphi), \widetilde{e_{j,k}} \cdot \varphi) \stackrel{(1.12)}{=} \sum_{j,k} \Big\{ -(\widetilde{e_{j,k}} \cdot D\varphi, \widetilde{e_{j,k}} \cdot \varphi)$$

$$
\begin{aligned}
& - 2(\nabla_{\widetilde{e_{j,k}}}\varphi, \widetilde{e_{j,k}} \cdot \varphi) \\
& + ((d+\delta)\widetilde{e_{j,k}}^{\flat} \cdot \varphi, \widetilde{e_{j,k}} \cdot \varphi)\Big\} \\
= \quad & -n(D\varphi,\varphi) + 2(D\varphi,\varphi) \\
& - \sum_{j,k}((\widetilde{e_{j,k}} \wedge d\widetilde{e_{j,k}}^{\flat}) \cdot \varphi, \varphi) \\
\leq \quad & -(n-2)(D\varphi,\varphi) + C\|\varphi\|^2,
\end{aligned}
$$

where $C := \max\limits_M |\sum_{j,k} \widetilde{e_{j,k}} \wedge d\widetilde{e_{j,k}}^{\flat}|$ is a finite nonnegative constant independent of $\varphi$. The assumption implies that, if $\varphi$ is a non-zero eigenvector for $D$ associated to the eigenvalue $\lambda$, then

$$
\begin{aligned}
(-(n-2)\lambda + C)\|\varphi\|^2 \quad & \geq \quad m\sum_{j,k}(\widetilde{e_{j,k}} \cdot \varphi, \widetilde{e_{j,k}} \cdot \varphi) \\
& = \quad nm\|\varphi\|^2,
\end{aligned}
$$

which leads to a contradiction for $\lambda$ large enough, QED.
As for the symmetry of $\mathrm{Spec}(D)$ about 0, it straightforward follows from Proposition 1.3.2 (i.e., from (1.10)) in the case where $n$ is even whereas for $n \equiv 1$ (4) it can be deduced from the existence of a real or quaternionic parallel structure on $\Sigma M$ anti-commuting with the Clifford multiplication and hence with the Dirac operator [91, Prop. p.31]. This concludes the proof. ☐

From Corollary 2.1.5 below, the Dirac spectrum of the $n$-dimensional real projective space $\mathbb{R}\mathrm{P}^n$ endowed with its round metric of sectional curvature 1 and one of its both spin structures is $\{\frac{n}{2} + n_1 + 2k,\ -\frac{n}{2} - n_2 - 2k, \quad k \in \mathbb{N}\}$, where $n_1$ is the mod 2 reduction of $\frac{n-3}{4}$ and $n_2$ that of $\frac{n+1}{4}$ (here the multiplicities are not taken into account). Thus the symmetry property of $\mathrm{Spec}(D)$ from Theorem 1.3.7.*iv*) breaks in dimension $n \equiv 3$ (4).

Eigenvectors for $D$ associated to the eigenvalue 0 are called *harmonic spinors*. Parallel spinors are harmonic but the converse is false in general. Moreover, unlike that of harmonic forms, the number of linearly independent harmonic spinors generally varies under a change of metric, see Section 6.2.

Turning to the square of the Dirac operator, its principal symbol is given by $\xi \mapsto -g(\xi,\xi)\mathrm{Id}$, which is exactly that of the rough Laplacian. The difference between both must therefore be a linear differential operator of order at most one. In fact, it turns out to be a very simple curvature expression.

**Theorem 1.3.8 ([223, 180])** *The Dirac operator $D$ of a Riemannian spin manifold $(M^n, g)$ satisfies the* Schrödinger-Lichnerowicz formula

$$
D^2 = \nabla^*\nabla + \frac{S}{4}\mathrm{Id}, \tag{1.15}
$$

*where $S$ is the scalar curvature of $(M^n, g)$.*

*Proof*: Fix a local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $TM$, then using the compatibility relations as well as (1.5) one has, for any $\varphi \in \Gamma(\Sigma M)$,

$$\begin{aligned}
D^2\varphi &= \sum_{j,k=1}^{n} e_j \cdot \nabla_{e_j}(e_k \cdot \nabla_{e_k}\varphi) \\
&= \sum_{j,k=1}^{n} e_j \cdot \nabla_{e_j}e_k \cdot \nabla_{e_k}\varphi + e_j \cdot e_k \cdot \nabla_{e_j}\nabla_{e_k}\varphi \\
&= -\sum_{j,k=1}^{n} e_j \cdot e_k \cdot \nabla_{\nabla_{e_j}e_k}\varphi + \sum_{j,k=1}^{n} e_j \cdot e_k \cdot \nabla_{e_j}\nabla_{e_k}\varphi \\
&= \sum_{j=1}^{n}(\nabla_{\nabla_{e_j}e_j} - \nabla_{e_j}\nabla_{e_j})\varphi \\
&\quad + \sum_{1\leq j<k\leq n} e_j \cdot e_k \cdot (\nabla_{e_j}\nabla_{e_k} - \nabla_{e_k}\nabla_{e_j} - \nabla_{\nabla_{e_j}e_k} + \nabla_{\nabla_{e_k}e_j})\varphi \\
&= \nabla^*\nabla\varphi - \sum_{1\leq j<k\leq n} e_j \cdot e_k \cdot (\nabla_{[e_j,e_k]} - [\nabla_{e_j}, \nabla_{e_k}])\varphi \\
&= \nabla^*\nabla\varphi - \frac{1}{2}\sum_{j,k=1}^{n} e_j \cdot e_k \cdot R^{\nabla}_{e_j,e_k}\varphi.
\end{aligned}$$

It now follows from Lemma 1.2.4 that

$$\begin{aligned}
\sum_{j,k=1}^{n} e_j \cdot e_k \cdot R^{\nabla}_{e_j,e_k}\varphi &= \frac{1}{2}\sum_{j=1}^{n} e_j \cdot \mathrm{Ric}(e_j) \cdot \varphi \\
&= -\frac{S}{2}\varphi,
\end{aligned}$$

which concludes the proof.

□

In particular the square of the Dirac operator coincides with the rough Laplacian acting on spinors as soon as the scalar curvature of the underlying manifold vanishes. This provides an answer to Dirac's original question on Euclidean space.

Applications of the Schrödinger-Lichnerowicz formula to eigenvalue estimates are discussed in Chapters 3 and 4. Vanishing theorems can also be obtained combining the Schrödinger-Lichnerowicz formula with the celebrated Atiyah-Singer index theorem [31], stating that the topological and the analytical index of any elliptic linear differential operator coincide. In the case of the Dirac operator, the index theorem reads as follows.

**Theorem 1.3.9 (M.F.Atiyah & I.M. Singer [31])** *Let $(M^n, g)$ be an even dimensional closed Riemannian spin manifold and $\mathrm{ind}(D^+)$ be the analytical index of the positive part of its Dirac operator. Then*

$$\mathrm{ind}(D^+) = \widehat{A}(M),$$

*where $\widehat{A}(M) = \int_M \widehat{A}[TM] \in \mathbb{Z}$ is the $\widehat{A}$-genus of $M$.*

Recall that, by definition, $\mathrm{ind}(D^+) := \dim(\mathrm{Ker}(D^+)) - \dim(\mathrm{Coker}(D^+))$. Since $D$ is formally self-adjoint (Proposition 1.3.4) and the formal adjoint of $D^\pm$ is $D^\mp$, one has $\mathrm{ind}(D^+) = \dim(\mathrm{Ker}(D^+)) - \dim(\mathrm{Ker}(D^-))$. The $\widehat{A}$-class of $TM$, denoted above by $\widehat{A}[TM]$, is the characteristic class associated to the Taylor expansion at 0 of the function $x \mapsto \frac{\sqrt{\frac{x}{2}}}{\sinh(\sqrt{\frac{x}{2}})}$. The $\widehat{A}$-genus of $M$ is by construction a rational number and it is already a highly non-trivial statement that, for spin manifolds, it must be an integer. Theorem 1.3.9 can be proved using either $K$-theoretical methods (see [178, Chap. 3] and references therein) or asymptotics for the heat kernel (see [61, Chap. 3] and references therein). Beware that, if the dimension of the underlying manifold is odd, then the index of its Dirac operator - as well as that of any elliptic linear differential operator - vanishes, see e.g. [178, Thm. 13.12].

We end this section with the remarkable property of conformal covariance of the Dirac operator.

**Proposition 1.3.10** *Let the spin structure be fixed and denote by $\overline{D}$ the Dirac operator of $M$ for the conformal metric $\overline{g} := e^{2u}g$, with $u \in C^\infty(M, \mathbb{R})$. Then there exists a unitary isomorphism between the spinor bundle of $(M^n, g)$ and that of $(M^n, \overline{g})$ (denoted in the whole text by $\varphi \mapsto \overline{\varphi}$) such that*

$$\overline{D}(e^{-\frac{n-1}{2}u}\overline{\varphi}) = e^{-\frac{n+1}{2}u}\overline{D\varphi} \tag{1.16}$$

*for every $\varphi \in \Gamma(\Sigma M)$, with $\overline{D} := D_{\overline{g}}$.*

*Proof*: The isometry $X \mapsto e^{-u}X$ from $(TM, g)$ onto $(TM, \overline{g})$ defines a principal bundle isomorphism $\mathrm{SO}_g(TM) \longrightarrow \mathrm{SO}_{\overline{g}}(TM)$ lifting to the spin level. More precisely, it induces a vector-bundle isomorphism $\Sigma_g M \longrightarrow \Sigma_{\overline{g}} M$, $\varphi \longmapsto \overline{\varphi}$, preserving the pointwise Hermitian inner product and sending $X \cdot \varphi$ onto $e^{-u}X \bar{\cdot} \overline{\varphi}$. As for the natural covariant derivative $\nabla$ on the spinor bundle, its local expression in terms of the Levi-Civita connection of $g$ on $TM$ (1.6) immediately implies for all $\varphi \in \Gamma(TM)$ and $X \in TM$:

$$\overline{\nabla}_X \overline{\varphi} = \overline{\nabla_X \varphi} - \frac{1}{2}\overline{X \cdot \mathrm{grad}(u) \cdot \varphi} - \frac{X(u)}{2}\overline{\varphi}. \tag{1.17}$$

Since $\{e^{-u}e_j\}_{1 \leq j \leq n}$ is a local o.n.b. of $TM$ for $\overline{g}$ as soon as $\{e_j\}_{1 \leq j \leq n}$ is one for $g$, we deduce that

$$\begin{aligned}
\overline{D}\overline{\varphi} &= \sum_{j=1}^n e^{-2u} e_j \bar{\cdot} \overline{\nabla}_{e_j} \overline{\varphi} \\
&\overset{(1.17)}{=} e^{-2u} \sum_{j=1}^n e_j \bar{\cdot} \left( \overline{\nabla_{e_j}\varphi} - \frac{1}{2}\overline{e_j \cdot \mathrm{grad}(u) \cdot \varphi} - \frac{e_j(u)}{2}\overline{\varphi} \right) \\
&= e^{-u} \sum_{j=1}^n \left( \overline{e_j \cdot \nabla_{e_j}\varphi} - \frac{1}{2}\overline{e_j \cdot e_j \cdot \mathrm{grad}(u) \cdot \varphi} - \frac{e_j(u)}{2}\overline{e_j \cdot \varphi} \right) \\
&= e^{-u}(\overline{D\varphi} + \frac{n-1}{2}\overline{\mathrm{grad}(u) \cdot \varphi}).
\end{aligned} \tag{1.18}$$

Hence

$$\begin{aligned}
\overline{D}(e^{-\frac{n-1}{2}u}\overline{\varphi}) &= e^{-u}(\overline{D(e^{-\frac{n-1}{2}u}\varphi)} + \frac{n-1}{2}e^{-\frac{n-1}{2}u}\overline{\mathrm{grad}(u)\cdot\varphi}) \\
&\overset{(1.11)}{=} -\frac{n-1}{2}e^{-\frac{n-1}{2}u}e^{-u}\overline{\mathrm{grad}(u)\cdot D\varphi} + e^{-\frac{n-1}{2}u}e^{-u}\overline{D\varphi} \\
&\quad +\frac{n-1}{2}e^{-\frac{n-1}{2}u}e^{-u}\overline{\mathrm{grad}(u)\cdot\varphi} \\
&= e^{-\frac{n+1}{2}u}\overline{D\varphi},
\end{aligned}$$

which concludes the proof. □

## 1.4 Spinors on hypersurfaces and coverings

In this section we discuss how spinors can be induced on submanifolds or quotients. We restrict ourselves to the case of Riemannian hypersurfaces and coverings, see e.g. [43, 108] and [204, 17, 122] for higher codimensional submanifolds and submersions or foliations respectively. The case of general homogeneous spaces, which is much more involved, is handled separately in Section 2.2. In order to simplify the notations, we denote in this survey, for the situation where $M$ is a hypersurface in some spin manifold $\widetilde{M}$, by "$\cdot$" the Clifford multiplication of $\widetilde{M}$ and by "$\underset{M}{\cdot}$" that of $M$.

**Proposition 1.4.1** *Let $\iota : M \longrightarrow \widetilde{M}$ be an immersed oriented Riemannian hypersurface in an $(n+1)$-dimensional Riemannian spin manifold. Let the unit normal $\nu \in \Gamma(T^{\perp}M)$ be chosen such that, for every local oriented basis $\{v_1,\ldots,v_n\}$ of $TM$, the local basis $\{v_1,\ldots,v_n,\nu\}$ of $T\widetilde{M}_{|_M}$ is oriented. Then the manifold $M$ is spin and carries an induced spin structure for which a unitary isomorphism exists*

$$\begin{array}{ccl}
\Sigma\widetilde{M}_{|_M} & \longrightarrow & \left| \begin{array}{ll} \Sigma M & \text{if } n \text{ is even} \\ \Sigma M \oplus \Sigma M & \text{if } n \text{ is odd} \end{array}\right. \\
\varphi & \longmapsto & \varphi
\end{array}$$

*where, for $n$ odd, the two copies of $\Sigma M$ correspond to the splitting $\Sigma\widetilde{M}_{|_M} = \Sigma^+\widetilde{M}_{|_M} \oplus \Sigma^-\widetilde{M}_{|_M}$. Moreover this isomorphism can be chosen so as to satisfy*

$$i\nu\cdot\varphi = \left| \begin{array}{ll} \omega_n^{\mathbb{C}} \underset{M}{\cdot} \varphi & \text{if } n \text{ is even} \\ \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}, & \text{if } n \text{ is odd} \end{array}\right. \tag{1.19}$$

$$X\cdot\nu\cdot\varphi = \left| \begin{array}{ll} X \underset{M}{\cdot} \varphi & \text{if } n \text{ is even} \\ (X \underset{M}{\cdot} \oplus - X \underset{M}{\cdot})\varphi & \text{if } n \text{ is odd} \end{array}\right. \tag{1.20}$$

*and*

$$\widetilde{\nabla}_X\varphi = \nabla_X\varphi + \frac{A(X)}{2}\cdot\nu\cdot\varphi \tag{1.21}$$

*for all $X \in TM$ and $\varphi \in \Gamma(\Sigma\widetilde{M}_{|_M})$, where $\omega_n^{\mathbb{C}}$ is the complex volume form (see (1.3) and $A := -\widetilde{\nabla}\nu$ denotes the Weingarten endormorphism field of $\iota$.*

*In particular the fundamental Dirac operators $D$ and $\widetilde{D}$ of $M$ and $\widetilde{M}$ respectively are related through*

$$\widetilde{D}\varphi = \nu \cdot \widetilde{\nabla}_\nu \varphi + \nu \cdot (D_2 - \frac{nH}{2})\varphi, \tag{1.22}$$

*where*

$$D_2 := \left| \begin{array}{ll} D & \text{if } n \text{ is even} \\ D \oplus -D & \text{if } n \text{ is odd} \end{array} \right.$$

*and $H := \frac{1}{n}\mathrm{tr}(A)$ is the mean curvature of $\iota$.*

*Proof*: Since $M$ has trivial normal bundle in $\widetilde{M}$, it is spin. This can be seen as a consequence of a more general result [200] or, alternatively, in the following way: the pull-back of $\mathrm{Spin}(T\widetilde{M})_{|_M}$ to $\mathrm{SO}(TM)$ over the map "completion by $\nu$"

$$\begin{array}{rcl} \mathrm{SO}(TM) & \longrightarrow & \mathrm{SO}(T\widetilde{M})_{|_M} \\ (e_1, \ldots, e_n) & \longmapsto & (e_1, \ldots, e_n, \nu) \end{array}$$

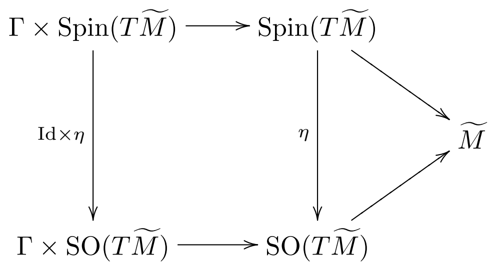
provides a 2-fold covering of $\mathrm{SO}(TM)$ which can easily be proved to be a $\mathrm{Spin}_n$-bundle hence a spin structure on $M$. The identity (1.20) is just the geometric translation of the canonical embedding of the Clifford algebra in dimension $n$ into that in dimension $n+1$, see e.g. [138, Prop. 2.7]. The local formula (1.6) defining the compatible covariant derivative combined with the classical Gauss-Weingarten identity $\widetilde{\nabla}_X Y = \nabla_X Y + g(A(X), Y)\nu$ (for all $X, Y \in \Gamma(TM)$) lead to (1.21). The last equality (1.22) is a straightforward consequence of both (1.20) and (1.21). $\square$

In particular a hypersurface in a spin manifold is spin as soon as it is orientable. For example the round sphere $\mathbb{S}^n$ is from its definition a hypersurface in $\mathbb{R}^{n+1}$, which is obviously spin, therefore it is also spin and carries an induced spin structure. This spin structure is unique if $n \geq 2$ since $\mathbb{S}^n$ is then simply-connected, however there exists another spin structure on $\mathbb{S}^1$, see Example 1.4.3.1 below.
Note also that, as a consequence of (1.19), (1.20) and (1.21), the Clifford action of $\nu$ onto $\Sigma M$ or $\Sigma M \oplus \Sigma M$ respectively (according to the parity of $n$) is $\nabla$-parallel, in particular it anti-commutes with $D_2$: for every $\varphi \in \Gamma(\Sigma M)$ (or in $\Gamma(\Sigma M \oplus \Sigma M)$ if $n$ is odd),

$$D_2(\nu \cdot \varphi) = -\nu \cdot D_2 \varphi. \tag{1.23}$$

Turning to coverings, we have the following proposition, which is also well-known (see e.g. [206, Lemma 7.3]).

**Proposition 1.4.2** *Let $\Gamma \times \widetilde{M} \to \widetilde{M}$ be a properly discontinuous free isometric and orientation-preserving action of a discrete group $\Gamma$ on a Riemannian spin manifold $(\widetilde{M}^n, g)$. Assume that the induced principal-bundle-action of $\Gamma$ on $\mathrm{SO}(T\widetilde{M})$ lifts to a principal-bundle-action on $\mathrm{Spin}(T\widetilde{M})$ such that the following* *diagram commutes:*




*Then the following holds:*

*i) The manifold* $M := {}^{\Gamma}\backslash^{\widetilde{M}}$ *is spin and carries an induced spin structure for which the* $\Gamma$*-action on* $\Sigma\widetilde{M}$ *provides a unitary isomorphism*

$$\Sigma M \cong {}^{\Gamma}\backslash^{\Sigma\widetilde{M}}$$

*preserving the Clifford multiplication and the natural compatible covariant derivative.*

*ii) This unitary isomorphism identifies the sections of* $\Sigma M$ *on* $M$ *with the* $\Gamma$*-equivariant sections of* $\Sigma\widetilde{M}$ *on* $\widetilde{M}$*, i.e.,*

$$\Gamma(\Sigma M) \cong \{\varphi \in \Gamma(\Sigma\widetilde{M}) \,|\quad \varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) \qquad \forall x \in \widetilde{M}, \forall \gamma \in \Gamma\}, \tag{1.24}$$

*where we also denote by "·" the action of* $\Gamma$ *on* $\Sigma\widetilde{M}$*. In particular, the eigenvectors of the Dirac operator on* $M$ *identify with those of the Dirac operator on* $\widetilde{M}$ *satisfying* (1.24)*.*

*iii) If* $\widetilde{M}$ *is simply-connected, then the spin structures on* $M$ *stand in one-to-one correspondence with* $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$*.*

*Proof*: The left-quotient ${}^{\Gamma}\backslash^{\mathrm{Spin}(T\widetilde{M})}$ obviously defines a spin structure on $M$. Since $\Gamma$ operates by principal-bundle-homomorphisms, its action commutes with the right action of the structure groups, therefore one straightforward obtains

$$\Sigma M = ({}^{\Gamma}\backslash^{\mathrm{Spin}(T\widetilde{M})}) \times_{\delta_n} \Sigma_n \cong {}^{\Gamma}\backslash^{(\mathrm{Spin}(T\widetilde{M}) \times_{\delta_n} \Sigma_n)} = {}^{\Gamma}\backslash^{\Sigma\widetilde{M}}.$$

For the same reason the Hermitian inner product of $\Sigma\widetilde{M}$ remains preserved by $\Gamma$ - hence the above identification can be assumed to be unitary - and so does the Clifford multiplication: for all $x \in \widetilde{M}$, $X \in T_x\widetilde{M}$, $\varphi \in \Sigma_x\widetilde{M}$ and $\gamma \in \Gamma$, $\gamma \cdot (X \cdot \varphi) = (d\gamma(X)) \cdot (\gamma \cdot \varphi)$. The equivariance condition follows from this observation. In the case where $\widetilde{M}$ is simply-connected one has

$$H^1(M, \mathbb{Z}_2) = \mathrm{Hom}(H_1(M), \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M), \mathbb{Z}_2) = \mathrm{Hom}(\Gamma, \mathbb{Z}_2),$$

where we have used $H_1(M) = {}^{\pi_1(M)}/_{[\pi_1(M), \pi_1(M)]}$. This concludes the proof. □

As noticed in Proposition 1.4.2.*iii*), the spin structure induced on the base space is in general not unique: multiplying the action of $\Gamma$ at the spin level by any group homomorphism $\Gamma \longrightarrow \{\pm 1\}$ provides another group action satisfying the assumptions of Proposition 1.4.2, however the spin structure downstairs changes. Note also that the pull-back of any spin structure on the base of a Riemannian covering is a spin structure on the total space and sections on the base can always be lifted to equivariant sections upstairs.

**Examples 1.4.3**

1. Let $\widetilde{M} := \mathbb{R}$ and $\Gamma := 2\pi\mathbb{Z}$ acting on $\widetilde{M}$ by translations. Since $\mathrm{Spin}(T\mathbb{R}) = \mathbb{R} \times \mathrm{Spin}_1 = \mathbb{R} \times \{\pm 1\}$ there are only two possible lifts of the $\Gamma$-action to the spin level which are determined by the image $(-1)^\delta$ of the generator $2\pi$ of $\Gamma$ in $\{\pm 1\}$, where $\delta \in \{0, 1\}$. We call the spin structure induced by $\delta = 0$ the *trivial spin structure* on $\mathbb{S}^1 = {}_{2\pi\mathbb{Z}}\backslash^{\mathbb{R}}$ and the one induced by $\delta = 1$ the *non-trivial* one.

2. More generally, consider a lattice $\Gamma \subset \widetilde{M} := \mathbb{R}^n$ of rank $n \geq 1$ and the corresponding torus $M := {}_{\Gamma}\backslash^{\mathbb{R}^n}$ with flat metric. The action of $\Gamma$ by translations induces the trivial group-homomorphism $\Gamma \longrightarrow \mathrm{SO}_n$, which obviously lifts to the spin level, hence $M$ is spin. Moreover, there exist $2^n$ group-homomorphisms $\Gamma \longrightarrow \{\pm 1\}$: fix a basis $(\gamma_1, \ldots, \gamma_n)$ of $\Gamma$, $\delta_1, \ldots, \delta_n \in \{0, 1\}$ and define the group-homomorphism $\varepsilon_{\delta_1,\ldots,\delta_n} : \Gamma \longrightarrow \{\pm 1\}$ by $\varepsilon_{\delta_1,\ldots,\delta_n}(\gamma_j) := (-1)^{\delta_j}$ for all $1 \leq j \leq n$. The spin structure on $M$ induced by $\varepsilon_{\delta_1,\ldots,\delta_n}$ is called $(\delta_1, \ldots, \delta_n)$-spin structure. We obtain in this way all spin structures on the torus.

3. Let $n \geq 2$ and $\Gamma$ be a finite subgroup of $\mathrm{SO}_{n+1}$ acting freely on $\mathbb{S}^n$. For $n$ even there is obviously no non-trivial such subgroup, thus we assume $n$ odd. On the round sphere both bundles $\mathrm{SO}(T\widetilde{M})$ and $\mathrm{Spin}(T\widetilde{M})$ canonically identify to $\mathrm{SO}_{n+1}$ and $\mathrm{Spin}_{n+1}$ respectively. Therefore, the existence of a lift of the action of $\Gamma$ to the spin level is equivalent to that of a group homomorphism $\epsilon : \Gamma \longrightarrow \mathrm{Spin}_{n+1}$ such that $\xi \circ \epsilon$ is the inclusion $\Gamma \subset \mathrm{SO}_{n+1}$. If this is fulfilled then from Proposition 1.4.2.*iii*) there are as many spin structures on ${}_{\Gamma}\backslash^{\mathbb{S}^n}$ as there are group homomorphisms $\Gamma \longrightarrow \{\pm 1\}$. For example consider $\Gamma = \mathbb{Z}_2 = \{\pm \mathrm{Id}\}$. Of course it preserves the orientation of $\mathbb{S}^n$ only if $n$ is odd. Furthermore, the pre-image of $-\mathrm{Id}$ under $\xi$ can be shown to be $\pm e_1 \cdot \ldots \cdot e_{n+1}$, which is involutive if and only if $n \equiv 0\ (4)$ or $n \equiv 3\ (4)$. Hence the $n$-dimensional real projective space $\mathbb{RP}^n$ is spin only for $n \equiv 3\ (4)$ and in that case it admits two spin structures.

## 1.5 Elliptic boundary conditions for the Dirac operator

We end this chapter by briefly describing four types of boundary conditions for the Dirac operator on manifolds $M$ with non-empty boundary $\partial M$ and discuss their ellipticity and self-adjointness. We mostly use the notations of [143], which is a good reference on the topic. A more general approach where less regularity is required can be found in the seminal paper [35].

Apart from the 1-dimensional case, the Dirac operator on a compact manifold with boundary is in general not Fredholm since it has infinite dimensional kernel; for example, given any domain $\Omega$ of $\mathbb{C}$, the kernel of the Dirac operator of $\Omega$ coincides with the direct sum of the space of holomorphic functions with that of anti-holomorphic functions on $\Omega$. The purpose of elliptic boundary conditions for $D$ is precisely to make it Fredholm. This is done in terms of pseudo-differential operators on the boundary. To avoid technicalities, we shortcut the original definition [143, p.380] of elliptic boundary condition providing the following one [143, Prop. 1]:

**Definition 1.5.1** *Let $(M^n, g)$ be a Riemannian spin manifold with non-empty boundary $\partial M$. Denote $\Sigma := \Sigma\partial M$ if $n$ is even and $\Sigma\partial M \oplus \Sigma\partial M$ if $n$ is odd.*

*i) An* elliptic boundary condition for $D$ *is a pseudo-differential operator $B$ : $\mathrm{L}^2(\Sigma) \longrightarrow \mathrm{L}^2(V)$, where $V$ is some Hermitian vector bundle on $\partial M$, such that the* boundary value problem

$$\left| \begin{array}{lll} D\varphi & = \Phi & \text{on } M \\ B(\varphi_{|\partial M}) & = \chi & \text{on } \partial M \end{array} \right. \tag{1.25}$$

*has smooth solutions up to a finite-dimensional kernel for any given smooth data $\Phi \in \Gamma(\Sigma M)$ and $\chi \in \Gamma(V)$ belonging to a certain subspace with finite codimension.*

*ii) An elliptic boundary condition for $D$ is called* self-adjoint *if and only if the restriction of $D$ onto $\{\varphi \in \Gamma(\Sigma M) \,|\, B(\varphi_{|\partial M}) = 0\}$ is symmetric.*

As a consequence of the Fredholm alternative, one may talk about the spectrum of $D$ if an elliptic boundary condition is fixed [143, Prop. 1]:

**Theorem 1.5.2** *Let $(M^n, g)$ be a compact Riemannian spin manifold with non-empty boundary $\partial M$. Let $B$ be an elliptic boundary condition for $D$. Then the eigenvalue problem*

$$\left| \begin{array}{lll} D\varphi & = \lambda\varphi & \text{on } M \\ B(\varphi_{|\partial M}) & = 0 & \text{on } \partial M \end{array} \right.$$

*has a discrete spectrum with finite dimensional eigenspaces in $\Gamma(\Sigma M)$, unless the spectrum is $\mathbb{C}$ itself. Moreover, if $B$ is self-adjoint, then the Dirac spectrum is real.*

In view of the study of spectral properties of the Dirac operator four kinds of boundary conditions have been mainly considered so far:

- *generalized Atiyah-Patodi-Singer* (gAPS) boundary condition (depending on a fixed $\beta \in \mathbb{R}$ that we omit in the notations) [78]: define $B := B_{\mathrm{gAPS}}$ to be the $\mathrm{L}^2$-orthogonal projection onto the subspace spanned by the eigenvectors of $D_2$ (which is $D_{\partial M}$ or $D_{\partial M} \oplus -D_{\partial M}$ according to the dimension) to eigenvalues not smaller than $\beta$. For $\beta = 0$ this condition is called the *Atiyah-Patodi-Singer* (APS) boundary condition and was originally introduced by Atiyah, Patodi and Singer to prove index theorems on manifolds with boundary, see e.g. [85, 147].

- Boundary condition associated to a *chirality* (CHI) operator (see e.g. [85, 147]): define the endomorphism-field $B_{\mathrm{CHI}} := \frac{1}{2}(\mathrm{Id} - \nu \cdot \mathcal{G})$ of $\Sigma$, where $\nu$ is a unit normal on $\partial M$ and $\mathcal{G}$ is a *chirality operator* (i.e., it is the restriction on $\partial M$ of an endomorphism-field $\mathcal{G}$ of $\Sigma M$ which is involutive, unitary, parallel and anti-commuting with the Clifford multiplication on $M$). Natural chirality operators appear in case $n$ is even (then define $\mathcal{G} := \omega_n^{\mathbb{C}} \cdot$, where $\omega_n^{\mathbb{C}}$ is the complex volume form of $M$, see (1.3)) or if $M$ is itself a spacelike hypersurface in a Lorentzian manifold (then define $\mathcal{G}$ to be the Clifford multiplication by a unit timelike normal vector field to $M$). Among others, this boundary condition has been used to prove positive mass theorems in the presence of black holes, see references in [143].

- MIT *bag* boundary condition (see e.g. [148]): define the endomorphism-field $B_{\mathrm{MIT}}$ of $\Sigma$ by $B_{\mathrm{MIT}} := \frac{1}{2}(\mathrm{Id} - i\nu\cdot)$. It was first introduced in the Lorentzian context by physicists at the MIT for the description of spin $\frac{1}{2}$-particles, see references in [143].

- *modified generalized Atiyah-Patodi-Singer*( (mgAPS) boundary condition (also depending on some fixed $\beta \in \mathbb{R}$) [78]: define $B_{\mathrm{mgAPS}} := B_{\mathrm{gAPS}}(\mathrm{Id} + \nu\cdot)$. In the particular case where $\beta = 0$ this condition is called the *modified Atiyah-Patodi-Singer* (mAPS) boundary condition [143].

To test the ellipticity of a boundary condition, practical criteria are available [143, Prop. 1]:

**Proposition 1.5.3** *Let $(M^n, g)$ be a compact Riemannian spin manifold with non-empty boundary $\partial M$. A pseudo-differential operator $B : \mathrm{L}^2(\Sigma) \longrightarrow \mathrm{L}^2(V)$, where $V$ is some Hermitian vector bundle on $\partial M$, is an elliptic boundary condition for $D$ if and only if its principal symbol $b : T^*\partial M \longrightarrow \mathrm{Hom}(\Sigma, V)$ satisfies the following two conditions:*

*a)* $\mathrm{Ker}(b(\xi)) \cap \mathrm{Ker}(i\xi \cdot \nu \cdot - g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id}) = \{0\}$

*b)* $\dim(\mathrm{Im}(b(\xi)) = \frac{\mathrm{rk}_{\mathbb{C}}(\Sigma M)}{2} = 2^{[\frac{n}{2}]-1}$

*for every $\xi \in T^*\partial M \setminus \{0\}$, where $\nu$ denotes the inner unit normal.*

The central result of this section is the following.

**Proposition 1.5.4** *Let $(M^n, g)$ be a compact Riemannian spin manifold with non-empty boundary $\partial M$. Then the* gAPS, CHI, MIT bag *and* mgAPS *boundary conditions are elliptic. Moreover, the spectrum of $D$ is a discrete unbounded sequence which is real for*

- *the* gAPS *and* mgAPS *boundary conditions with $\beta \leq 0$,*

- *the* CHI *boundary condition,*

*and which is contained in the upper half of $\mathbb{C}$ for the* MIT bag *boundary condition.*

*Proof*: We consider the four conditions separately.
• gAPS *boundary condition*: It can be shown (see e.g. [62, Prop. 14.2]) that the principal symbol of $B_{\mathrm{gAPS}}$ is given on any $\xi \in T^*\partial M \setminus \{0\}$ by $b_{\mathrm{gAPS}}(\xi) = \frac{1}{2}(i\xi \cdot \nu \cdot$

$+g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id})$, i.e., it is the (pointwise) orthogonal projection onto the eigenspace of the Clifford-multiplication by $i\xi$ (on $\partial M$, which corresponds to $i\xi\cdot\nu\cdot$, cf. (1.20)) to the eigenvalue $g(\xi,\xi)^{\frac{1}{2}}$. For $\xi\neq 0$ the image of $b_{\mathrm{gAPS}}(\xi)$ is obviously $2^{[\frac{n}{2}]-1}$-dimensional. On the other hand, $\mathrm{Ker}(b_{\mathrm{gAPS}}(\xi))=\mathrm{Ker}(i\xi\cdot\nu\cdot+g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id})$, so that the criterium *a*) of Proposition 1.5.3 is also fulfilled, hence the gAPS boundary condition is elliptic.

We now have to show that, for any $\varphi,\psi\in\Gamma(\Sigma M)$ satisfying $B_{\mathrm{gAPS}}(\varphi_{|\partial M})=B_{\mathrm{gAPS}}(\psi_{|\partial M})=0$ then $\int_M\langle D\varphi,\psi\rangle v_g=\int_M\langle\varphi,D\psi\rangle v_g$ holds. But from (1.14) and Green's formula we know that, for all $\varphi,\psi\in\Gamma(\Sigma M)$,

$$\int_M\langle D\varphi,\psi\rangle-\langle\varphi,D\psi\rangle v_g=\int_{\partial M}\langle\varphi,\nu\cdot\psi\rangle v_g^{\partial M}. \tag{1.26}$$

Therefore we have to prove that $B_{\mathrm{gAPS}}(\varphi_{|\partial M})=B_{\mathrm{gAPS}}(\psi_{|\partial M})=0$ implies the vanishing of $\int_{\partial M}\langle\varphi,\nu\cdot\psi\rangle v_g^{\partial M}$. Denote by $(\cdot\,,\cdot)_M:=\int_M\langle\cdot\,,\cdot\rangle v_g^M$ the $\mathrm{L}^2$-inner product on $\Gamma(\Sigma M)$, $(\cdot\,,\cdot)_{\partial M}:=\int_{\partial M}\langle\cdot\,,\cdot\rangle v_g^{\partial M}$ that on $\Gamma(\Sigma\partial M)$ and for any real numbers $c<d$ by $\pi_{\geq c}$ (resp. $\pi_{>c}$, $\pi_{<c}$, $\pi_{\leq c}$, $\pi_{[c,d]}$) the $\mathrm{L}^2$-orthogonal projection onto the subspace spanned by the eigenvectors of $D_2$ associated to the eigenvalues lying in the interval $[c,+\infty[$ (resp. $]c,+\infty[$, $]-\infty,c[$, $]-\infty,c]$, $[c,d]$). Since $B_{\mathrm{gAPS}}=\pi_{\geq\beta}$, we have, for $\beta\leq 0$ and all $\varphi,\psi\in\Gamma(\Sigma M)$ satisfying $B_{\mathrm{gAPS}}(\varphi_{|\partial M})=B_{\mathrm{gAPS}}(\psi_{|\partial M})=0$:

$$\begin{aligned}(\varphi,\nu\cdot\psi)_{\partial M}&=\Big(\pi_{<\beta}(\varphi),\pi_{<\beta}(\nu\cdot\psi)\Big)_{\partial M}\\ &\overset{(1.23)}{=}\Big((\pi_{<\beta}(\varphi),\nu\cdot\underbrace{\pi_{>-\beta}(\psi)}_{0}\Big)_{\partial M}\\ &=0.\end{aligned}$$

• CHI *boundary condition*: The endomorphism-field $\nu\cdot\mathcal{G}$ of $\Sigma$ is by definition of $\mathcal{G}$ unitary, Hermitian and involutive, therefore $B_{\mathrm{CHI}}$ is nothing but the pointwise orthogonal projection onto its eigenspace to the eigenvalue $-1$, which is $2^{[\frac{n}{2}]-1}$-dimensional. Moreover since it is a differential operator of zero order its principal symbol is the operator itself, therefore criterium *b*) of Proposition 1.5.3 is fulfilled. On the other hand, for any $\xi\in T^*\partial M\setminus\{0\}$, the endomorphisms $i\xi\cdot\nu\cdot$ and $\nu\cdot\mathcal{G}$ of $\Sigma$ obviously anti-commute, so that, for any $\varphi\in\mathrm{Ker}(\mathrm{Id}-\nu\cdot\mathcal{G})\cap\mathrm{Ker}(i\xi\cdot\nu\cdot-g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id})$,

$$\begin{aligned}g(\xi,\xi)^{\frac{1}{2}}\varphi&=i\xi\cdot\nu\cdot\varphi\\ &=i\xi\cdot\nu\cdot\nu\cdot\mathcal{G}\varphi\\ &=-\nu\cdot\mathcal{G}(i\xi\cdot\nu\cdot\varphi)\\ &=-g(\xi,\xi)^{\frac{1}{2}}\nu\cdot\mathcal{G}\varphi\\ &=-g(\xi,\xi)^{\frac{1}{2}}\varphi,\end{aligned}$$

which implies $\varphi=0$. Hence the criterium *a*) of Proposition 1.5.3 is also fulfilled. This shows the ellipticity of the CHI boundary condition.

Let now $\varphi,\psi\in\Gamma(\Sigma M)$ satisfying $B_{\mathrm{CHI}}(\varphi_{|\partial M})=B_{\mathrm{CHI}}(\psi_{|\partial M})=0$, then

$$\begin{aligned}(\varphi,\nu\cdot\psi)_{\partial M}&=(\nu\cdot\mathcal{G}\varphi,\nu\cdot\psi)_{\partial M}\\ &=(\varphi,\mathcal{G}\psi)_{\partial M}\\ &=-(\varphi,\nu\cdot\mathcal{G}^2\psi)_{\partial M}\\ &=-(\varphi,\nu\cdot\psi)_{\partial M},\end{aligned}$$

hence $(\varphi, \nu \cdot \psi)_{\partial M} = 0$ and the spectrum of $D$ must therefore be real.
• MIT bag *boundary condition*: The endomorphism-field $i\nu\cdot$ of $\Sigma$ is unitary, Hermitian and involutive, so that $B_{\mathrm{MIT}}$ is the pointwise orthogonal projection onto its eigenspace to the eigenvalue $-1$, which is $2^{[\frac{n}{2}]-1}$-dimensional. Moreover, for any $\xi \in T^*\partial M \setminus \{0\}$, the endomorphisms $i\nu\cdot$ and $i\xi \cdot \nu\cdot$ of $\Sigma$ anti-commute, so that the same arguments as for the CHI boundary condition apply for the ellipticity.
Let now $\varphi, \psi \in \Gamma(\Sigma M)$ satisfying $B_{\mathrm{MIT}}(\varphi_{|\partial M}) = B_{\mathrm{MIT}}(\psi_{|\partial M}) = 0$, then this time $(\varphi, \nu \cdot \psi)_{\partial M}$ does not vanish in general. However (1.14) with $\varphi = \psi$ gives

$$2i\Im m\left(\int_M \langle D\psi, \psi\rangle v_g\right) = \int_{\partial M} \langle \psi, \nu \cdot \psi\rangle v_g^{\partial M} = i\int_{\partial M} |\psi|^2 v_g^{\partial M},$$

therefore any eigenvalue $\lambda$ of $D$ with associated (non-zero) eigenvector $\psi$ must satisfy $\Im m(\lambda) = \frac{\int_{\partial M} |\psi|^2 v_g^{\partial M}}{2\int_M |\psi|^2 v_g}$. If $\psi_{|\partial M} = 0$ then a unique continuation property for the Dirac operator [62, Sec. 1.8] would imply $\psi = 0$ on $M$, contradiction. Therefore $\Im m(\lambda) > 0$.
• mgAPS *boundary condition*: Since $\mathrm{Id} + \nu\cdot$ is an isomorphism-field of $\Sigma$, the principal symbol of $B_{\mathrm{mgAPS}}$ evaluated on any vector $\xi \in T^*\partial M \setminus \{0\}$ is $b_{\mathrm{mgAPS}}(\xi) = b_{\mathrm{gAPS}}(\xi) \circ (\mathrm{Id} + \nu\cdot)$, hence it has rank $2^{[\frac{n}{2}]-1}$. Moreover,

$$\mathrm{Ker}(b_{\mathrm{mgAPS}}(\xi)) = (\mathrm{Id} - \nu\cdot)\mathrm{Ker}(b_{\mathrm{gAPS}}(\xi)) = (\mathrm{Id} - \nu\cdot)\mathrm{Ker}(i\xi \cdot \nu \cdot + g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id}).$$

Let $\varphi \in \mathrm{Ker}(b_{\mathrm{mgAPS}}(\xi)) \cap \mathrm{Ker}(i\xi \cdot \nu \cdot - g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id})$, then there exists a $\psi \in \mathrm{Ker}(i\xi \cdot \nu \cdot + g(\xi,\xi)^{\frac{1}{2}}\mathrm{Id})$ with $\varphi = (\mathrm{Id} - \nu\cdot)\psi$ so that

$$\begin{aligned} g(\xi,\xi)^{\frac{1}{2}}(\mathrm{Id} - \nu\cdot)\psi &= g(\xi,\xi)^{\frac{1}{2}}\varphi \\ &= i\xi \cdot \nu \cdot \varphi \\ &= (\mathrm{Id} + \nu\cdot) i\xi \cdot \nu \cdot \psi \\ &= -g(\xi,\xi)^{\frac{1}{2}}(\mathrm{Id} + \nu\cdot)\psi, \end{aligned}$$

from which one deduces that $\psi = 0$ hence $\varphi = 0$. This shows the ellipticity of the mgAPS boundary condition.
Let now $\varphi, \psi \in \Gamma(\Sigma M)$ satisfying $B_{\mathrm{mgAPS}}(\varphi_{|\partial M}) = B_{\mathrm{gAPS}}(\psi_{|\partial M}) = 0$, then with the notations introduced above for the gAPS boundary condition and for $\beta \leq 0$ one has:

$$\begin{aligned} 2(\varphi, \nu \cdot \psi)_{\partial M} &= \Big(\{\mathrm{Id} + \nu\cdot\}\varphi, \{\mathrm{Id} + \nu\cdot\}\nu \cdot \psi\Big)_{\partial M} \\ &= \Big(\pi_{<\beta}(\{\mathrm{Id} + \nu\cdot\}\varphi), \pi_{<\beta}(\{\mathrm{Id} + \nu\cdot\}\nu \cdot \psi)\Big)_{\partial M} \\ &\overset{(1.23)}{=} \Big(\pi_{<\beta}(\{\mathrm{Id} + \nu\cdot\}\varphi), \nu \cdot \underbrace{\pi_{>-\beta}(\{\mathrm{Id} + \nu\cdot\}\psi)}_{0}\Big)_{\partial M} \\ &= 0, \end{aligned}$$

which shows that the spectrum of $D$ under the mgAPS boundary condition is real and concludes the proof. □

# Chapter 2

# Explicit computations of spectra

In this chapter we present the few known closed Riemannian spin manifolds whose Dirac spectrum - or at least some eigenvalues - can be explicitly computed.

## 2.1 Spectrum of some non-negatively curved space-forms

We begin with the examples where no machinery is required. The simplest ones are the flat tori. The description of the $(\delta_1, \ldots, \delta_n)$-spin structure on the $n$-torus is explained in Example 1.4.3.2.

**Theorem 2.1.1 (T. Friedrich [89])** *For a positive integer $n$ let $\Gamma \subset \mathbb{R}^n$ be a lattice and $M = \mathbb{T}^n := {}_{\Gamma}\backslash^{\mathbb{R}^n}$ the corresponding $n$-dimensional torus. Fix a basis $(\gamma_1, \ldots, \gamma_n)$ of $\Gamma$ and $\delta_1, \ldots, \delta_n \in \{0, 1\}$.*
*Then the spectrum of the Dirac operator of $\mathbb{T}^n$ endowed with the induced flat metric and the $(\delta_1, \ldots, \delta_n)$-spin structure is given by*

$$\Big\{ \pm 2\pi|\gamma^* + \frac{1}{2}\sum_{j=1}^{n} \delta_j \gamma_j^*|, \quad \gamma^* \in \Gamma^* \Big\},$$

*where $\Gamma^* := \{\theta \in (\mathbb{R}^n)^* \,|\, \theta(\Gamma) \subset \mathbb{Z}\}$ is the dual lattice and $(\gamma_1^*, \ldots, \gamma_n^*)$ the basis of $\Gamma^*$ dual to $(\gamma_1, \ldots, \gamma_n)$. Furthermore, if non-zero, the eigenvalue provided by $\gamma^*$ has multiplicity $2^{[\frac{n}{2}]-1}$. In case $\delta_1 = \ldots = \delta_n = 0$ the multiplicity of the eigenvalue $0$ is $2^{[\frac{n}{2}]}$.*

Beware that the multiplicities add if the corresponding eigenvalues are equal. Thus, the multiplicity of the eigenvalue $\pm 2\pi|\gamma^* + \frac{1}{2}\sum_{j=1}^{n} \delta_j \gamma_j^*|$ is always at least $2^{[\frac{n}{2}]}$ (even for $n = 1$): if it is non zero, then $\gamma'^* := -\gamma^* - \sum_{j=1}^{n} \delta_j \gamma_j^* \in \Gamma^*$ provides the same eigenvalue and $\gamma'^* \neq \gamma^*$.

*Proof of* Theorem 2.1.1: For any $f \in C^\infty(\mathbb{R}^n, \Sigma_n)$, the equivariance condition (1.24) reads

$$f(x + \gamma_j) = (-1)^{\delta_j} f(x)$$

for all $x \in \mathbb{R}^n$ and $1 \leq j \leq n$. Given $\gamma^* \in \Gamma^*$, we denote by $\theta_\gamma$ the constant 1-form $\gamma^* + \frac{1}{2}\sum_{j=1}^n \delta_j \gamma_j^*$ on $\mathbb{R}^n$. For an arbitrary orthonormal basis $(\sigma_l)_{1 \leq l \leq 2^{[\frac{n}{2}]}}$ of $\Sigma_n$ - which we trivially extend onto $\mathbb{R}^n$ as sections of $\Sigma(\mathbb{R}^n) = \mathbb{R}^n \times \Sigma_n$ - and $1 \leq l \leq n$, we define the spinor field

$$\phi_{\gamma,l} := e^{2i\pi\theta_\gamma}\sigma_l \tag{2.1}$$

on $\mathbb{R}^n$. It obviously satisfies the equivariance condition and, for any $X \in \mathbb{R}^n$,

$$\begin{aligned} \nabla_X \phi_{\gamma,l} &= X(\phi_{\gamma,l}) \\ &= 2i\pi\theta_\gamma(X) e^{2i\pi\theta_\gamma}\sigma_l \\ &= 2i\pi\theta_\gamma(X)\phi_{\gamma,l}, \end{aligned}$$

so that, choosing a local orthonormal basis $(e_k)_{1 \leq k \leq n}$ of $\mathbb{R}^n$

$$\begin{aligned} D\phi_{\gamma,l} &= \sum_{k=1}^n e_k \cdot \nabla_{e_k}\phi_{\gamma,l} \\ &= 2i\pi \sum_{k=1}^n \theta_\gamma(e_k) e_k \cdot \phi_{\gamma,l} \\ &= 2i\pi\theta_\gamma \cdot \phi_{\gamma,l}. \end{aligned}$$

If $\theta_\gamma = 0$, which only happens if $\gamma^* = 0$ and $\delta_1 = \ldots = \delta_n = 0$, the spinor $\phi_{\gamma,l} = \sigma_l$ provides an eigenvector of $D$ associated to the eigenvalue 0. Moreover, as a consequence of the Schrödinger-Lichnerowicz formula (1.15), the kernel of $D$ consists of parallel spinors - as it does whenever the scalar curvature of the closed manifold vanishes. We deduce in that case that 0 is an eigenvalue of $D$ with multiplicity $2^{[\frac{n}{2}]}$.
If $\theta_\gamma \neq 0$ and $n = 1$ then $i\frac{\theta_\gamma}{|\theta_\gamma|}\cdot = \mathrm{Id}$ or $-\mathrm{Id}$ on $\Sigma_1 = \mathbb{C}$, therefore $D\phi_{\gamma,l} = 2\pi|\theta_\gamma|\phi_{\gamma,l}$ or $-2\pi|\theta_\gamma|\phi_{\gamma,l}$, i.e., $\phi_{\gamma,l}$ is a non-zero eigenvector of $D$ associated to the eigenvalue $2\pi|\theta_\gamma|$ or $-2\pi|\theta_\gamma|$. Both eigenvalues occur because, as noticed above, $\theta_{-\gamma^*-\sum_{j=1}^n \delta_j\gamma_j^*} = -\theta_\gamma$. Each eigenvalue has multiplicity 1.
If $\theta_\gamma \neq 0$ and $n \geq 2$ we consider the Clifford action of $i\frac{\theta_\gamma}{|\theta_\gamma|}$ on $\Sigma_n$. It is involutive, parallel and unitary, hence it induces the orthogonal and parallel splitting

$$\Sigma_n = \mathrm{Ker}(i\frac{\theta_\gamma}{|\theta_\gamma|} \cdot -\mathrm{Id}) \oplus \mathrm{Ker}(i\frac{\theta_\gamma}{|\theta_\gamma|} \cdot +\mathrm{Id}),$$

where both spaces on the r.h.s. have the same dimension since the Clifford actions of two orthogonal vectors anti-commute. We replace in that case $(\sigma_l)_{1 \leq l \leq 2^{[\frac{n}{2}]}}$ by an orthonormal basis $(\sigma_1^+, \ldots, \sigma^+_{2^{[\frac{n}{2}]-1}}, \sigma_1^-, \ldots, \sigma^-_{2^{[\frac{n}{2}]-1}})$ of $\Sigma_n$, where each $(\sigma_1^\epsilon, \ldots, \sigma^\epsilon_{2^{[\frac{n}{2}]-1}})$ is a constant orthonormal basis of $\mathrm{Ker}(i\frac{\theta_\gamma}{|\theta_\gamma|} \cdot -\epsilon\mathrm{Id}) \subset \Sigma_n$ for $\epsilon \in \{\pm 1\}$. We redenote the $\phi_{\gamma,l}$ of (2.1) by $\phi^\epsilon_{\gamma,l}$. The above computations apply with $\sigma_l^\epsilon$ instead of $\sigma_l$, so that

$$\begin{aligned} D\phi^\epsilon_{\gamma,l} &= 2i\pi(-i\epsilon|\theta_\gamma|)\phi^\epsilon_{\gamma,l} \\ &= 2\pi\epsilon|\theta_\gamma|\phi^\epsilon_{\gamma,l}, \end{aligned}$$

i.e., $\phi^\epsilon_{\gamma,l}$ is a non-zero eigenvector of $D$ associated to the eigenvalue $2\pi\epsilon|\theta_\gamma|$. Since the sections $\phi^\epsilon_{\gamma,1}, \ldots, \phi^\epsilon_{\gamma,2^{[\frac{n}{2}]-1}}$ are linearly independent, the multiplicity

of the eigenvalue $2\pi\epsilon|\theta_\gamma|$ is at least $2^{[\frac{n}{2}]-1}$.
To conclude the proof, it remains to remember that the $\{e^{i\gamma^*}, \gamma^* \in \Gamma^*\}$ form a Hilbert basis of $\mathrm{L}^2(\mathbb{T}^n, \mathbb{C})$ and therefore so do the $\phi^{(\epsilon)}_{\gamma,l}$ in $\mathrm{L}^2(\Sigma\mathbb{T}^n)$. □

**Notes 2.1.2**

1. For $n = 1$, Theorem 2.1.1 reads as follows: the Dirac spectrum of the circle $\mathbb{S}^1(L)$ of length $L > 0$ (for $L = 2\pi$ we just write $\mathbb{S}^1$) and the $\delta$-spin structure, where $\delta \in \{0, 1\}$, is $\frac{2\pi}{L}(\frac{\delta}{2} + \mathbb{Z})$. Furthermore, each eigenvalue is simple.

2. It is remarkable that the kernel of the Dirac operator of $(\mathbb{T}^n, g_{\mathrm{flat}})$ is not reduced to 0 for the $(0, \ldots, 0)$-spin structure (called sometimes the trivial spin structure) whereas it is for all other ones. Flat tori are thus the most simple-minded examples of closed manifolds with non-zero harmonic spinors for some spin structure. Moreover, as already noticed in the proof of Theorem 2.1.1, the kernel of $D$ actually consists of parallel spinors. Therefore, $(\mathbb{T}^n, g_{\mathrm{flat}})$ admits a $2^{[\frac{n}{2}]}$-dimensional space of parallel spinors for the trivial spin structure and no non-zero one otherwise. The reader interested in basic results as well as the classification of complete Riemannian spin manifolds with parallel spinors should refer to Section A.4.

Only few spectra of closed flat manifolds are known, although such manifolds are always covered by flat tori as a consequence of Bieberbach's theorems. This is due to the high complexity of the groups involved, which makes the search for equivariant eigenvectors very technical. Up to now, only dimension 3 (F. Pfäffle [212]) and some particular cases in higher dimensions (R. Miatello and R. Podestá [194]) have been handled using representation-theoretical methods, see Section 2.2.

A rather different technique leads to the Dirac spectrum of round spheres.

**Theorem 2.1.3 (S. Sulanke [232], see also [41, 75, 234, 235])** *Consider, for $n \geq 2$, the round sphere $M = \mathbb{S}^n := \{x \in \mathbb{R}^{n+1}, |x| = 1\}$ with its canonical metric $g$ of constant sectional curvature 1 and its canonical spin structure.
Then the spectrum of the Dirac operator is $\{\pm(\frac{n}{2} + k), k \in \mathbb{N}\}$ and each eigenvalue $\pm(\frac{n}{2} + k)$ has multiplicity $2^{[\frac{n}{2}]} \cdot \binom{n+k-1}{k}$.*

*Proof*: We present here C. Bär's proof [41, Sec. 2], which has the advantage to get to the result in a very elementary way. It is based on the knowledge of the spectrum of the scalar Laplacian and on the trivialization of the spinor bundle of $\mathbb{S}^n$ through either $-\frac{1}{2}$- or $\frac{1}{2}$-Killing spinors, see Example A.1.3.2.
Let $\varphi$ be a non-zero $\frac{\varepsilon}{2}$-Killing spinor on $(\mathbb{S}^n, \mathrm{can})$, where $\varepsilon \in \{\pm 1\}$. Then $D\varphi = \frac{\varepsilon}{2}\sum_{j=1}^n e_j \cdot e_j \cdot \varphi = -\frac{n\varepsilon}{2}\varphi$, so that, for every $f \in C^\infty(\mathbb{S}^n, \mathbb{R})$,

$$
\begin{aligned}
D^2(f\varphi) &\overset{(1.13)}{=} fD^2\varphi - 2\nabla_{\mathrm{grad}(f)}\varphi + (\Delta f)\varphi \\
&= \frac{n^2}{4}f\varphi - \varepsilon\mathrm{grad}(f) \cdot \varphi + (\Delta f)\varphi
\end{aligned}
$$

$$\begin{aligned}&\stackrel{(1.11)}{=} \frac{n^2}{4}f\varphi - \varepsilon(D(f\varphi) - fD\varphi) + (\Delta f)\varphi\\ &= (\frac{n^2}{4} - \frac{n}{2})f\varphi - \varepsilon D(f\varphi) + (\Delta f)\varphi,\end{aligned}$$

that is, $((D + \frac{\varepsilon}{2}\mathrm{Id})^2 - \frac{1}{4}\mathrm{Id})(f\varphi) = (\frac{n^2}{4} - \frac{n}{2})f\varphi + (\Delta f)\varphi$, or, equivalently,

$$(D + \frac{\varepsilon}{2}\mathrm{Id})^2(f\varphi) = (\Delta f + (\frac{n-1}{2})^2 f)\varphi. \tag{2.2}$$

The spectrum of the scalar Laplacian $\Delta$ on $(\mathbb{S}^n, \mathrm{can})$ is given by (see e.g. [77]) $\{k(n+k-1)\,|\,k \in \mathbb{N}\}$, where the eigenvalue $k(n+k-1)$ appears with multiplicity $\frac{n+2k-1}{n+k-1} \cdot \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$. Since the spinor bundle $\Sigma\mathbb{S}^n$ of $\mathbb{S}^n$ is trivialized by $\frac{\varepsilon}{2}$-Killing spinors one deduces from (2.2) that, if $\{f_k\}_{k\in\mathbb{N}}$ denotes a $\mathrm{L}^2$-orthonormal basis of eigenfunctions of $\Delta$ on $\mathbb{S}^n$ and $\{\varphi_j\}_{1\le j\le 2^{[\frac{n}{2}]}}$ a trivialization of $\Sigma\mathbb{S}^n$ through a pointwise orthonormal basis, then $\{f_k\varphi_j\,|\,k \in \mathbb{N}, 1 \le j \le 2^{[\frac{n}{2}]}\}$ provides a complete $\mathrm{L}^2$-orthonormal basis of $\mathrm{L}^2(\Sigma\mathbb{S}^n)$ made out of eigenvectors of $(D+\frac{\varepsilon}{2}\mathrm{Id})^2$ associated to the eigenvalues $(\frac{n-1}{2}+k)^2$ with $k \in \mathbb{N}$, each of those having multiplicity $2^{[\frac{n}{2}]} \cdot \frac{n+2k-1}{n+k-1} \cdot \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$. Therefore the spectrum of $D$ is contained in $-\frac{\varepsilon}{2} \pm (\frac{n-1}{2} + \mathbb{N})$, where the multiplicities remain to be determined. We name the possible eigenvalues of $D$

$$\begin{aligned}\lambda_k^+ &:= \frac{n}{2} + k\\ \lambda_{-k-1}^+ &:= -\frac{n}{2} - k + 1\\ \lambda_k^- &:= -\frac{n}{2} - k\\ \lambda_{-k-1}^- &:= \frac{n}{2} + k - 1,\end{aligned}$$

with $k \in \mathbb{N}$, and denote by $m(\cdot)$ their corresponding multiplicity (both $\varepsilon = \pm 1$ have to be taken into account). It can already be deduced from the splitting

$$\begin{aligned}\mathrm{Ker}((D + \frac{\varepsilon}{2}\mathrm{Id})^2 - (\frac{n-1}{2} + k)^2\mathrm{Id}) &= \mathrm{Ker}(D + (\frac{\varepsilon}{2} - \frac{n-1}{2} - k)\mathrm{Id})\\ &\bigoplus \mathrm{Ker}(D + (\frac{\varepsilon}{2} + \frac{n-1}{2} + k)\mathrm{Id})\end{aligned}$$

that, for every $k \in \mathbb{N}$,

$$m(\lambda_k^\pm) + m(\lambda_{-k-1}^\pm) = 2^{[\frac{n}{2}]} \cdot \frac{n+2k-1}{n+k-1} \cdot \begin{pmatrix} n+k-1 \\ k \end{pmatrix}.$$

Next we show by induction on $k$ that $m(\lambda_k^\pm) = 2^{[\frac{n}{2}]} \cdot \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$.

For $k = 0$, both $\lambda_0^+$ and $\lambda_0^-$ have multiplicity $2^{[\frac{n}{2}]}$ because, as we have seen above, $\frac{\varepsilon}{2}$-Killing spinors are eigenvectors for $D$ associated to the eigenvalue $-\frac{n\varepsilon}{2}$. Alternatively the eigenvalues $\lambda_{-1}^\pm = \mp(\frac{n}{2} - 1)$ cannot appear because otherwise Friedrich's inequality (3.1) would be violated.

If the result is true for $k$, then the identity just above implies

$$\begin{aligned} m(\lambda^{\pm}_{k+1}) &= 2^{[\frac{n}{2}]}\cdot\frac{n+2k+1}{n+k}\cdot\left(\begin{array}{c} n+k \\ k+1 \end{array}\right) - m(\lambda^{\pm}_{-k-2}) \\ &= 2^{[\frac{n}{2}]}\cdot\frac{n+2k+1}{n+k}\cdot\left(\begin{array}{c} n+k \\ k+1 \end{array}\right) - m(\lambda^{\mp}_{k}) \\ &= 2^{[\frac{n}{2}]}\cdot\Big\{\frac{n+2k+1}{n+k}\cdot\left(\begin{array}{c} n+k \\ k+1 \end{array}\right) - \left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\Big\} \\ &= 2^{[\frac{n}{2}]}\cdot\left(\begin{array}{c} n+k \\ k+1 \end{array}\right), \end{aligned}$$

which was to be shown. □

On spaceforms of positive curvature the Dirac spectrum can be determined thanks to a handy formula which reads as follows. First assume such a spaceform $(M^n, g) = (\Gamma\backslash\mathbb{S}^n, \mathrm{can})$ to be spin, which is equivalent to $n$ odd and the existence of a group homomorphism $\epsilon : \Gamma \longrightarrow \mathrm{Spin}_{n+1}$ such that $\xi \circ \epsilon$ is the inclusion map $\Gamma \subset \mathrm{SO}_{n+1}$, see Example 1.4.3.3. Proposition 1.4.2 states that the eigenvectors of the Dirac operator on $(M^n, g)$ are exactly the $\Gamma$-equivariant eigenvectors of the Dirac operator on $(\mathbb{S}^n, \mathrm{can})$. In particular the Dirac spectrum of $(M^n, g)$ is included in that of $(\mathbb{S}^n, \mathrm{can})$, so that it is enough to find the multiplicity $m(\cdot)$ of each eigenvalue $\pm(\frac{n}{2} + k)$. To that extent one encodes them into the two following formal power series:

$$F_{\pm}(z) := \sum_{k=0}^{+\infty} m(\pm(\frac{n}{2} + k))z^k.$$

**Theorem 2.1.4 (C. Bär [41])** *For $n \geq 3$ odd, let $(M^n, g) := (\Gamma\backslash\mathbb{S}^n, \mathrm{can})$ be a Riemannian spin spaceform of constant sectional curvature 1 and with spin structure fixed by $\epsilon$. Then the Dirac eigenvalues of $(M^n, g)$ are contained in $\{\pm(\frac{n}{2} + k),\ k \in \mathbb{N}\}$ with multiplicities given by*

$$F_{\pm}(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^{\mp}(\epsilon(\gamma)) - \chi^{\pm}(\epsilon(\gamma))\cdot z}{\det(\mathrm{Id}_{\mathbb{R}^{n+1}} - z\cdot\gamma)},$$

*where $\chi^{\pm} := \mathrm{tr}(\delta^{\pm}_{n+1}) : \mathrm{Spin}_{n+1} \longrightarrow \mathbb{C}$ is the character of $\delta^{\pm}_{n+1}$.*

The proof of Theorem 2.1.4 relies on a similar formula for the Laplace eigenvalues by A. Ikeda, see [41, Sec. 3]. As an application of Theorem 2.1.4, one obtains the Dirac spectrum of real projective spaces. We keep the notations of Example 1.4.3.3.

**Corollary 2.1.5** *For $n \equiv 3\ (4)$ let $M^n := \mathbb{R}\mathrm{P}^n = \mathbb{Z}_2\backslash\mathbb{S}^n$ be the $n$-dimensional real projective space of constant sectional curvature 1 and with spin structure fixed by $\epsilon(-\mathrm{Id}) = (-1)^{\delta} e_1 \cdot \ldots \cdot e_{n+1}$, where $\delta \in \{0, 1\}$. Then the spectrum of its Dirac operator is*

$$\{\frac{n}{2} + k,\quad k \in \mathbb{N} \cap (\delta + \frac{n-3}{4} + 2\mathbb{Z})\} \cup \{-\frac{n}{2} - k,\quad k \in \mathbb{N} \cap (\delta + \frac{n+1}{4} + 2\mathbb{Z})\},$$

*each eigenvalue corresponding to $k$ having multiplicity $2^{[\frac{n}{2}]}\cdot\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)$.*

*Proof*: On the one hand, $\chi^{\pm}(\epsilon(\mathrm{Id})) = \mathrm{tr}(\delta_{n+1}^{\pm}(1)) = 2^{\frac{n-1}{2}}$, on the other hand,

$$\begin{aligned}
\chi^{\pm}(\epsilon(-\mathrm{Id})) &= (-1)^{\delta}\mathrm{tr}(\delta_n^{\pm}(e_1 \cdot \ldots \cdot e_{n+1})) \\
&= (-1)^{\delta+\frac{n+1}{4}}\mathrm{tr}(\delta_n^{\pm}(\omega_n^{\mathbb{C}})) \\
&\stackrel{(1.4)}{=} \pm(-1)^{\delta+\frac{n+1}{4}}\mathrm{tr}(\mathrm{Id}_{\Sigma_n^{\pm}}) \\
&= \pm(-1)^{\delta+\frac{n+1}{4}}2^{\frac{n-1}{2}},
\end{aligned}$$

so that Theorem 2.1.4 implies that

$$\begin{aligned}
F_{\pm}(z) &= \frac{1}{2}\sum_{\gamma\in\Gamma}\frac{\chi^{\mp}(\epsilon(\gamma)) - \chi^{\pm}(\epsilon(\gamma))\cdot z}{\det(\mathrm{Id}_{\mathbb{R}^{n+1}} - z\cdot\gamma)} \\
&= \frac{1}{2}\Big(\frac{2^{\frac{n-1}{2}} - 2^{\frac{n-1}{2}}z}{(1-z)^{n+1}} + \frac{\mp(-1)^{\delta+\frac{n+1}{4}}2^{\frac{n-1}{2}} \mp (-1)^{\delta+\frac{n+1}{4}}2^{\frac{n-1}{2}}z}{(1+z)^{n+1}}\Big) \\
&= 2^{\frac{n-1}{2}}\Big(\frac{1}{2(1-z)^n} \mp \frac{(-1)^{\delta+\frac{n+1}{4}}}{2(1+z)^n}\Big) \\
&= 2^{\frac{n-1}{2}}\sum_{k=0}^{+\infty}\frac{1\mp(-1)^{k+\delta+\frac{n+1}{4}}}{2}\begin{pmatrix} n+k-1 \\ k \end{pmatrix} z^k,
\end{aligned}$$

from which the result follows. □

Apart from the examples above, no Dirac spectrum can be computed in such an elementary way. This is in particular the case for closed hyperbolic manifolds (spaceforms of negative curvature), see also Theorem 2.2.3 below. However, it remains theoretically possible to do the computations on homogeneous spaces, where there exists a representation-theoretical method to express the Dirac operator. This is the object of the next section.

## 2.2 Spectrum of some other homogeneous spaces

Let us first introduce a few notations and recall basic facts. Denote by $M := {}^{G}\!/\!_{H}$ an $n$-dimensional homogeneous space and by $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of $G$ (resp. of $H$). In that case the existence as well as the set of spin structures on $M$ can be read off the isotropy representation of $M$, which is defined as the Lie-group-homomorphism $\alpha : H \to \mathrm{GL}({}^{\mathfrak{g}}\!/\!_{\mathfrak{h}})$ induced by the restriction of the adjoint map Ad of $G$ onto $H$. It is indeed well-known that $M$ carries a homogeneous Riemannian metric (i.e., a metric invariant under the left $G$-action) or an orientation if and only if $\alpha(H)$ is compact or connected respectively. Assuming both conditions $\alpha$ becomes a map $H \longrightarrow \mathrm{SO}({}^{\mathfrak{g}}\!/\!_{\mathfrak{h}})$. The existence of a homogeneous spin structure (i.e., a spin structure on which the left $G$-action on $\mathrm{SO}(TM)$ lifts) on $M$ is then equivalent to that of a lift $\widetilde{\alpha}$ of $\alpha$ into $\mathrm{Spin}({}^{\mathfrak{g}}\!/\!_{\mathfrak{h}})$, the set of spin structures standing then in one-to-one correspondence with that of such $\widetilde{\alpha}$'s, that is, with the set of group-homomorphisms $H \longrightarrow \{\pm 1\}$, see [37, Lemma 1]. For example, if $H$ is connected, then there can exist only one homogeneous spin structure on $M$. Moreover, if $G$ is simply-connected, then all spin structures are obtained in such a way.

To describe the Dirac operator on $M$, one has to look at the left action of $G$ onto the space of sections of $\Sigma M$, which can be identified with the space of equivariant maps $G \longrightarrow \Sigma_n$. This left action induces a unitary representation of $G$ onto the space of $\mathrm{L}^2$ sections on $M$, which can be split into irreducible and finite-dimensional components since $G$ is compact. The Dirac operator preserves that splitting and can be determined with the help of the following result based on the Frobenius reciprocity principle (see reference in [37]).

**Theorem 2.2.1 (see e.g. [37])** *Let $M := {}^{G}\!/_{H}$ be an $n$-dimensional Riemannian homogeneous spin manifold with $G$ compact and simply-connected. Let $\mathfrak{p}$ be an $\mathrm{Ad}(H)$-invariant supplementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$ and fix a p.o.n.b $\{X_1, \ldots, X_n\}$ of $\mathfrak{p}$. Choose a spin structure on $M$ and let $\widetilde{\alpha} : H \longrightarrow \mathrm{Spin}_n$ be the corresponding lift of the isotropy representation. Denote by $\Sigma_{\widetilde{\alpha}} M \longrightarrow M$ the spinor bundle of $M$ associated with $\widetilde{\alpha}$. Let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$ (we identify an element of $\widehat{G}$ with one of its representants).*

1. *The space $\mathrm{L}^2(M, \Sigma_{\widetilde{\alpha}} M)$ splits under the unitary left action of $G$ into a direct Hilbert sum*
$$\overline{\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)}$$
*where $V_\gamma$ is the space of the representation $\gamma$ (i.e., $\gamma : G \longrightarrow \mathrm{U}(V_\gamma)$) and*
$$\begin{aligned} \mathrm{Hom}_H(V_\gamma, \Sigma_n) \quad := \quad & \Big\{ f \in \mathrm{Hom}(V_\gamma, \Sigma_n) \text{ s.t. } \forall h \in H, \\ & \qquad f \circ \gamma(h) = (\delta_n \circ \widetilde{\alpha})(h) \circ f \Big\}. \end{aligned}$$

2. *The Dirac operator $D$ of $M$ preserves each summand in the splitting above; more precisely, if $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, then for every $\gamma \in \widehat{G}$, the restriction of $D$ to $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ is given by $\mathrm{Id} \otimes D_\gamma$, where, for every $A \in \mathrm{Hom}_H(V_\gamma, \Sigma_n)$,*
$$\begin{aligned} D_\gamma(A) \quad = \quad & -\sum_{k=1}^{n} e_k \cdot A \circ d_e\gamma(X_k) \\ & + \left( \sum_{i=1}^{n} \beta_i e_i + \sum_{i<j<k} \alpha_{ijk} e_i \cdot e_j \cdot e_k \right) \cdot A, \end{aligned} \tag{2.3}$$
*and*
$$\begin{aligned} \beta_i \quad &:= \quad \frac{1}{2} \sum_{j=1}^{n} \langle [X_j, X_i]_{\mathfrak{p}}, X_j \rangle \\ \alpha_{ijk} \quad &:= \quad \frac{1}{4} \left( \langle [X_i, X_j]_{\mathfrak{p}}, X_k \rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i \rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j \rangle \right) \end{aligned}$$
*(here $X_{\mathfrak{p}}$ denotes the image of $X \in \mathfrak{g}$ under the projection $\mathfrak{g} \longrightarrow \mathfrak{p}$ with kernel $\mathfrak{h}$).*

Of course, if $H$ is a discrete subgroup of the Lie group $G$, then Theorem 2.2.1 contains Proposition 1.4.2 (just transform the left-action into a right-one).

Theorem 2.2.1 can for instance be applied to the computation of the Dirac spectrum of $\mathbb{S}^{2m+1} = \widetilde{\mathrm{U}}_{m+1}/\widetilde{\mathrm{U}}_m$ with Berger metric, where $\widetilde{\mathrm{U}}_m$ stands for the universal covering of the unitary group $\mathrm{U}_m$. Recall that, if $n$ is odd, then the round sphere $\mathbb{S}^n$ is the total space of an $\mathbb{S}^1$-bundle called the Hopf-fibration $\mathbb{S}^n \longrightarrow \mathbb{C}\mathrm{P}^{\frac{n-1}{2}}$, where $\mathbb{C}\mathrm{P}^{\frac{n-1}{2}}$ is the complex projective space of complex dimension $\frac{n-1}{2}$. In particular one may decompose the round metric on $\mathbb{S}^n$ along the $\mathbb{S}^1$-fibres and their orthogonal complement; we denote this decomposition by $\mathrm{can} = g_{\mathbb{S}^1} \oplus g_\perp$.

**Theorem 2.2.2 (C. Bär [42])** *For $n = 2m+1 \geq 3$, let $M^n := \mathbb{S}^n$ with Berger metric $g_t = tg_{\mathbb{S}^1} \oplus g_\perp$ for some real $t > 0$. Then the Dirac operator of $(M^n, g_t)$ has the following eigenvalues:*

*i)* $\frac{1}{t}(\frac{m+1}{2} + k) + \frac{tm}{2}$, $k \in \mathbb{N}$, *with multiplicity* $\binom{m+k}{k}$.

*ii)* $(-1)^{m+1}\big(\frac{1}{t}(\frac{m+1}{2} + k) + \frac{tm}{2}\big)$, $k \in \mathbb{N}$, *with multiplicity* $\binom{m+k}{k}$.

*iii)* $(-1)^j \frac{t}{2} \pm \sqrt{[\frac{t}{2}(m-1-2j) + \frac{1}{t}(a_1 - a_2 + \frac{m-1}{2} - j)]^2 + 4(m + a_1 - j)(1 + a_2 + j)}$, $a_1, a_2 \in \mathbb{N}$, $j \in \{0, 1 \ldots, m-1\}$, *with multiplicity* $\frac{(m+a_1)!(m+a_2)!(m+1+a_1+a_2)!}{a_1!a_2!m!j!(m-1-j)!(m+a_1-j)(1+a_2+j)}$.

The reader interested in the proof of Theorem 2.2.2 or in the computation of the Dirac spectrum of 3-dimensional lens spaces $\mathbb{S}^3/\mathbb{Z}_k$ should refer to [42, Sec. 3] or to [36] respectively.

In general, the formula (2.3) describing the Dirac operator as a direct sum of endomorphisms in finite-dimensional spaces cannot be better explicited. In case $M$ is symmetric, (2.3) simplifies in the sense that the square of $D$ is given by

$$D^2 = \Omega_G + \frac{S}{8}\mathrm{Id}, \tag{2.4}$$

where $\Omega_G$ is the Casimir operator of $G$ and $S$ is the scalar curvature of $M$, see e.g. [91, Prop. p.87]. However, on general homogeneous spaces, obtaining the spectrum explicitly still remains very difficult, see e.g. [106, Sec. 3].

In case $M$ is the quotient of a non-compact Lie group $G$, Theorem 2.2.1 does not apply since no Frobenius reciprocity is available. Moreover, the irreducible representations of $G$ are much harder to classify, so that the knowledge of the spectrum is generally out of reach. We quote the only example where this was successfully carried out and which is a quotient of the projective special linear group of $\mathbb{R}^2$, that we denote $\mathrm{PSL}_2(\mathbb{R})$. Define a Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$ to be a discrete one. The signature of a Fuchsian subgroup $\Gamma$ is the tuple of non-negative integers $(g; m_1, \ldots, m_r)$ such that $\Gamma$ is presented by

$$\begin{aligned}\Gamma \quad = \quad & < A_1, B_1, \ldots A_g, B_g, X_1, \ldots, X_r \mid \\ & X_1^{m_1} = \ldots = X_r^{m_r} = X_1 \ldots X_r A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} = 1 > .\end{aligned}$$

**Theorem 2.2.3 (J. Seade and B. Steer [224])** *Let $M := {}^{\mathrm{PSL}_2(\mathbb{R})}\!/_{\Gamma}$, where $\Gamma$ is a co-compact Fuchsian subgroup of signature $(g; m_1, \ldots, m_r)$ in $\mathrm{PSL}_2(\mathbb{R})$. Consider the orientation and the 1-parameter-family of left-invariant metrics $(g_t)_{t>0}$ on $M$ for which $(\frac{1}{t}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ is a positively-oriented orthonormal basis of $T_1 M$.*
*Then the Dirac eigenvalues of $(M, g_t)$ endowed with its trivial (left-invariant) spin structure are:*

*i)* $-\frac{t}{2} + \frac{1}{t}$ *with multiplicity* 2.

*ii)* $-\frac{t}{2} - \frac{2k-1}{t}$, $k \geq 1$, *with multiplicity*

$$\frac{2k-1}{2\pi}\mathrm{Vol}(M) - \sum_{j=1}^{r} \frac{1}{m_j} \sum_{l=1}^{m_j - 1} \frac{\sin((2k-1)m_j\pi)}{\sin(\frac{l\pi}{m_j})}.$$

*iii)* $-\frac{t}{2} \pm \sqrt{(2n-1)^2(1+t^{-2}) - (2k-1)^2}$, $k \geq 1$, $n > k$, *with the same multiplicity as in ii).*

*iv)* $-\frac{t}{2} \pm \sqrt{(2n-1)^2(1+t^{-2}) - s^2}$, $n \in \mathbb{Z}$, $s \in \Lambda$, *where $\Lambda$ is some countable subset of $]-1, 1[ \cup i\mathbb{R}$.*

The set $\Lambda$ depends on representation-theoretical data and cannot be explicited in general. For $t = 1$, the metric $g_t$ has constant negative sectional curvature. This is up to the knowledge of the author the only compact hyperbolic manifold whose Dirac spectrum has been computed.

To summarise, we reproduce - with small changes - the list in [45] of all homogeneous spaces of which Dirac spectrum has already been computed:

| **space** | **description** | **references** |
|---|---|---|
| $G$ | simply-connected compact Lie groups | [86] |
| $\mathbb{R}^n/\mathbb{Z}^n$ | flat tori | [89] |
| $\mathbb{R}^3/\Gamma$ | 3-dim. (flat) Bieberbach manifolds | [212] |
| $\mathbb{R}^n/\Gamma$ | $n$-dim. (flat) Bieberbach manifolds | [194] |
| $\mathbb{S}^n$ | round spheres | [232] |
| $\mathbb{S}^n/\Gamma$ | spherical spaceforms | [41] |
| $\mathbb{S}^{2m+1}$ | spheres with Berger metric | [152, 42] |
| $\mathbb{S}^3/\mathbb{Z}_k$ | 3-dim. lens spaces with Berger metric | [37] |
| $\mathbb{S}^3/\mathrm{Q}_8$ | $\mathbb{S}^3$ through the group of quaternions, with Berger metric | [106] |
| $H^3/\Gamma$ | 3-dim. Heisenberg manifolds | [17] |
| $\mathrm{PSL}_2(\mathbb{R})/\Gamma$ | ($\Gamma$ Fuchsian) | [224] |
| $\mathbb{C}\mathrm{P}^{2m+1}$ | complex projective spaces | [71, 72, 228] |
| $\mathbb{H}\mathrm{P}^m$ | quaternionic projective spaces | [69, 195] |
| $\mathbb{O}\mathrm{P}^2$ | Cayley projective plane | [240] |
| $\mathrm{Gr}_2(\mathbb{R}^{2m})$ | real 2-Grassmannians | [230, 231] |
| $\mathrm{Gr}_{2p}(\mathbb{R}^{2m})$ | real $2p$-Grassmannians | [225] |
| $\mathrm{Gr}_2(\mathbb{C}^{m+2})$ | complex 2-Grassmannians | [196] |
| $\mathrm{G}_2/\mathrm{SO}_4$ | - | [225, 226] |

## 2.3 Small eigenvalues of some symmetric spaces

In case of a symmetric space $M := {}^{G}\!/_{H}$ where $G$ and $H$ have the same rank, a formula due to R. Parthasarathy (see reference in [197]) allows to express certain parts of the Casimir operator $\Omega_G$ (see (2.4)) in terms of representation-theoretical data:

**Theorem 2.3.1 (J.-L. Milhorat [197, 198])** *Let $M := {}^{G}\!/_{H}$ be a spin compact simply-connected irreducible symmetric space with $G$ compact and simply-connected, endowed with the metric $\langle \cdot\,, \cdot \rangle$ induced by the Killing form of $G$ sign-changed. Assume that $G$ and $H$ have the* same rank *and fix a spin structure on ${}^{G}\!/_{H}$. Let $\beta_k$, $k = 1, \ldots, p$, be the $H$-dominant weights occurring in the decomposition into irreducible components of the spin representation under $H$.*
*Then the square of the first eigenvalue of $D$ is*

$$\frac{n}{8} + 2 \min_{1\leq k\leq p} \|\beta_k\|^2 = \frac{n}{8} + 2 \min_{w\in W_G} \|w \cdot \delta_G - \delta_H\|^2,$$

*where $\|\cdot\|$ is the norm associated to $\langle \cdot\,, \cdot \rangle$, $W_G$ is the Weyl group of $G$ and $\delta_G$ (resp. $\delta_H$) is the half-sum of the positive roots of $G$ (resp. $H$).*

Theorem 2.3.1 has been applied by J.-L. Milhorat in [198] to compute the smallest eigenvalue $\lambda_1(D^2)$ for the following symmetric spaces (where $S$ denotes the scalar curvature of $M$ and $E_p$ the exceptional simple Lie group of rank $p$):

| $M = {}^{G}\!/_{H}$ | $\dim(M)$ | $\lambda_1(D^2)$ |
|---|---|---|
| ${}^{\mathrm{Spin}_{m+4}}\!/_{(\mathrm{Spin}_m \cdot \mathrm{Spin}_4)}$ ($m$ even) | $4m$ | $\frac{m^2+6m-4}{m(m+2)} \cdot \frac{m}{2} = \frac{m^2+6m-4}{m(m+2)} \cdot \frac{S}{4}$ |
| ${}^{\mathrm{E}_6}\!/_{(\mathrm{SU}_6 \cdot \mathrm{SU}_2)}$ | 40 | $\frac{41}{6} = \frac{41}{30} \cdot \frac{S}{4}$ |
| ${}^{\mathrm{E}_7}\!/_{(\mathrm{Spin}_{12} \cdot \mathrm{SU}_2)}$ | 64 | $\frac{95}{9} = \frac{95}{72} \cdot \frac{S}{4}$ |
| ${}^{\mathrm{E}_8}\!/_{(\mathrm{E}_7 \cdot \mathrm{SU}_2)}$ | 112 | $\frac{269}{15} = \frac{269}{210} \cdot \frac{S}{4}$ |

The quotient $\frac{S}{4}$ has been each time factorized out in order to compare the dimension-depending coefficient standing before with $\frac{n}{n-1}$, which is the corresponding one in Friedrich's inequality (3.1).

# Chapter 3

# Lower eigenvalue estimates on closed manifolds

In this chapter we assume that $M$ has empty boundary.

## 3.1 Friedrich's inequality

The most general sharp lower bound for the Dirac spectrum has been proved by T. Friedrich in [88] and is now known under the name "Friedrich's inequality". For the concept of Killing spinor we refer to Section A.1.

**Theorem 3.1.1 (T. Friedrich [88])** *Any eigenvalue $\lambda$ of $D$ on an $n(\geq 2)$-dimensional closed Riemannian spin manifold $(M^n, g)$ satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M(S), \tag{3.1}$$

*where $S$ is the scalar curvature of $M$.*
*Moreover (3.1) is an equality for some eigenvalue $\lambda$ if and only if there exists a non-zero real Killing spinor on $(M^n, g)$.*

*Proof*: It follows from the Schrödinger-Lichnerowicz formula (1.15) that, for any $\varphi \in \Gamma(\Sigma M)$,

$$\int_M \langle D^2\varphi, \varphi\rangle v_g = \int_M \langle \nabla^*\nabla\varphi, \varphi\rangle v_g + \int_M \frac{S}{4}|\varphi|^2 v_g.$$

By definition of $\nabla^*\nabla$ and since $D$ is formally self-adjoint we can write

$$\int_M |D\varphi|^2 v_g = \int_M |\nabla\varphi|^2 v_g + \int_M \frac{S}{4}|\varphi|^2 v_g. \tag{3.2}$$

Decompose now w.r.t. any local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $TM$

$$\nabla\varphi = \underbrace{\nabla\varphi + \frac{1}{n}\sum_{j=1}^n e_j^* \otimes e_j \cdot D\varphi}_{=P\varphi\,\in\,\mathrm{Ker}(\mu)} - \underbrace{\frac{1}{n}\sum_{j=1}^n e_j^* \otimes e_j \cdot D\varphi}_{\in\,\mathrm{Ker}(\mu)^\perp},$$

where $P$ is the so-called Penrose operator of $(M^n, g)$, see also Appendix A. We deduce that $|\nabla\varphi|^2 = |P\varphi|^2 + \frac{1}{n}|D\varphi|^2$ (identity (A.11) in Appendix A). Replacing $|\nabla\varphi|^2$ in (3.2) one obtains

$$\int_M |D\varphi|^2 v_g = \frac{1}{n}\int_M |D\varphi|^2 v_g + \int_M |P\varphi|^2 v_g + \int_M \frac{S}{4}|\varphi|^2 v_g,$$

that is,

$$\int_M \left(|D\varphi|^2 - \frac{n}{4(n-1)}S|\varphi|^2\right) v_g = \frac{n}{n-1}\int_M |P\varphi|^2 v_g. \tag{3.3}$$

Choose $\varphi$ to be a non-zero eigenvector for $D$ associated to the eigenvalue $\lambda$. From $|P\varphi|^2 \geq 0$ one straightforward obtains the inequality (3.1).
If (3.1) is an equality for some eigenvalue $\lambda$ then (3.3) implies $P\varphi = 0$ for any non-zero eigenvector $\varphi$ for $D$ associated to $\lambda$, hence any such $\varphi$ is a (necessarily real) Killing spinor on $(M^n, g)$. Conversely, if $(M^n, g)$ carries a non-zero $\alpha$-Killing spinor $\varphi$, then since $M$ is compact $\alpha$ must be real. Moreover, on the one hand $\varphi$ is an eigenvector for $D$ associated to the eigenvalue $-n\alpha$, on the other hand we know from Proposition A.4.1 that the scalar curvature of $(M^n, g)$ must be $S = 4n(n-1)\alpha^2$, in particular it must be non-negative. Therefore such a $\varphi$ must be an eigenvector for $D$ associated to the eigenvalue $\sqrt{\frac{nS}{4(n-1)}}$ or $-\sqrt{\frac{nS}{4(n-1)}}$. This shows the equivalence in the limiting-case and concludes the proof. ☐

Another method for the proof of (3.1), which is actually T. Friedrich's in [88], relies on the *modified connection*

$$\widetilde{\nabla}_X\psi := \nabla_X\psi + \frac{\lambda}{n}X\cdot\psi$$

for every $X \in TM$, where $D\psi = \lambda\psi$: Compute $|\widetilde{\nabla}\psi|^2$ (which plays the role of $|P\psi|^2$ above), integrate and apply the Schrödinger-Lichnerowicz formula. Alternatively but still along the same idea, it can be directly deduced from the Cauchy-Schwarz inequality that $|D\varphi| \leq \sqrt{n}|\nabla\varphi|$ for every section $\varphi$, from which (3.1) follows.

As a consequence of Theorem 3.1.1, if the scalar curvature $S$ of $(M^n, g)$ is positive then its Dirac operator has trivial kernel - whatever the spin structure is. This had been already noticed by A. Lichnerowicz in [180] where he had obtained as a straightforward application of (3.2) the following estimate:

$$\lambda^2 \geq \frac{1}{4}\inf_M(S). \tag{3.4}$$

It follows from (3.4) combined with the Atiyah-Singer index theorem [31] (see Theorem 1.3.9) that a Riemannian manifold with positive scalar curvature must have vanishing topological index. In particular, if the manifold has non-vanishing $\widehat{A}$-genus, then it cannot carry any Riemannian manifold with positive scalar curvature. In other words, there exists a topological obstruction to the existence of metrics with positive scalar curvature on closed spin manifolds, at least in even dimensions. The reader interested in further results in that topic - such as

Gromov-Lawson's work - should refer to [178] or to [127]. The existence of Riemannian metrics for which the Dirac kernel is non-zero is discussed in Section 6.2. Moreover, we mention another closely related application of the Atiyah-Singer index theorem to geometry via the Schrödinger-Lichnerowicz formula, namely to the so-called scalar curvature rigidity issue which asks for the possibility of increasing the scalar curvature without shrinking the distances of a given metric on a fixed background manifold. For example this is not possible on the round sphere (M. Llarull [186, Thm. B]) nor on any connected closed Kähler manifold with non-negative Ricci curvature (S. Goette and U. Semmelmann [112, Thm. 0.1]), we refer to [113] and [185] for the case of symmetric spaces and references.

Although it requires the non-negativity of $S$ to be non-trivial, Friedrich's inequality (3.1) provides fine information of geometrical nature on the Dirac spectrum. Indeed $S$ stands for a very weak curvature invariant of a given Riemannian manifold. This shows for example a difference of behaviour with other differential operators such as the scalar Laplacian $\Delta$: by a result of A. Lichnerowicz [179], any non-zero eigenvalue $\lambda$ of $\Delta$ satisfies

$$\lambda \geq \frac{n}{n-1} \inf_M(\mathrm{Ric}),$$

where Ric is the Ricci curvature tensor of $(M^n, g)$, which is a stronger curvature invariant. In case $\inf_M(S) \leq 0$ Friedrich's inequality (3.1) can be improved in different ways using various techniques, see Sections 3.3 to 3.7.

Besides, (3.1) is sharp since e.g. $M := \mathbb{S}^n$ $(n \geq 2)$ admits non-zero Killing spinors, see Example A.1.3.2. For the classification of Riemannian spin manifolds carrying non-zero real Killing spinors we refer to Theorems A.4.2 and A.4.3 in Appendix A.

## 3.2 Improving Friedrich's inequality in presence of a parallel form

O. Hijazi [132] and A. Lichnerowicz [181, 182] noticed that equality in (3.1) cannot hold on those $M$ admitting a non-zero *parallel $k$-form* for some $k \in \{1, \ldots, n-1\}$. This suggests (3.1) could be enhanced under this assumption.

The idea of proof for Theorems 3.2.1, 3.2.4 and 3.2.6 can be summarised as follows (see [67]): given any eigenvector $\varphi$ of $D$ to the eigenvalue $\lambda$, decompose $|\nabla\varphi|^2$ in a sharper way than for the proof of Friedrich's inequality, using the splitting of $\Sigma M$ induced by the Clifford action of the parallel form.

**Theorem 3.2.1 (B. Alexandrov, G. Grantcharov and S. Ivanov [5])** *Any eigenvalue $\lambda$ of $D$ on an $n(\geq 3)$-dimensional closed Riemannian spin manifold $(M^n, g)$ admitting a non-zero* parallel 1-form *satisfies*

$$\lambda^2 \geq \frac{n-1}{4(n-2)} \inf_M(S), \tag{3.5}$$

*where $S$ is the scalar curvature of $M$.*
*Moreover if* (3.5) *is an equality for some eigenvalue $\lambda$, then the universal cover of $M$ is a Riemannian product of the form $\mathbb{R}\times N$, where $N$ admits a real Killing spinor.*

We shall prove a more general result:

**Theorem 3.2.2 (A. Moroianu and L. Ornea [208])** *Inequality* (3.5) *holds as soon as $M^n$ $(n \geq 3)$ admits a non-zero* harmonic 1-form of constant length. *Furthermore, if it is an equality for some eigenvalue $\lambda$, then this form is parallel.*

*Proof*: Let $\xi$ be the dual vector field to the harmonic 1-form of constant length. We may assume that $g(\xi,\xi)=1$ on $M$. Define the following Penrose-like operator

$$T_X\varphi := \nabla_X\varphi + \frac{1}{n-1}(X - g(X,\xi)\xi)\cdot D\varphi - \frac{1}{n-1}(ng(X,\xi) + X\cdot\xi\cdot)\nabla_\xi\varphi$$

for all $X \in TM$ and $\varphi \in \Gamma(\Sigma M)$. In case $\xi$ is parallel this operator can be described as the sum of the orthogonal projections of $\nabla\varphi$ onto the kernels of the Clifford multiplications $T^*M\otimes\Sigma_\pm M \xrightarrow{\mu_\pm} \Sigma M$, where $\Sigma_\pm M := \mathrm{Ker}(i\xi\cdot\mp\mathrm{Id}_{\Sigma M})$, see [5, eq. (4)] for another equivalent expression (note however that it does not exactly coincide with the $T$ defined in [208]). Nevertheless $\xi$ need not be parallel in order for $T$ to play its role for the estimate as we shall see in the proof.
Fix a local orthonormal frame $\{e_j\}_{1\leq j\leq n}$ of $TM$. For any $\varphi \in \Gamma(\Sigma M)$, we have

$$\begin{aligned}
|T\varphi|^2 &= \sum_{j=1}^n |T_{e_j}\varphi|^2 \\
&= |\nabla\varphi|^2 + \frac{1}{n-1}|D\varphi|^2 + \frac{n}{n-1}|\nabla_\xi\varphi|^2 \\
&\quad -\frac{2}{n-1}(|D\varphi|^2 + \Re e\,(\langle\xi\cdot D\varphi, \nabla_\xi\varphi\rangle)) \\
&\quad -\frac{2}{n-1}(n|\nabla_\xi\varphi|^2 - \Re e\,(\langle\xi\cdot\nabla_\xi\varphi, D\varphi\rangle)) \\
&\quad -\frac{2}{n-1}\Re e\,(\langle\xi\cdot\nabla_\xi\varphi, D\varphi\rangle)) \\
&= |\nabla\varphi|^2 - \frac{1}{n-1}|D\varphi|^2 - \frac{n}{n-1}|\nabla_\xi\varphi|^2 + \frac{2}{n-1}\Re e\,(\langle\xi\cdot\nabla_\xi\varphi, D\varphi\rangle)).
\end{aligned}$$

Now we can express the last term on the r.h.s. through the other ones, a trick due to the authors of [208]: namely, since $\xi$ is assumed to be harmonic, i.e., closed and co-closed, the identity (1.12) reads $D(\xi\cdot\varphi) = -\xi\cdot D\varphi - 2\nabla_\xi\varphi$ and hence

$$\Re e\,(\langle\xi\cdot\nabla_\xi\varphi, D\varphi\rangle) = |\nabla_\xi\varphi|^2 + \frac{1}{4}(|D\varphi|^2 - |D(\xi\cdot\varphi)|^2),$$

from which we obtain

$$|T\varphi|^2 = |\nabla\varphi|^2 - \frac{1}{n-1}|D\varphi|^2 - \frac{n-2}{n-1}|\nabla_\xi\varphi|^2 + \frac{1}{2(n-1)}(|D\varphi|^2 - |D(\xi\cdot\varphi)|^2).$$

Integrating this identity over $M$ and applying the Schrödinger-Lichnerowicz formula (1.15) we have

$$\begin{aligned}\int_M |T\varphi|^2 v_g &= \frac{n-2}{n-1}\int_M |D\varphi|^2 v_g - \frac{1}{4}\int_M S|\varphi|^2 v_g - \frac{n-2}{n-1}\int_M |\nabla_\xi\varphi|^2 v_g \\ &+\frac{1}{2(n-1)}\int_M |D\varphi|^2 - |D(\xi\cdot\varphi)|^2 v_g.\end{aligned}$$

But choosing $\varphi$ to be eigen for $D$ for the smallest (in absolute value) eigenvalue $\lambda$, the min-max principle (see Lemma 5.0.2) implies

$$\begin{aligned}\int_M |D(\xi\cdot\varphi)|^2 v_g &\geq \lambda^2\int_M |\xi\cdot\varphi|^2 v_g \\ &= \lambda^2\int_M |\varphi|^2 v_g \\ &= \int_M |D\varphi|^2 v_g,\end{aligned}$$

hence

$$\left(\frac{n-2}{n-1}\lambda^2 - \frac{1}{4}\inf_M(S)\right)\int_M |\varphi|^2 v_g \geq \int_M |T\varphi|^2 v_g + \frac{n-2}{n-1}\int_M |\nabla_\xi\varphi|^2 v_g \qquad (3.6)$$

and the inequality (3.5) follows.
If (3.5) is an equality for some eigenvalue $\lambda$, then (3.6) implies $T\varphi = 0$ and $\nabla_\xi\varphi = 0$, that is,

$$\nabla_X\varphi = -\frac{\lambda}{n-1}(X - g(X,\xi)\xi)\cdot\varphi \qquad (3.7)$$

for any eigenvector $\varphi$ associated to $\lambda$ and any $X \in TM$. In particular its length must be constant on $M$. As in [208] we next show that $\xi$ must be parallel. For this purpose we compute the curvature tensor on such a $\varphi$: let $X, Y \in TM$, then

$$\begin{aligned}R_{X,Y}\varphi &= \nabla_{[X,Y]}\varphi - [\nabla_X,\nabla_Y]\varphi \\ &= \frac{\lambda}{n-1}\left((g(X,\nabla_Y\xi) - g(Y,\nabla_X\xi))\xi + g(X,\xi)\nabla_Y\xi - g(Y,\xi)\nabla_X\xi\right)\cdot\varphi \\ &+\frac{\lambda^2}{(n-1)^2}\Big((X - g(X,\xi)\xi)\cdot(Y - g(Y,\xi)\xi) \\ &\qquad - (Y - g(Y,\xi)\xi)\cdot(X - g(X,\xi)\xi)\Big)\cdot\varphi,\end{aligned}$$

hence using (1.9) and the fact that $g(\xi,\xi) = 1$ on $M$ we obtain

$$\begin{aligned}\frac{1}{2}\mathrm{Ric}(\xi)\cdot\varphi &= \sum_{j=1}^n e_j\cdot R^\nabla_{\xi,e_j}\varphi \\ &= \frac{\lambda}{n-1}\sum_{j=1}^n\Big((g(\xi,\nabla_{e_j}\xi) - g(e_j,\nabla_\xi\xi))e_j\cdot\xi \\ &\qquad + (g(\xi,\xi)\nabla_{e_j}\xi - g(e_j,\xi)\nabla_\xi\xi)\cdot e_j\cdot\varphi\Big) \\ &= \frac{\lambda}{n-1}\left(-2\nabla_\xi\xi\cdot\xi + \sum_{j=1}^n\nabla_{e_j}\xi\cdot e_j\right)\cdot\varphi.\end{aligned}$$

The last sum vanishes because of (1.2) together with $\xi$ being closed and co-closed. On the other hand $d\xi = 0$ means that $g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X) = 0$ for all $X, Y \in TM$, hence for $X = \xi$ one obtains - using once again $g(\xi,\xi) = 1$ - that $g(\nabla_\xi\xi, Y) = 0$ for all $Y \in TM$, i.e., $\nabla_\xi\xi = 0$. This shows $\mathrm{Ric}(\xi) = 0$. From Bochner's formula for the Laplace operator on 1-forms (see e.g. [178, Cor. 8.3 p.156]) one deduces that $\nabla\xi = 0$, i.e., that $\xi$ is parallel.
We now prove the limiting-case in Theorem 3.2.1. If $\xi$ is parallel then the universal cover of $M$ must be a Riemannian product of the form $\mathbb{R} \times N$. W.r.t. the pull-back spin structure the lift of $\varphi$ to $\mathbb{R} \times N$ also satisfies (3.7) provided $\xi$ is replaced by $\frac{\partial}{\partial t}$. Since each $\{t\} \times N$ sits totally geodesically in $\mathbb{R} \times N$ the Gauss-type formula (1.21) implies that the induced spinor field on $N$ is a real Killing spinor for one of the constants $\pm\frac{\lambda}{n-1}$. □

Beware that the necessary condition for (3.5) to be an equality is not sufficient, since e.g. $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ with flat metric carries non-zero parallel spinors whereas (flat) $\mathbb{T}^3 = {}_{\mathbb{Z}^3}\backslash^{\mathbb{R}^3}$ only admits such spinors in case it carries the trivial spin structure (i.e., the spin structure induced by the trivial lift of the $\mathbb{Z}^3$-action to the spin level, see Proposition 1.4.2). In fact (3.5) is an equality if and only if there exists a $\pi_1(M)$-equivariant solution - in the sense of (1.24) - to (3.7) on the universal cover $\mathbb{R} \times N$ of $M$.

Although the (real) Killing-spinor-equation is completely understood (see Theorems A.4.2 and A.4.3), the list of all local Riemannian products on which (3.5) is sharp is not entirely known. B. Alexandrov, G. Grantcharov and S. Ivanov have shown in [5] that, under this assumption and if $n \neq 7$ is odd, then $M$ is diffeomorphic - but not necessarily isometric - to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

It is moreover important to note that the hypothesis in Theorem 3.2.2 on the length being constant cannot be removed: C. Bär and M. Dahl showed in [49] that in dimension $n \geq 3$ Friedrich's inequality (3.1) cannot be improved with the help of topological assumptions. Namely there exists on any given compact spin manifold $M^n$ admitting a metric with positive scalar curvature a smooth family of Riemannian metrics $(g_t)_{t>0}$ with $S_{g_t} \geq n(n-1)$ and

$$\frac{n^2}{4} \leq \lambda_1(D_{g_t}^2) \leq \frac{n^2}{4} + t,$$

where $D_{g_t}$ stands for the Dirac operator to the metric $g_t$ on $M$. In other words, one can get as close as one wants to the equality in Friedrich's inequality (3.1) on any such manifold. Note that the set of compact spin manifolds with positive first Betti number and admitting a metric with positive scalar curvature is non-empty since it contains e.g. $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $n \geq 3$.

The generalization of Theorem 3.2.1 to locally reducible Riemannian manifolds was achieved by B. Alexandrov, extending earlier work by E.C. Kim [158]:

**Theorem 3.2.3 (B. Alexandrov [4])** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional closed Riemannian spin manifold with positive scalar curvature $S$. Assume that $TM$ splits orthogonally into*

$$TM = \bigoplus_{j=1}^{k} T_j,$$

*where $T_j$ is a parallel distribution of dimension $n_j$ and $n_1 \leq \ldots \leq n_k$. Then any eigenvalue $\lambda$ of $D$ satisfies*

$$\lambda^2 \geq \frac{n_k}{4(n_k-1)} \inf_M(S). \tag{3.8}$$

*Moreover, if* (3.8) *is an equality for some eigenvalue $\lambda$, then the universal cover of $M$ is isometric to $M_1 \times \ldots \times M_k$, where $M_j$ is a closed $n_j$-dimensional Riemannian spin manifold admitting a non-zero real non-parallel Killing spinor for $j = k$, a non-zero parallel spinor if $n_j < n_k$ and a non-zero real Killing spinor if $n_j = n_k$.*

Note that (3.8) contains both (3.1) and (3.5) and that $n_k > 1$ because of the assumption $S > 0$. Moreover, for any integers $1 \leq n_1 \leq \ldots \leq n_p < n_{p+1} = \ldots = n_k$, all Riemannian products of the form $M_1 \times \ldots \times M_k$, where $M_j$ is an $n_j$-dimensional closed Riemannian spin manifold admitting a non-zero parallel spinor for $j \leq p$ and a non-zero real Killing spinor for $j \geq p+1$ which is furthermore non-parallel for $j = k$, satisfy the equality in (3.8) w.r.t. the product spin structure.

*Sketch of proof of* Theorem 3.2.3: The proof follows the lines of that of Theorem 3.2.1. Define the Penrose-like operator $T$: for any $\varphi \in \Gamma(\Sigma M)$ and any $X \in TM$,

$$T_X\varphi := \nabla_X\varphi + \sum_{j=1}^{k} \frac{1}{n_j}\pi_j(X) \cdot D_{[j]}\varphi,$$

where $\pi_j : TM \to T_j$ is the orthogonal projection, $D_{[j]}\varphi := \sum_{l=1}^{n_j} e_{l,j} \cdot \nabla_{e_{l,j}}\varphi$ and $(e_{1,j}, \ldots, e_{n_j,j})$ denotes a local orthonormal frame of $T_j$, for every $j \in \{1, \ldots, k\}$. A short computation gives

$$|T\varphi|^2 = |\nabla\varphi|^2 - \sum_{j=1}^{k} \frac{1}{n_j}|D_{[j]}\varphi|^2.$$

On the other hand, it is an exercise to show that $D^2 = \sum_{j=1}^{k} D_{[j]}^2$ and that $D_{[j]}$ is formally self-adjoint, so that, after integration and application of the Schrödinger-Lichnerowicz formula (1.15), one obtains

$$\|D\varphi\|^2 = \frac{n_k}{n_k-1}\|T\varphi\|^2 + \sum_{j=1}^{k} \frac{n_k-n_j}{n_j(n_k-1)}\|D_{[j]}\varphi\|^2 + \frac{n_k}{4(n_k-1)}(S\varphi,\varphi). \tag{3.9}$$

Choosing $\varphi$ to be an eigenvector for $D$ associated to the eigenvalue $\lambda$ leads to the inequality. If this inequality is an equality for some $\lambda$, then (3.9) implies that, for any non-zero eigenvector $\varphi$ for $D$ associated to the eigenvalue $\lambda$, one has $T\varphi = 0$, $D_{[j]}\varphi = 0$ as soon as $n_j < n_k$ and $S$ is constant on $\{x \in M \mid \varphi(x) \neq 0\}$. In case $n_j < n_k$ one deduces that $\nabla_{\pi_j(X)}\varphi = 0$ for every $X$. It remains to prove that, on the universal cover of $M$, which is a Riemannian product of the form $M_1 \times \ldots \times M_k$ by assumption, the lift of $\varphi$ induces a real Killing spinor on each $M_j$, which is parallel if $n_j < n_k$ and non-parallel for $j = k$. We refer to [4, Sec. 2] for the details. □

For 2-forms, the canonical class of manifolds to be considered consists of that of Kähler manifolds, i.e., of triples $(M^n, g, J)$ where $(M^n, g)$ is a Riemannian manifold and $J$ a parallel almost Hermitian structure on $TM$. Recall that $J \in \Gamma(\mathrm{End}(TM))$ is called almost Hermitian if and only if $J^2 = -\mathrm{Id}_{TM}$ and $g(J(X), J(Y)) = g(X, Y)$ for all $X, Y \in TM$. In this case $n$ is even and the Kähler-form $\Omega$ is parallel, where $\Omega$ is defined by

$$\Omega(X, Y) := g(J(X), Y)$$

for all $X, Y \in TM$. K.-D. Kirchberg was the first to enhance Friedrich's inequality (3.1) on Kähler manifolds:

**Theorem 3.2.4 (K.-D. Kirchberg [160])** *Any eigenvalue $\lambda$ of $D$ on an $n(\geq 4)$-dimensional closed* Kähler *spin manifold $(M^n, g, J)$ satisfies*

$$\lambda^2 \geq \left| \begin{array}{ll} \frac{n+2}{4n} \inf_M(S) & \text{if } \frac{n}{2} \text{ is odd} \\ \\ \frac{n}{4(n-2)} \inf_M(S) & \text{if } \frac{n}{2} \text{ is even} \end{array} \right. , \tag{3.10}$$

*where $S$ is the scalar curvature of $M$. Moreover, in the case where $S > 0$,* (3.10) *is an equality for some eigenvalue $\lambda$ if and only if there exists non-zero sections $\psi, \phi$ of $\Sigma M$ satisfying*

$$\left| \begin{array}{ll} \nabla_X \psi & = -\frac{\lambda}{n+2}(X + iJ(X)) \cdot \phi \\ \nabla_X \phi & = -\frac{\lambda}{n+2}(X - iJ(X)) \cdot \psi \end{array} \right. \tag{3.11}$$

*for all $X \in TM$ if $\frac{n}{2}$ is odd and a non-zero section $\psi$ of $\Sigma M$ satisfying*

$$\left| \begin{array}{ll} D^2\psi & = \lambda^2 \psi \\ \nabla_X \psi & = -\frac{1}{n}(X + iJ(X)) \cdot D\psi \\ \Omega \cdot \psi & = -2i\psi \\ \Omega \cdot D\psi & = 0 \end{array} \right. \tag{3.12}$$

*for all $X \in TM$ if $\frac{n}{2}$ is even.*

*Proof*: We follow the proof given in [135, 137], see also [229, Sec. 3] or [154]. We may assume that $S > 0$ on $M$ (otherwise the estimate is trivial). Set, for every $X \in TM$, $p_\pm(X) := \frac{1}{2}(X \mp iJ(X)) \in TM \otimes \mathbb{C}$. In the whole proof we shall redenote $m := \frac{n}{2}$. Given a pointwise orthonormal basis $(e_1, \ldots, e_n)$ of $TM$ such that $e_{j+m} = J(e_j)$ for every $1 \leq j \leq m$, define $z_j := p_+(e_j)$ and $\overline{z}_j := p_-(e_j)$ for all $1 \leq j \leq m$. Then $(z_1, \ldots, z_m)$ and $(\overline{z}_1, \ldots, \overline{z}_m)$ are bases of $T^{1,0}M := p_+(TM)$ and $T^{0,1}M := p_-(TM)$ respectively satisfying

$$z_j \cdot z_k = -z_k \cdot z_j, \qquad \overline{z}_j \cdot \overline{z}_k = -\overline{z}_k \cdot \overline{z}_j, \qquad z_j \cdot \overline{z}_k + \overline{z}_k \cdot z_j = -\delta_{jk}$$

for all $1 \leq j, k \leq m$. With those notations, it is elementary to show that the ranked-$2^m$-vector bundle $(\oplus_{r=0}^m \Lambda^r T^{1,0}M) \cdot \overline{z}_1 \cdot \ldots \cdot \overline{z}_m$ becomes a non-trivial Clifford submodule of the Clifford algebra bundle, which is independent of the basis originally chosen. Therefore it can be identified with the spinor bundle $\Sigma M$ itself. Moreover, setting $\Sigma_r M := (\Lambda^r T^{1,0}M) \cdot \overline{z}_1 \cdot \ldots \cdot \overline{z}_m$, it is an exercise to prove that, w.r.t. the Clifford action of the Kähler-form $\Omega$,

$$\Sigma_r M = \mathrm{Ker}(\Omega \cdot -i(2r - m)\mathrm{Id})$$

and that the Clifford action by $\Omega$ is skew-Hermitian and parallel. As a consequence, one obtains the orthogonal and parallel decomposition

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M. \tag{3.13}$$

By construction $p_\pm(X) \cdot \Sigma_r M \subset \Sigma_{r\pm 1} M$, for every $X \in TM$, where we set $\Sigma_r M := 0$ as soon as $r \notin \{0, 1, \ldots, m\}$. In particular the Dirac operator $D$ does not preserve (3.13): setting $D_\pm := \sum_{j=1}^n p_\pm(e_j) \cdot \nabla_{e_j}$, we have $D = D_+ + D_-$ with $D_\pm : \Gamma(\Sigma_r M) \to \Gamma(\Sigma_{r\pm1} M)$ for every $r \in \{0, 1 \ldots, m\}$. Nevertheless, a more precise study of $D_\pm$ shows that $D_+ \circ D_+ = D_- \circ D_- = 0$, so that (3.13) is preserved by $D^2 = D_+ \circ D_- + D_- \circ D_+$. Beware that the operators $D_\pm$ have nothing to do with the $D^\pm$ of Proposition 1.3.2.
For any $r \in \{0, 1 \ldots, m\}$ and $\varphi \in \Gamma(\Sigma_r M)$ define

$$T_X^{(r)} \varphi := \nabla_X \varphi + \frac{1}{2(r+1)} p_-(X) \cdot D_+ \varphi + \frac{1}{2(m-r+1)} p_+(X) \cdot D_- \varphi$$

for all $X \in TM$. In other words, $T^{(r)}\varphi$ is the orthogonal projection of $\nabla\varphi$ onto the kernel of the Clifford multiplication $\mu : T^*M \otimes \Sigma_r M \longrightarrow \Sigma_{r-1} M \oplus \Sigma_{r+1} M$. Elementary computations show that

$$\sum_{j=1}^n p_+(e_j) \cdot p_-(e_j) \cdot = i\Omega \cdot -m\mathrm{Id} \qquad \text{and} \qquad \sum_{j=1}^n p_-(e_j) \cdot p_+(e_j) \cdot = -i\Omega \cdot -m\mathrm{Id},$$

in particular $\sum_{j=1}^n p_+(e_j) \cdot p_-(e_j) \cdot \varphi = -2r\varphi$ and $\sum_{j=1}^n p_-(e_j) \cdot p_+(e_j) \cdot \varphi = -2(m-r)\varphi$. We deduce for the norms that

$$\begin{aligned}
|T^{(r)}\varphi|^2 &= \sum_{j=1}^n |T_{e_j}^{(r)}\varphi|^2 \\
&= \sum_{j=1}^n |\nabla_{e_j}\varphi|^2 \\
&\quad + \sum_{j=1}^n \frac{1}{4(r+1)^2} |p_-(e_j) \cdot D_+\varphi|^2 + \frac{1}{4(m-r+1)^2} |p_+(e_j) \cdot D_-\varphi|^2 \\
&\quad + 2\sum_{j=1}^n \Re e \left( \frac{1}{2(r+1)} \langle \nabla_{e_j}\varphi, p_-(e_j) \cdot D_+\varphi \rangle \right) \\
&\quad + 2\sum_{j=1}^n \Re e \left( \frac{1}{2(m-r+1)} \langle \nabla_{e_j}\varphi, p_+(e_j) \cdot D_-\varphi \rangle \right) \\
&\quad + 2\sum_{j=1}^n \Re e \left( \frac{1}{4(r+1)(m-r+1)} \langle p_-(e_j) \cdot D_+\varphi, p_+(e_j) \cdot D_-\varphi \rangle \right) \\
&= |\nabla\varphi|^2 \\
&\quad - \frac{1}{4(r+1)^2} \sum_{j=1}^n \langle p_+(e_j) \cdot p_-(e_j) \cdot D_+\varphi, D_+\varphi \rangle \\
&\quad - \frac{1}{4(m-r+1)^2} \sum_{j=1}^n \langle p_-(e_j) \cdot p_+(e_j) \cdot D_-\varphi, D_-\varphi \rangle
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{r+1}|D_+\varphi|^2 - \frac{1}{m-r+1}|D_-\varphi|^2 \\
= \;& |\nabla\varphi|^2 \\
& + \frac{1}{2(r+1)}|D_+\varphi|^2 + \frac{1}{2(m-r+1)}|D_-\varphi|^2 \\
& -\frac{1}{r+1}|D_+\varphi|^2 - \frac{1}{m-r+1}|D_-\varphi|^2,
\end{aligned}$$

that is,

$$|T^{(r)}\varphi|^2 = |\nabla\varphi|^2 - \frac{1}{2(r+1)}|D_+\varphi|^2 - \frac{1}{2(m-r+1)}|D_-\varphi|^2. \tag{3.14}$$

Let $r \in \{0, 1 \ldots, m\}$ be the smallest integer for which $\mathrm{Ker}(D^2 - \lambda^2 \mathrm{Id}) \cap \Gamma(\Sigma_r M) \neq 0$. Let $\psi \in \Gamma(\Sigma_r M)$ be a non-zero eigenvector for $D^2$ associated to the eigenvalue $\lambda^2$. Since $[D^2, D_\pm] = 0$, both $D_+\psi_r$ and $D_-\psi_r$ lie in $\mathrm{Ker}(D^2 - \lambda^2\mathrm{Id})$, in particular $D_-\psi_r = 0$ by the choice of $r$. Independently, there exists on $\Sigma M$ a parallel field $j$ of complex antilinear automorphisms commuting with the Clifford multiplication by vectors (see e.g. [104, Lemma 1]), in particular $[\Omega\cdot, j] = [D, j] = 0$, so that $j(\mathrm{Ker}(D^2 - \lambda^2\mathrm{Id})) \subset \mathrm{Ker}(D^2 - \lambda^2\mathrm{Id})$ and $j(\Sigma_l M) = \Sigma_{m-l} M$ for every $l \in \{0, \ldots, m\}$. Thus the existence of $j$ imposes $r \leq m - r$, hence $r \leq \frac{m-1}{2}$ for $m$ odd. If $m$ is even, then $r = \frac{m}{2}$ cannot happen since otherwise $D_+\psi = D_-\psi = D\psi = 0$ would hold, which would contradict (3.1) together with $S > 0$, therefore $r \leq \frac{m-2}{2}$ for $m$ even.
We are now ready to prove the estimate. Integrating (3.14) and using Schrödinger-Lichnerowicz' formula (1.15), we obtain

$$\begin{aligned}
\|T^{(r)}\psi\|^2 &= (D^2\psi, \psi) - (\frac{S}{4}\psi, \psi) - \frac{1}{2(r+1)}\|D\psi\|^2 \\
&\leq (\frac{2r+1}{2(r+1)}\lambda^2 - \frac{1}{4}\inf_M(S))\|\psi\|^2,
\end{aligned}$$

from which one deduces that $\lambda^2 \geq \frac{2(r+1)}{4(2r+1)} \inf_M(S)$. The r.h.s. of that inequality decreases with $r$, so that it is bounded from below by the corresponding expression for $r = \frac{m-1}{2}$ in case $m$ is odd and for $r = \frac{m-2}{2}$ in case $m$ is even. Inequality (3.10) follows.
Assume now (3.10) to be an equality for some eigenvalue $\lambda$. If $\inf_M(S) = 0$ then $(M^n, g)$ has a non-zero parallel spinor, as already proved in Theorem 3.1.1. If $\inf_M(S) > 0$, then for any eigenvector $\psi$ for $D$ associated to the eigenvalue $\lambda$, one has on the one hand $\psi = \psi_{\frac{m-1}{2}} + \psi_{\frac{m+1}{2}}$ if $m$ is odd and $\psi = \psi_{\frac{m-2}{2}} + \psi_{\frac{m}{2}} + \psi_{\frac{m+2}{2}}$ if $m$ is even, on the other hand $T^{(r)}\psi_r = 0$ for $r = \frac{m\pm1}{2}$ and $r = \frac{m\pm2}{2}$ for $m$ odd and even respectively. Redenoting $\psi_{\frac{m-1}{2}}$ by $\psi$ and $\psi_{\frac{m+1}{2}}$ by $\phi$, we obtain (3.11) in case $m$ is odd. If $m$ is even then redenoting $\psi_{\frac{m-2}{2}}$ by $\psi$ one obtains (3.12). Conversely, mimiking the proof of Proposition A.4.1, it is elementary to show that, if (3.11) is satisfied by some non-zero $(\psi, \phi)$, then $\psi + \phi$ is an eigenvector for $D$ associated to the eigenvalue $\lambda$ and $S = \frac{4n\lambda^2}{n+2}$, therefore (3.10) is an equality. Similarly, if $\frac{n}{2}$ is even and (3.12) is satisfied by some non-zero $\psi$, then the scalar curvature of $(M^n, g)$ is equal to $\frac{4(n-2)\lambda^2}{n}$, therefore (3.10) is an equality. □

A pair of spinors $(\psi, \phi)$ satisfying (3.11) for some non-zero real number $\lambda$ is called a real Kählerian Killing spinor. As for Killing spinors (see Proposition A.4.1), it is not too difficult to show that, if a non-zero real Kählerian Killing spinor exists on a given complete Kähler spin manifold and associated to some (non-zero) real $\lambda$, then this manifold has odd complex dimension and is Einstein with positive scalar curvature (in particular it is closed). However, the precise classification of those Kähler spin manifolds carrying non-trivial real Kählerian Killing spinors is more technical, even if it turns out to provide simpler results. The idea to achieve it, due to A. Moroianu [203], can be summarised as follows: Show the existence of a suitable $\mathbb{S}^1$-bundle over such a manifold where the pull-back of the real Kählerian Killing spinor induces a non-zero real Killing spinor; then show that, among the possible holonomies listed in C. Bär's classification (Theorem A.4.3), only those associated to a so-called regular 3-Sasaki structure can occur on that $\mathbb{S}^1$-bundle. We refer to [203] for details and mention that, before [203] was published, partial results had been obtained by K.-D. Kirchberg [160, 161, 162] and O. Hijazi [135], see references in [203].
The even-complex-dimensional case turns out to be more involved since the underlying manifold is no more Einstein. In dimension $n = 4$, arguments from complex geometry and based on Kirchberg's work [163, Thm. 15] allowed T. Friedrich [90, Thm. 2] to prove that, if $(M^4, g, J)$ carries a non-zero spinor $\psi$ satisfying (3.12), then up to rescaling the metric $(M^4, g, J)$ must be holomorphically isometric either to $\mathbb{S}^2 \times \mathbb{S}^2$ or to $\mathbb{S}^2 \times \mathbb{T}^2$, both with product metric and spin structure, where $\mathbb{T}^2$ carries a flat metric and the trivial spin structure. In higher dimensions, if a non-zero spinor $\psi$ exists satisfying (3.12), then A. Moroianu showed [206] that the Ricci tensor of $(M^n, g)$ is parallel and has exactly two eigenvalues. This implies that the universal cover of $M$ is holomorphically isometric to the Riemannian product $N \times \mathbb{R}^2$, where $N$ is a closed Kähler spin manifold admitting a non-zero real Kählerian Killing spinor, see [206] for details. This result had been formulated by A. Lichnerowicz [184] where there remained however gaps in the proof.
We formulate the precise statements on the characterization of the limiting-case of (3.10).

**Theorem 3.2.5** *Let $(M^n, g, J)$ be a closed $n(\geq 4)$-dimensional Kähler spin manifold with positive scalar curvature.*

1. *If $\frac{n}{2}$ is odd, then* (3.10) *is an equality for some eigenvalue $\lambda$ of $D$ if and only if $(M^n, g, J)$ is holomorphically isometric to $\mathbb{C}\mathrm{P}^{\frac{n}{2}}$ in case $\frac{n}{2} \equiv 1$ (4) or to the twistor-space of a quaternionic Kähler manifold with positive scalar curvature in case $\frac{n}{2} \equiv 3$ (4)* (K.-D. Kirchberg [161] for $n = 6$, A. Moroianu [203] for $n \geq 6$).

2. *If $\frac{n}{2}$ is even, then* (3.10) *is an equality for some eigenvalue $\lambda$ of $D$ if and only if $(M^n, g, J)$ is isometric to $\Gamma \backslash N \times \mathbb{R}^2$, where $N$ is a simply-connected closed Kähler manifold admitting a non-zero real Kählerian Killing spinor $(\psi, \phi)$ and $\Gamma$ is generated by $(\gamma_j, \tau_j)$, $j = 1, 2$, where the $\tau_j$'s are translations of $\mathbb{R}^2$ and the $\gamma_j$'s are commuting holomorphic isometries of $N$ preserving its spin structure and $(\psi, \phi)$* (T. Friedrich [90] for $n = 4$, A. Moroianu [206] for $n \geq 8$).

Beware that, in case $\frac{n}{2}$ is even, the Kähler manifold must not be holomorphically isometric to the quotient $\Gamma\backslash N \times \mathbb{R}^2$ endowed with the Kähler structure induced from that of $N$. A simple criterion for holomorphicity is given in [206, Lemma 7.6].

The last class of manifolds with a non-trivial parallel form having been handled is that of quaternionic-Kähler manifolds, which carry a canonical parallel 4-form. The following theorem was proved by O. Hijazi and J.-L. Milhorat [139, 140, 141] for $n = 8$, 12 and by W. Kramer, U. Semmelmann and G. Weingart in the general case [167, 168]:

**Theorem 3.2.6** *Any eigenvalue $\lambda$ of $D$ on an $n(\geq 8)$-dimensional closed* quaternionic-Kähler *spin manifold $(M^n, g)$ with positive scalar curvature satisfies*

$$\lambda^2 \geq \frac{n+12}{4(n+8)} S, \tag{3.15}$$

*where $S$ is the scalar curvature of $M$. Moreover, this inequality is an equality for some eigenvalue $\lambda$ if and only if $(M^n, g)$ is isometric to the quaternionic projective space $\mathbb{HP}^{\frac{n}{4}}$.*

Here it should be noticed that every quaternionic Kähler manifold of even quaternionic dimension is spin whereas only the quaternionic projective space is spin if $\frac{n}{4}$ is odd; moreover, every quaternionic Kähler manifold is Einstein, hence has constant scalar curvature, see references in [168].

*Sketch of proof of* Theorem 3.2.6: We follow the proof detailed in [168], which relies on the representation theory of $\mathrm{Sp}_1 \times \mathrm{Sp}_k$. Denote $\frac{n}{4}$ by $m$. A quaternionic structure on $(M^n, g)$ is given by a triple $(I, J, K)$ of parallel orthogonal endomorphisms of $TM$ with $I^2 = J^2 = K^2 = -\mathrm{Id}_{TM}$ and $IJ = -JI = K$. Each of those endormorphisms is a Kähler structure on $TM$ with associated Kähler form, so that one may define the so-called fundamental form

$$\Omega := \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K$$

on $TM$. The 4-form $\Omega$ is parallel and can be shown to act on $\Sigma M$ so as to split it into

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M,$$

with $\Sigma_r M := \mathrm{Ker}(\Omega \cdot -(6m - 4r(r+2))\mathrm{Id}) \subset \Sigma M$ [140]. As in the Kähler case, the Clifford multiplication sends $T^*M \otimes \Sigma_r M$ into $\Sigma_{r-1}M \oplus \Sigma_{r+1}M$. Decomposing $\mathrm{Ker}(\mu)_{|_{T^*M\otimes\Sigma_r M}}$ into irreducible components under $\mathrm{Sp}_1 \times \mathrm{Sp}_{m-1}$, one obtains four twistor operators associated to the orthogonal projections of $\nabla^2\varphi$ onto the irreducible components [168, p.745]. Taking $\varphi \in \Gamma(\Sigma_r M)$ to be an eigenvector for $D^2$ and applying Schrödinger-Lichnerowicz' formula (1.15) lead to the desired inequality, see [168, Sec. 4] for the rather technical proof where the authors determine all Weitzenböck formulas involving Dirac and twistor operators.

The equality case in the inequality is sharp for $\mathbb{HP}^m$ [195]. Conversely, if it is sharp, then the spinor bundle of $M$ carries a particular (non-zero) section called quaternionic Killing spinor [167, p.340]. This spinor induces a non-zero real Killing spinor on the total space of the $SO_3$-principal bundle associated to the quaternionic Kähler structure and for a suitable metric [167, p.344]. Then Bär's classification (Theorem A.4.3) of manifolds with real Killing spinors forces $M$ to be isometric to $\mathbb{HP}^m$, we refer to [167, Sec. 7] for the details. □

## 3.3 Improving Friedrich's inequality in a conformal way

N. Hitchin [152] noticed in the Riemannian setting (as well as H. Baum [52] in the pseudo-Riemannian one) that the fundamental Dirac operator is *conformally covariant*, see Proposition 1.3.10. This was the starting point for the following result.

**Theorem 3.3.1 (O. Hijazi [132])** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional closed Riemannian spin manifold and $u \in C^\infty(M, \mathbb{R})$, then any eigenvalue $\lambda$ of $D$ on $(M^n, g)$ satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M (\overline{S} e^{2u}), \tag{3.16}$$

*where $\overline{S}$ is the scalar curvature of $(M^n, \overline{g} := e^{2u} g)$. Moreover,* (3.16) *is an equality for some eigenvalue $\lambda$ if and only if $u$ is constant and $(M^n, g)$ carries a non-zero real Killing spinor.*

*Proof*: We use the notations of Proposition 1.3.10. For any $\varphi \in \Gamma(\Sigma M)$ one has from (3.3) applied to $\overline{\psi} := e^{-\frac{n-1}{2}u}\overline{\varphi}$,

$$\int_M \left( |\overline{D\psi}|^2 - \frac{n}{4(n-1)} \overline{S}\, |\overline{\psi}|^2 \right) v_{\overline{g}} = \frac{n}{n-1} \int_M |\overline{P}\,\overline{\psi}|^2 v_{\overline{g}} \geq 0.$$

Proposition 1.3.10 states that $\overline{D\psi} = e^{-\frac{n+1}{2}u}\overline{D\varphi}$, so that choosing $\varphi$ to be a non-zero eigenvector for $D$ associated to the eigenvalue $\lambda$ one obtains $\overline{D}\,\overline{\psi} = \lambda e^{-u}\overline{\psi}$ and the inequality (3.16). Furthermore, if (3.16) is an equality for some eigenvalue $\lambda$ of $D$ on $(M^n, g)$, then for any non-zero eigenvector $\varphi$ for $D$ associated to $\lambda$, the identity $\overline{P}\,\overline{\psi} = 0$ holds, where $\overline{\psi} := e^{-\frac{n-1}{2}u}\overline{\varphi}$. This implies in turn

$$\overline{\nabla}_X \overline{\psi} = -\frac{\lambda e^{-u}}{n} X \,\overline{\cdot}\, \overline{\psi}$$

for every $X \in TM$. Elementary computations as in the proof of Proposition A.4.1 but carried out on $(M^n, \overline{g})$ (see e.g. [138, Prop. 5.12]) show that necessarily $du = 0$, thus $u$ is constant and therefore $\varphi$ is a real Killing spinor on $(M^n, g)$. The converse statement follows from the characterization of the equality case in (3.1). This concludes the proof. □

**Corollary 3.3.2** *Any eigenvalue $\lambda$ of $D$ on an $n$-dimensional closed Riemannian spin manifold $(M^n, g)$ satisfies:*

*i)* (C. Bär [38]) *For $n = 2$,*

$$\lambda^2 \geq \frac{2\pi\chi(M^2)}{\text{Area}(M^2, g)}, \tag{3.17}$$

*where* $\chi(M^2)$ *is the Euler characteristic of* $M^2$*. Moreover,* (3.17) *is an equality for some eigenvalue* $\lambda$ *of* $D$ *if and only if* $(M^2, g)$ *is isometric either to* $\mathbb{S}^2$ *with constant curvature metric or to* $\mathbb{T}^2$ *with flat metric and trivial spin structure.*

*ii)* (O. Hijazi [132]) *For* $n \geq 3$,

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1, \tag{3.18}$$

*where* $\mu_1$ *denotes the first eigenvalue of the scalar conformal Laplace operator* $4\frac{n-1}{n-2}\Delta + S$*. Moreover,* (3.18) *is an equality for some eigenvalue* $\lambda$ *of* $D$ *if and only if* $(M^n, g)$ *carries a non-zero real Killing spinor.*

*Proof*: We deduce both (3.18) and (3.17) from (3.16) and from the following transformation formula for scalar curvature after conformal change of the metric:

$$\overline{S}e^{2u} = S + 2(n-1)\Delta u - (n-1)(n-2)|\mathrm{grad}(u)|^2, \tag{3.19}$$

for $\overline{g} := e^{2u}g$ and $u \in C^\infty(M, \mathbb{R})$. This is applied to a conformal metric $\overline{g}$ for which $\overline{S}e^{2u}$ is constant on $M$.
*i)* Let $u_0 \in C^\infty(M, \mathbb{R})$ solve $\Delta u_0 = \frac{\int_M S v_g}{2\mathrm{Area}(M^2,g)} - \frac{S}{2}$ (such a solution exists because the r.h.s. has vanishing integral). Since in dimension $n = 2$ the formula (3.19) reads $\overline{S}e^{2u_0} = S + 2\Delta u_0$ one obtains $\overline{S}e^{2u_0} = \frac{\int_M S v_g}{\mathrm{Area}(M^2,g)}$. Applying now the Gauss-Bonnet Theorem, (3.16) becomes

$$\lambda^2 \geq \frac{1}{2}\frac{\int_M S v_g}{\mathrm{Area}(M^2, g)} = \frac{2\pi\chi(M^2)}{\mathrm{Area}(M^2, g)},$$

which is (3.17). If now (3.17) is an equality for some eigenvalue $\lambda$ of $D$, then by construction of $u_0$ the scalar curvature $S$ must be constant and non-negative. Since we explicitly know the Dirac spectra of $\mathbb{S}^2$ (see Theorem 2.1.3) and of $\mathbb{T}^2$ (see Theorem 2.1.1), we can compare the smallest non-negative Dirac eigenvalue with the lower bound in (3.17) and state that equality always occurs for $S > 0$ and occurs on $\mathbb{T}^2$ only when fixing the trivial spin structure.
*ii)* We can assume that $\mu_1 > 0$. From Courant's nodal domain theorem (see e.g. [77, p.19]) every non-zero eigenfunction $h_0$ for $L := 4\frac{n-1}{n-2}\Delta + S$ associated to its smallest eigenvalue $\mu_1$ cannot vanish, hence may be assumed to be positive. In particular $\mu_1 = h_0^{-1}Lh_0$. But in dimension $n \geq 3$ the formula (3.19) can be rewritten under the following form: for any positive smooth function $h$ on $M^n$,

$$\overline{S}h^{\frac{4}{n-2}} = h^{-1}Lh,$$

where $\overline{S}$ is the scalar curvature of $(M^n, \overline{g} := h^{\frac{4}{n-2}}g)$. Thus, choosing $\overline{g}_0 := h_0^{\frac{4}{n-2}}g$ on $M^n$, one obtains $\overline{S}h_0^{\frac{4}{n-2}} = \mu_1$, which together with Theorem 3.3.1 implies (3.18). The characterization of the equality case in (3.18) follows from Theorem 3.3.1 as well. □

Another proof of (3.18) involving Kato type inequalities can be found in [74].

Inequality (3.18) improves Friedrich's inequality (3.1) for $n \geq 3$ since obviously $\mu_1 \geq \inf_M (S)$. It also proves the existence in dimension $n \geq 3$ of an explicit conformal lower bound for the spectrum of $D^2$:

**Corollary 3.3.3 (O. Hijazi [134])** *For any Riemannian metric $g$ on a closed $n(\geq 3)$-dimensional spin manifold $M^n$,*

$$\lambda_1(D^2_{M,g})\mathrm{Vol}(M,g)^{\frac{2}{n}} \geq \frac{n}{4(n-1)}Y(M,[g]), \tag{3.20}$$

*where $\lambda_1(D^2_{M,g})$ denotes the smallest non-negative eigenvalue of $D^2$ associated to the metric $g$ and $Y(M,[g])$ is the Yamabe invariant of $M$ w.r.t. the conformal class of $g$.*

*Proof*: Recall that the Yamabe invariant of $M^n$ w.r.t. $[g]$ is the conformal invariant defined by

$$Y(M,[g]) := \inf_{f\in C^\infty(M,\mathbb{R})\setminus\{0\}} \Big\{ \frac{\int_M (4\frac{n-1}{n-2}\Delta_g f + S_g f) f v_g}{(\int_M f^{\frac{2n}{n-2}} v_g)^{\frac{n-2}{n}}} \Big\}.$$

Hölder's inequality gives $\int_M f^2 v_g \leq (\int_M f^{\frac{2n}{n-2}} v_g)^{\frac{n-2}{n}} \cdot \mathrm{Vol}(M^n,g)^{\frac{2}{n}}$. Assuming $Y(M,[g]) > 0$ (otherwise (3.20) is trivially satisfied), one obtains

$$\mu_1 \mathrm{Vol}(M^n,g)^{\frac{2}{n}} \geq Y(M,[g]),$$

which with (3.18) implies the result. □

Note however that (3.18) is not itself conformal. We also mention that inequality (3.18) can be combined with lower bounds of $\mu_1$ to provide an estimate of $\lambda_1(D^2_{M,g})$ in terms of the total $Q$-curvature in dimension $n = 4$ and of the first eigenvalue of the so-called Branson-Paneitz operator in dimension $n \geq 5$, see [149, Sec. 4]. The *a priori* existence of a qualitative conformal lower bound for the Dirac spectrum was proved independently by J. Lott [187] using the boundedness of particular Sobolev embeddings. More precisely, if the Dirac operator of a given closed Riemannian spin manifold $(M^n, g)$ is invertible, then there exists a positive constant $c$ depending only on the conformal class of $g$ such that [187, Prop. 1]

$$\lambda_1(D^2_{M,\overline{g}})\mathrm{Vol}(M,\overline{g})^{\frac{2}{n}} \geq c \tag{3.21}$$

for any metric $\overline{g}$ conformal to $g$ on $M^n$.

C. Bär's estimate (3.17) gives a topologically invariant lower bound on the Dirac spectrum. Surprisingly enough this contrasts with the situation of the scalar Laplacian on $\mathbb{S}^2$ for which this invariant provides an upper bound for the first non-zero eigenvalue in one of the corresponding estimates established by J. Hersch (see reference in [45]) and which reads

$$\lambda_1(\Delta_{\mathbb{S}^2,g})\mathrm{Area}(\mathbb{S}^2,g) \leq 8\pi, \tag{3.22}$$

where $\lambda_1(\Delta_{\mathbb{S}^2,g})$ denotes the smallest positive eigenvalue of the scalar Laplace operator $\Delta$ on $(\mathbb{S}^2, g)$. For lower bounds in higher genus, where (3.17) is trivial, see Section 3.6.

As noticed in the proof of Corollary 3.3.2, one has from the Gauss-Bonnet Theorem in dimension 2 the following identity: $\frac{2\pi\chi(M^2)}{\mathrm{Area}(M^2,g)} = \frac{2}{4(2-1)}\frac{\int_{M^2} S v_g}{\mathrm{Area}(M^2,g)}$. Can one improve Friedrich's inequality (3.1) in dimension $n \geq 3$ by replacing $\inf_M(S)$ by $\frac{1}{\mathrm{Vol}(M,g)}\int_M S v_g$? B. Ammann and C. Bär showed [18] that this is not the case at all: on any compact spin manifold of dimension $n \geq 3$ and for any positive integer $k$ there exists a sequence of Riemannian metrics for which the $k^{\text{th}}$ Dirac eigenvalue remains bounded whereas the averaged total scalar curvature tends to infinity. We refer to [45] for a detailed and illustrated proof.

## 3.4 Improving Friedrich's inequality with the e-nergy-momentum tensor

The main idea to prove inequality (3.1) was to split the spinorial Levi-Civita connection in a clever way so as to make the term which is dropped off after integration and application of the Schrödinger-Lichnerowicz formula as small as possible. This led to the introduction of the Penrose operator. In an equivalent way, this means deforming the Levi-Civita connection in the direction of $\mathrm{Id}_{TM}$, i.e., defining $T_X\varphi := \nabla_X\varphi + fX\cdot\varphi$ for some real or complex-valued function $f$ to be fixed later, see T. Friedrich's method of proof in Section 3.1. O. Hijazi's idea for the following result is to introduce a different Penrose-like operator, deforming the Levi-Civita connection in the direction of some symmetric 2-tensor tensor $T^\psi$ associated to an eigenvector $\psi$:

**Theorem 3.4.1 (O. Hijazi [136])** *Let $\lambda$ be an eigenvalue of the fundamental Dirac operator on a closed $n(\geq 2)$-dimensional closed Riemannian spin manifold $(M^n, g)$ and $\psi$ be a non-zero eigenvector for $D$ to the eigenvalue $\lambda$. Then*

$$\lambda^2 \geq \inf_{M_\psi}\left(\frac{S}{4} + |T^\psi|^2\right), \tag{3.23}$$

*where $T^\psi(X,Y) := \frac{1}{2}\Re e\left(\langle X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \frac{\psi}{|\psi|^2}\rangle\right)$ for all $X, Y \in TM$ and $M_\psi := \{x \in M \,|\, \psi(x) \neq 0\}$. If furthermore (3.23) is an equality then $\psi$ solves*

$$\nabla_X\psi = -T^\psi(X)\cdot\psi \tag{3.24}$$

*for all $X \in TM$.*

*Proof*: Define the following modified connection $\widehat{\nabla}$ on $\Sigma M$ (and outside the zero set of $\psi$, of which measure vanishes) by

$$\widehat{\nabla}_X\psi := \nabla_X\psi + T^\psi(X)\cdot\psi$$

for every $X \in TM$. We compute in a local orthonormal frame $\{e_j\}_{1\leq j\leq n}$:

$$\begin{aligned}
|\widehat{\nabla}\psi|^2 &= \sum_{j=1}^n |\widehat{\nabla}_{e_j}\psi|^2 \\
&= \sum_{j=1}^n |\nabla_{e_j}\psi|^2 + |T^\psi(e_j)|^2|\psi|^2 + 2\Re e\left(\langle\nabla_{e_j}\psi, T^\psi(e_j)\cdot\psi\rangle\right) \\
&= |\nabla\psi|^2 + |T^\psi|^2|\psi|^2 - 2\sum_{j,k=1}^n T^\psi(e_j,e_k)\Re e\left(\langle e_k\cdot\nabla_{e_j}\psi, \psi\rangle\right) \\
&= |\nabla\psi|^2 - |T^\psi|^2|\psi|^2
\end{aligned}$$

since from its definition the tensor $T^\psi$ is symmetric. Integrating and applying the Schrödinger-Lichnerowicz formula (1.15) leads straightforward to the inequality, of which limiting-case implies $\widehat{\nabla}\psi = 0$. This concludes the proof. □

The tensor $T^\psi$ is sometimes called the *energy-momentum tensor* associated to $\psi$, see e.g. [50, Sec. 6] for a justification of its name.

The lower bound in (3.23) for the eigenvalue $\lambda$ has the obvious disadvantage to depend on the eigenvector $\psi$ to $\lambda$, hence Theorem 3.4.1 does not directly provide a geometric lower bound for the Dirac spectrum. Note however that (3.23) improves Friedrich's inequality (3.1) whatever the tensor $T^\psi$ could be: one can indeed write $T^\psi = (T^\psi)^0 + \frac{\mathrm{tr}_g(T^\psi)}{n} g$, where $(T^\psi)^0$ denotes the trace-free part of $T^\psi$. Since $D\psi = \lambda\psi$ one has in any local o.n.b. $\{e_j\}_{1\leq j\leq n}$ of $TM$:

$$\begin{aligned} \mathrm{tr}_g(T^\psi) &= \sum_{j=1}^n \Re e \left( \langle e_j \cdot \nabla_{e_j}\psi, \frac{\psi}{|\psi|^2}\rangle \right) \\ &= \Re e \left( \langle D\psi, \frac{\psi}{|\psi|^2}\rangle \right) \\ &= \lambda, \end{aligned}$$

so that $|T^\psi|^2 = |(T^\psi)^0|^2 + \frac{\lambda^2}{n}$ and

$$\lambda^2 \geq \inf_{M_\psi} \Big( \frac{n}{4(n-1)} S + \underbrace{\frac{n}{n-1} |(T^\psi)^0|^2}_{\geq 0} \Big),$$

which implies (3.1).

One can also remark that the proof of Theorem 3.4.1 only needs $\psi$ to be eigen for $D^2$, which is weaker than $\psi$ be eigen for $D$. However the comparison with Friedrich's inequality as just above is in this case not available.

Closed spin manifolds carrying a non-zero eigenvector $\psi$ of $D$ satisfying (3.24) have not been completely classified yet. From the above comparison with Friedrich's inequality (3.1) they contain all manifolds carrying non-zero real Killing spinors. Recent works [122, 107], where examples of manifolds are given where (3.23) is sharp but not (3.1) (e.g. Heisenberg manifolds, see [107, Ex. 6.4]), show that they form a strictly larger family.

Besides we mention that Theorem 3.4.1 was generalized by T. Friedrich and E.-C. Kim [96, Lemma 5.1] and by G. Habib [122, Thm. 2.2.1] (see also [125]). More recently, an analogous ansatz was successfully carried out by T. Friedrich and E.C. Kim [97, Thm. 1.1] where the lower bound for the Dirac spectrum depends on the spectrum of a Dirac-type operator associated to a so-called nondegenerate Codazzi tensor, we refer to [97] for details.

## 3.5 Improving Friedrich's inequality with other curvature components

In case the scalar curvature of a compact spin manifold is not everywhere positive one can try to look for lower eigenvalue bounds involving the Ricci and Weyl components of the curvature tensor. The proof of the following theorems relies on the application of the Schrödinger-Lichnerowicz formula (1.15) after a suitable choice of Penrose-like operator involving those tensors, see e.g. [165] for the highly technical details.

**Theorem 3.5.1 (T. Friedrich and K.-D. Kirchberg [99])** *Any eigenvalue $\lambda$ of $D$ on an $n(\geq 2)$-dimensional closed Riemannian spin manifold $(M^n, g)$* with divergence-free curvature tensor, vanishing scalar curvature and nowhere-vanishing Ricci-curvature *satisfies:*

$$\lambda^2 > \frac{1}{4} \frac{\inf_M |\mathrm{Ric}|^2}{\sqrt{\frac{n-1}{n}} \inf_M |\mathrm{Ric}| - \kappa_0},$$

*where $\kappa_0$ denotes the smallest eigenvalue of the Ricci tensor* Ric *on $M$.*

Examples of closed Riemannian manifolds satisfying the assumptions of Theorem 3.5.1 and where the lower bound can be explicitly computed can be found among the following families, see [99, Ex. 1-4] and [164, Ex. 4.1 & 4.2]: (local) Riemannian products of Einstein manifolds, warped products of $\mathbb{S}^1$ with an Einstein manifold with positive scalar curvature, warped products on Riemannian surfaces, conformally flat manifolds. Note however that Einstein manifolds themselves or manifolds whose Ricci tensor vanishes somewhere cannot be handled by Theorem 3.5.1. This was the motivation of T. Friedrich and K.-D. Kirchberg for obtaining a lower bound involving the Weyl tensor only. The best result in this direction was obtained by K.-D. Kirchberg, generalizing an earlier one by T. Friedrich and himself [98, Thm. 3.1]:

**Theorem 3.5.2 (K.-D. Kirchberg [165])** *Any eigenvalue $\lambda$ of $D$ on an $n(\geq 2)$-dimensional closed Riemannian spin manifold $(M^n, g)$* with divergence-free Weyl-tensor *and $\mu > 0$ satisfies:*

$$\lambda^2 \geq \frac{1}{8(n-1)} \left( (2n-1) \inf_M(S) + \sqrt{\inf_M(S)^2 + \frac{n}{n-1}(\frac{4\nu_0}{\mu})^2} \right), \qquad (3.25)$$

*where $\nu_0 \geq 0$ and $\mu$ are conformal invariants depending on the Weyl tensor only.*

Recall that every Einstein Riemannian manifold has divergence-free Weyl tensor. In case $\inf_M(S) > 0$ inequality (3.25) obviously enhances Friedrich's inequality (3.1). In case $\inf_M(S) \leq 0$ it is easy to see that the lower bound in (3.25) is positive if and only if $\nu_0 > \frac{(n-1)\mu}{2} |\inf_M(S)|$. However, it is up to now not known if (3.25) can be an equality [165, Rem. 4.2.ii)].

Theorems 3.5.1 and 3.5.2 actually follow from a whole series of estimates [99, 165] involving curvature tensors and that can be applied to produce fine vanishing theorems for the kernel of the Dirac operator.

## 3.6 Improving Friedrich's inequality on surfaces of positive genus

C. Bär's inequality (3.17) does not give any information on the spectrum of $D$ on compact Riemannian surfaces with nonpositive Euler characteristic, i.e., with positive genus. Estimates on such surfaces have to depend on the choice of spin structure, as the example of the 2-torus already shows: for its trivial spin structure (i.e., for the spin structure coming from the trivial lift of the lattice-action to the spin level, see Proposition 1.4.2) it admits harmonic spinors - for flat hence any metrics because of (1.16) - but not for any other spin structure [89].

The first estimate to have been proved is a qualitative one and dates back to J. Lott's work [187] providing lower bounds for general conformally covariant elliptic self-adjoint linear differential operators. In the case of surfaces it states that, if the Dirac operator of a given closed oriented surface $(M^2, g)$ is invertible, then there exists a positive constant $c$ such that, for any metric $\overline{g}$ conformal to $g$ on $M^2$ (see (3.21)):

$$\lambda_1(D^2_{M,\overline{g}})\mathrm{Area}(M, \overline{g}) \geq c.$$

The constant $c$ expresses the boundedness of particular Sobolev embeddings hence cannot be made explicit in general.

The first successful attempt in looking for a geometric estimate is due to B. Ammann [14]. His lower bound, which was proved for the 2-torus, involves the so-called *spinning systole* spin-sys$(M)$ of a closed oriented surface $M$ with positive genus which is defined to be the minimum of the lengths of all noncontractible loops (in our convention, loops are simply closed curves) along which the induced spin structure is non trivial. Recall that the systole of $M$ is defined to be the minimum of the lengths of all noncontractible loops in $M$.

**Theorem 3.6.1 (B. Ammann [14])** *Let $g$ be an arbitrary Riemannian metric on the 2-torus $M := \mathbb{T}^2$ carrying a non-trivial spin structure. Assume that $\|K_g\|_{\mathrm{L}^1(\mathbb{T}^2,g)} < 4\pi$, where $K_g$ ist the Gauss curvature of $(\mathbb{T}^2, g)$. Then there exists for each $p > 1$ a constant $C_p > 0$ depending on $\|K_g\|_{\mathrm{L}^1(\mathbb{T}^2,g)}$, $\|K_g\|_{\mathrm{L}^p(\mathbb{T}^2,g)}$, the area and the systole of $(\mathbb{T}^2, g)$ such that any eigenvalue $\lambda$ of $D$ satisfies*

$$\lambda^2 \geq \frac{\sup\limits_{p>1} C_p}{\text{spin-sys}(\mathbb{T}^2)^2}.$$

*Moreover this inequality is an equality if and only if $g$ is flat, the lattice is generated by an orthogonal pair and the spin structure is the $(1, 0)$- or $(0, 1)$-one.*

*Sketch of proof of* Theorem 3.6.1: The proof of the inequality combines the following steps. First one chooses a flat metric $g_0 := e^{2u}g$ in the conformal class of $g$. Using the min-max principle (see e.g. Lemma 5.0.2) it can be easily proved that $\lambda^2 \geq e^{2\max(u)}\lambda_0^2$, where $\lambda_0 > 0$ is the smallest Dirac eigenvalue (in absolute value) on $(\mathbb{T}^2, g_0)$ and for the same spin structure. Now the Dirac spectrum of $(\mathbb{T}^2, g_0)$ for any spin structure is explicitly known (see Theorem 2.1.1), in particular the following equality holds

$$\lambda_0^2 \mathrm{Area}(\mathbb{T}^2, g_0) = 4\pi^2 \|\chi\|^2_{\mathrm{L}^2(\mathbb{T}^2, g_0)},$$

where $\chi \in H^1(\mathbb{T}^2, \mathbb{Z}_2)$ is the cohomology class representing the spin structure (it is non-zero if the spin structure is non-trivial). Obviously $\mathrm{Area}(\mathbb{T}^2, g) \geq e^{-2\min(u)}\mathrm{Area}(\mathbb{T}^2, g_0)$ so that one obtains a lower bound of $\lambda$ in terms of the area of $(\mathbb{T}^2, g)$, of the $\mathrm{L}^2$-norm of $\chi$ and of the so-called oscillation $\mathrm{osc}(u) := \max(u) - \min(u)$ of $u$ on $\mathbb{T}^2$. On the other hand the $\mathrm{L}^2$-norm of $\chi$ can be proved to be only dependent of the conformal class of $g$ and can be estimated against an expression involving the spinning systole, the area and $\mathrm{osc}(u)$ [14, Sec. 4]. What remains - the whole work - is to estimate $\mathrm{osc}(u)$ against the desired geometric data. For an illustrated proof of this Sobolev-type inequality we refer to [14, Sec. 6]. The limiting-case occurs if and only if $\mathrm{osc}(u) = 0$ (i.e., $g$ is flat) and the estimate of $\mathrm{osc}(u)$ is sharp, which yields strong conditions on the lattice defining the flat metric $g$, see [14]. □

Another and completely different approach was developed by B. Ammann and C. Bär in [19]. It aimed at obtaining a lower bound in terms of a geometric invariant called the *spin-cut diameter* $\delta(M)$. This is a positive number which is associated to the surface $M$ and its spin structure. The idea is simple: apply (3.17) to the surface obtained from the genus $g$ surface $M$ by cutting $g$ suitable loops out of $M$. Here "suitable" means the following: on the one hand one has to choose the loops such that the resulting surface $\widetilde{M}$ is diffeomorphic to an open subset of $\mathbb{S}^2$ - actually to a 2-sphere with $2g$ disks removed; this is the case as soon as the $\mathbb{Z}_2$-homology classes associated to those loops form a basis of $H_1(M, \mathbb{Z}_2)$. On the other hand the cut-out-process must also respect the spin structures in the sense that the restrictions of the original one and of the one from $\mathbb{S}^2$ have to coincide on $\widetilde{M}$. This however is only possible if the so-called *Arf-invariant* (which associates to each spin structure on $M$ the number 1 or $-1$, see [19, Def. p.430]) of the spin structure of $M$ is 1 [19, Cor. 3.3]. The spin-cut diameter can then be defined from the distances between the cut-out loops, see [19, Def. p.433]. For $M = \mathbb{T}^2$ the Arf-invariant of the trivial spin structure is $-1$ and it is 1 for the other ones. In the latter case, extending by means of suitable cut-offs an eigenvector on $M$ to the $\mathbb{S}^2$ obtained by adding two disks to the gluing of a finite number of copies of $\widetilde{M}$ (which is then a cylinder) one can prove the following result [19, Sec. 5]:

**Theorem 3.6.2 (B. Ammann and C. Bär [19])** *Let $M := \mathbb{T}^2$ be the 2-torus with arbitrary Riemannian metric and non-trivial spin structure. Then any eigenvalue $\lambda$ of $D$ satisfies*

$$|\lambda| \geq \sup_{\substack{k\in\mathbb{N} \\ k\neq 0}} \left( -\frac{2}{k\delta(\mathbb{T}^2)} + \sqrt{\frac{\pi}{k\mathrm{Area}(\mathbb{T}^2)} + \frac{2}{k^2\delta(\mathbb{T}^2)^2}} \right), \qquad (3.26)$$

*where $\delta(\mathbb{T}^2)$ is the spin-cut diameter of $\mathbb{T}^2$ associated to this spin structure.*

The supremum in the lower bound is attained for $k = \left[\frac{4(\sqrt{2}+1)\mathrm{Area}(\mathbb{T}^2)}{\pi\delta(\mathbb{T}^2)^2}\right]$ or $k = \left[\frac{4(\sqrt{2}+1)\mathrm{Area}(\mathbb{T}^2)}{\pi\delta(\mathbb{T}^2)^2}\right]+1$. It is positive and for the boundary $\mathbb{T}^2_\varepsilon$ of an $\varepsilon$-tubular neighbourhood of a circle of radius $\frac{1}{\varepsilon}$ it is asymptotic to $\sqrt{\frac{\pi}{\mathrm{Area}(\mathbb{T}^2_\varepsilon)}}$ when $\varepsilon$ tends to 0. Therefore Theorem 3.6.2 can be viewed as a generalization of Corollary 3.3.2.*ii*) for $\mathbb{T}^2$ with non-trivial spin structure.

In genus $g \geq 1$ one can apply the same argument to the $\mathbb{S}^2$ obtained by adding disks to a clever gluing of $2g+1$ copies of $\widetilde{M}$ and prove [19, Sec. 6]:

**Theorem 3.6.3 (B. Ammann and C. Bär [19])** *Let $M$ be a closed Riemannian surface of positive genus $g$ with spin structure whose Arf-invariant equals 1. Then any eigenvalue $\lambda$ of $D$ satisfies*

$$|\lambda| \geq \frac{2}{2g+1} \cdot \sqrt{\frac{\pi}{\mathrm{Area}(M)}} - \frac{1}{\delta(M)}. \tag{3.27}$$

Although the lower bound need this time not be positive there exist examples for which it is: as above, consider an $\varepsilon$-tubular neighbourhood $M_\varepsilon$ of a closed plane curve with exactly $g-1$ intersections and such that, w.r.t. any allowed spin structure, $\delta(M_\varepsilon) \underset{\varepsilon\to 0}{\sim} \frac{\mathrm{cst}}{\varepsilon}$ (fix for instance the diameter equal to $\frac{1}{\varepsilon}$). Then the lower bound is asymptotic to $\frac{2}{2g+1} \cdot \sqrt{\frac{\pi}{\mathrm{Area}(M_\varepsilon)}}$ for $\varepsilon \to 0$. In the case where $g = 1$ the $k$-dependent expression in the r.h.s. of (3.26) is for $k = 2$ greater than the r.h.s. of (3.27), so that (3.26) is better than (3.27).
Combining Theorems 3.6.2 and 3.6.3 with the extrinsic upper bound (5.19) for the smallest Dirac eigenvalue for surfaces embedded in $\mathbb{R}^3$ one obtains a lower bound of the Willmore functional, see [19, Thm. 7.1]. Besides, we mention that Theorems 3.6.2 and 3.6.3 can be extended to complete surfaces with finite area [19, Thm. 8.1].

## 3.7 Improving Friedrich's inequality on bounding manifolds

In case $M$ bounds a compact spin manifold $\widetilde{M}$ the input of extrinsic geometrical data - such as the mean curvature of $M$ in $\widetilde{M}$ - can improve Friedrich's inequality (3.1). The main step consists in solving a suitable boundary value problem, see also Chapter 4. The following theorem was proved by O. Hijazi, S. Montiel and X. Zhang in [146] for $c = 0$ and by O. Hijazi, S. Montiel and A. Roldán in [144] for $c < 0$. Recall that $D_2$ is the operator acting on the sections of $\Sigma := \Sigma M$ or $\Sigma := \Sigma M \oplus \Sigma M$ and which is defined by $D_2 := D$ for $n$ even or $D_2 := D \oplus -D$ for $n$ odd respectively, see Proposition 1.4.1.

**Theorem 3.7.1** *Let $M^n = \partial\widetilde{M}$, where $\widetilde{M}$ is a compact Riemannian spin manifold. Assume that, for a constant $c \leq 0$, the scalar curvature $\widetilde{S}$ of $\widetilde{M}$ and the* *mean curvature* $H$ *of* $M$ *in* $\widetilde{M}$ *w.r.t. the inner normal satisfy* $\widetilde{S} \geq (n+1)nc$ *and* $H \geq \sqrt{-c}$ *respectively. Then for any eigenvalue* $\lambda$ *of* $D$,

$$|\lambda| \geq \frac{n}{2} \inf_M (\sqrt{H^2 + c}). \tag{3.28}$$

*Moreover* (3.28) *is an equality if and only if* $H$ *is constant, the manifold* $(\widetilde{M}, g)$ *admits a non-trivial* $\frac{\sqrt{c}}{2}$*- or* $-\frac{\sqrt{c}}{2}$*-Killing spinor and the eigenspace of* $D_2$ *to the eigenvalue* $\frac{n}{2}\inf_M(\sqrt{H^2+c})$ *coincides with* $(\mathcal{K}_0)_{|_M}$ *for* $c = 0$ *and with*

$$\begin{array}{ll} \bigoplus_{j=0,1} (\mathrm{Id} + (-1)^j i(H - \sqrt{H^2+c})\nu\cdot)\mathcal{K}_{(-1)^j\frac{\sqrt{c}}{2}}(\widetilde{M}, g)_{|_M} & \textit{if } H > \sqrt{-c} \\ \bigoplus_{j=0,1} \mathcal{K}_{(-1)^j\frac{\sqrt{c}}{2}}(\widetilde{M}, g)_{|_M} & \textit{if } H = \sqrt{-c} \end{array}$$

*for* $c < 0$*, where* $\mathcal{K}_{\pm\frac{\sqrt{c}}{2}}(\widetilde{M}, g)$ *denotes the space of* $\pm\frac{\sqrt{c}}{2}$*-Killing spinors on* $(\widetilde{M}, g)$.

*Proof in the case* $c = 0$: Denote by $\widetilde{\nabla}$ the spinorial Levi-Civita connection of $\widetilde{M}$. The Schrödinger-Lichnerowicz formula for the Dirac operator $\widetilde{D}$ of $(\widetilde{M}, g)$ and elementary computations as in Section 1.3 show that, for any $\varphi \in \Gamma(\Sigma\widetilde{M})$,

$$\begin{aligned} |\widetilde{D}\varphi|^2 &= \Re e\left(\langle \widetilde{D}^2\varphi, \varphi\rangle\right) + \mathrm{div}_{\widetilde{M}}(V) \\ &\overset{(1.15)}{=} \Re e\left(\langle \widetilde{\nabla}^*\widetilde{\nabla}\varphi, \varphi\rangle\right) + \frac{\widetilde{S}}{4}|\varphi|^2 + \mathrm{div}_{\widetilde{M}}(V) \\ &= |\widetilde{\nabla}\varphi|^2 + \frac{\widetilde{S}}{4}|\varphi|^2 + \mathrm{div}_{\widetilde{M}}(V+W), \end{aligned} \tag{3.29}$$

where $V$ and $W$ are the vector fields on $\widetilde{M}$ defined by the relations $g(V, X) := \Re e\left(\langle X\cdot\widetilde{D}\varphi, \varphi\rangle\right)$ and $g(W, X) := \Re e\left(\langle \widetilde{\nabla}_X\varphi, \varphi\rangle\right)$ for all $X \in T\widetilde{M}$ respectively (remember that we denote by "$\cdot$" the Clifford multiplication of $\widetilde{M}$ and not that of $M$). Splitting $|\widetilde{\nabla}\varphi|^2$ as in (A.11) one comes to

$$\frac{\widetilde{S}}{4}|\varphi|^2 - \frac{n}{n+1}|\widetilde{D}\varphi|^2 = -|\widetilde{P}\varphi|^2 - \mathrm{div}_{\widetilde{M}}(V+W),$$

where $\widetilde{P}$ is the Penrose operator of $(\widetilde{M}^{n+1}, g)$, see Appendix A. Let $\nu$ be the inner unit normal vector field of $M$ in $\widetilde{M}$. Integrating the last identity and applying Green's formula one obtains

$$\begin{aligned} \int_{\widetilde{M}} \left(\frac{\widetilde{S}}{4}|\varphi|^2 - \frac{n}{n+1}|\widetilde{D}\varphi|^2\right) v_g^{\widetilde{M}} &= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\ &\quad - \int_{\widetilde{M}} \mathrm{div}_{\widetilde{M}}(V+W) v_g^{\widetilde{M}} \\ &= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} - \int_M g(V+W, \nu) v_g \\ &= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \end{aligned}$$

$$
\begin{aligned}
&- \int_M \Re e\left(\langle \nu\cdot\widetilde{D}\varphi,\varphi\rangle + \langle\widetilde{\nabla}_\nu\varphi,\varphi\rangle\right) v_g \\
\overset{(1.22)}{=} \quad & - \int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\
& + \int_M \Re e\left(\langle D_2\varphi,\varphi\rangle - \frac{nH}{2}|\varphi|^2\right) v_g.
\end{aligned}
$$

Since $D_2$ is formally self-adjoint (Proposition 1.3.4) one comes to

$$
\begin{aligned}
\int_{\widetilde{M}} \left(\frac{\widetilde{S}}{4}|\varphi|^2 - \frac{n}{n+1}|\widetilde{D}\varphi|^2\right) v_g^{\widetilde{M}} &= - \int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\
&\quad + \int_M \left(\langle D_2\varphi,\varphi\rangle - \frac{nH}{2}|\varphi|^2\right) v_g. \qquad (3.30)
\end{aligned}
$$

Let $\lambda$ be an eigenvalue of $D$. The spectrum of $D_2$ being the symmetrized of that of $D$ w.r.t. the origin (for $n$ even it follows from (1.10) that the spectrum of $D$ is already symmetric w.r.t. the origin) there always exists a non-zero eigenvector $\psi$ for $D_2$ associated to the eigenvalue $|\lambda|$. The crucial point is now the existence of a smooth solution $\phi$ to the boundary value problem with APS-boundary condition

$$
\left|\begin{array}{lll} \widetilde{D}\phi & = 0 & \text{on } \widetilde{M} \\ \pi_{\geq 0}\phi & = \psi & \text{on } M, \end{array}\right. \qquad (3.31)
$$

where $\pi_{\geq 0} : \Gamma(\Sigma) \to \Gamma(\Sigma)$ denotes the $\mathrm{L}^2$-orthogonal projection onto the eigenspaces of $D_2$ to nonnegative eigenvalues, see Section 1.5 and Chapter 4. Since $\widetilde{S} \geq 0$ and $H \geq 0$ the identity (3.30) with $\varphi := \phi$ implies

$$
\begin{aligned}
0 \leq \int_{\widetilde{M}} \frac{\widetilde{S}}{4}|\phi|^2 v_g^{\widetilde{M}} &= \int_{\widetilde{M}} \left(\frac{\widetilde{S}}{4}|\phi|^2 - \frac{n}{n+1}|\widetilde{D}\phi|^2\right) v_g^{\widetilde{M}} \\
&\leq \int_M \left(\langle D_2\phi,\phi\rangle - \frac{nH}{2}|\phi|^2\right) v_g \\
&\leq \int_M \left(\langle D_2\pi_{\geq 0}\phi,\pi_{\geq 0}\phi\rangle - \frac{nH}{2}|\pi_{\geq 0}\phi|^2\right) v_g \\
&= \int_M (|\lambda| - \frac{nH}{2})|\psi|^2 v_g
\end{aligned}
$$

from which the inequality follows.
In case the lower bound is attained the mean curvature $H$ of $M$ must be constant, $\phi = \pi_{\geq 0}\phi$ on $M$ and $\widetilde{P}\phi = 0$ on $\widetilde{M}$, where $\phi$ is any section of $\Sigma\widetilde{M}$ solving (3.31) for any given eigenvector $\psi$ of $D_2$ associated to the eigenvalue $|\lambda|$. In particular $\psi = \phi_{|_M}$ with $\widetilde{\nabla}\phi = 0$ on $\widetilde{M}$, i.e., every eigenvector $\psi$ of $D_2$ associated to the eigenvalue $|\lambda|$ must be the restriction on $M$ of parallel spinor on $\widetilde{M}$ (note that the existence of a non-zero parallel spinor on $\widetilde{M}$ implies $\widetilde{S} = 0$, see Proposition A.4.1). Moreover (1.22) already implies that the restriction of any parallel spinor onto a hypersurface with constant mean curvature is an eigenvector of $D$ to the eigenvalue $\frac{nH}{2}$. Therefore the eigenspace of $D_2$ associated to the eigenvalue $\frac{nH}{2}$ exactly coincides with $(\mathcal{K}_0)_{|_M}$. The other implication is trivial.

For the proof in the case $c < 0$, which is based on the same argument for a Schrödinger operator associated to $D$, we refer to [144]. □

In case $\widetilde{M} \subset \widetilde{M}^{n+1}(c)$, where $\widetilde{M}^{n+1}(c)$ is a spaceform with constant curvature $c \leq 0$, Gauss' equations imply in particular $(\frac{n}{2})^2(H^2 + c) \geq \frac{n}{4(n-1)}S$, hence (3.28) improves (3.1) under the supplementary assumption $H \geq \sqrt{-c}$.

There exists a conformal version of (3.28) in terms of the so-called Yamabe relative invariant, see [148].

The characterization of the equality case in (3.28) provides a short proof of Alexandrov's theorem (see reference in [146]) on constant mean curvature embedded hypersurfaces in the Euclidean and hyperbolic spaces respectively [146, Thm. 8]:

**Theorem 3.7.2 (A.D. Alexandrov)** *Every closed embedded hypersurface with constant mean curvature in $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ is a round geodesic hypersphere.*

*Proof in the case* $c = 0$: Let $M$ be such a hypersurface. It is embedded so that on the one hand it bounds a compact domain $\widetilde{M}$ (in particular it is orientable hence spin, see Proposition 1.4.1); on the other hand, it can be shown that necessarily $H \geq 0$ by a result of S. Montiel and A. Ros (see reference in [146]). Moreover the assumption $H$ constant implies that (3.28) is an equality, in which case every non-zero eigenvector of $D_2$ associated to the eigenvalue $\frac{nH}{2}$ must be the restriction onto $M$ of a parallel spinor on $\widetilde{M}$ (Theorem 3.7.1). But considering the spinor field

$$x \longmapsto \varphi_x := \nu_x \cdot \phi + Hx \cdot \phi,$$

on $\widetilde{M}$, where $\nu$ is the inner unit normal and $\phi$ a parallel spinor on $\widetilde{M} \subset \mathbb{R}^{n+1}$, one notices that

$$\begin{aligned} D_2\varphi &= D_2(\nu \cdot \phi) + HD_2(x \cdot \phi) \\ &\overset{(1.19)}{=} -\nu \cdot D_2\phi + HD_2(x \cdot \phi) \\ &\overset{(1.22)}{=} -\nu \cdot D_2\phi + H(\frac{nH}{2}x \cdot \phi - \widetilde{\nabla}_\nu(x \cdot \phi) - \nu \cdot \widetilde{D}(x \cdot \phi)) \\ &= -\frac{nH}{2}\nu \cdot \phi + H(\frac{nH}{2}x \cdot \phi - \nu \cdot \phi + (n+1)\nu \cdot \phi) \\ &= \frac{nH}{2}(\nu \cdot \phi + Hx \cdot \phi) \\ &= \frac{nH}{2}\varphi, \end{aligned}$$

i.e., $\varphi$ is an eigenvector for $D_2$ associated to the eigenvalue $\frac{nH}{2}$. Therefore $\varphi$ must be either identically zero or non-zero and parallel. The first possibility already implies that $M$ must be a geodesic sphere. The second one means that, for every $X \in TM$,

$$0 = \widetilde{\nabla}_X\varphi = -A(X) \cdot \phi + HX \cdot \phi.$$

Since $\phi$ has no zero on $\widetilde{M}$ one deduces that $A = H\mathrm{Id}_{TM}$, i.e., that $M$ must be totally umbilical in $\widetilde{M}$ hence in $\mathbb{R}^{n+1}$. This concludes the proof for $c = 0$. For

$c < 0$ we refer to [144]. □

Another clever application of Theorem 3.7.1 is:

**Theorem 3.7.3 (O. Hijazi and S. Montiel [142])** *Let $(\widetilde{M}^{n+1}, g)$ be a complete Riemannian spin manifold with nonnegative Ricci curvature, mean convex boundary $\partial\widetilde{M}$ and nonnegative Einstein-tensor along the normal direction of $\partial\widetilde{M}$. Then $(\widetilde{M}^{n+1}, g)$ is isometric to a Euclidean ball.*

As a corollary, any Ricci-flat complete spin manifold with boundary isometric to the round sphere $\mathbb{S}^n$ is already isometric to a Euclidean ball. Recently rigidity results have been obtained by S. Raulot [219] under weaker assumptions on the boundary. We also mention that it remains open whether analogous estimates on the boundary of positively-curved domains can be obtained.

# Chapter 4

# Lower eigenvalue estimates on compact manifolds with boundary

The study of the Dirac operator on compact spin manifolds $M$ with boundary was initiated by Atiyah, Patodi and Singer [30] in the search for index theorems on such manifolds, see e.g. [62] which is a comprehensive reference on the subject. In order to be able to talk about eigenvalues of the Dirac operator of $M$ in this context, elliptic boundary conditions have to be introduced as we have seen in Section 1.5. Following [85] a whole bunch of spectral properties of the Dirac operator have recently been proved using varied boundary conditions, leading sometimes to very beautiful geometric results, such as those already presented in Section 3.7. In this chapter we mainly show that, under any of the four boundary conditions introduced in Section 1.5, some kind of Friedrich's inequality (3.1) holds on $M$ although the lower bound is not always attained according to the boundary condition chosen. For readers interested in more details and references we suggest [143].

## 4.1 Case of the gAPS boundary condition

Remember that the generalized Atiyah-Patodi-Singer (gAPS) boundary condition is defined by $B_{\mathrm{gAPS}} := \pi_{\geq\beta}$, i.e., it is the $\mathrm{L}^2$-orthogonal projection onto the (Hilbert) direct sum of the eigenspaces of the boundary Dirac operator $D$ to the eigenvalues not smaller than a fixed $\beta \leq 0$. The following theorem is a particular case of [78, Thm. 3.1] which is itself a generalization of [147, Thm. 4] (see also [201, Thm. 16.5]).

**Theorem 4.1.1** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional compact Riemannian spin manifold with non-empty boundary $\partial M$. Assume that $\partial M$ has* non-negative mean curvature *w.r.t. the inner normal. Then any eigenvalue $\lambda$ of the fundamental Dirac operator $D$ of $(M^n, g)$ under the* gAPS *boundary condition satisfies*

$$\lambda^2 > \frac{n}{4(n-1)} \inf_M(S),$$

*where $S$ is the scalar curvature of $(M^n, g)$.*

*Proof*: There exists a non-zero smooth solution $\varphi \in \Gamma(\Sigma M)$ to the boundary value problem

$$\left| \begin{array}{lll} D\varphi & = \lambda\varphi & \text{on } M \\ \pi_{\geq\beta}(\varphi_{|\partial M}) & = 0 & \text{on } \partial M. \end{array} \right.$$

Since $\pi_{\geq\beta}(\varphi_{|\partial M}) = 0$ and $\varphi_{|\partial M} \neq 0$ (otherwise a unique continuation property for the Dirac operator $D$ [62, Sec. 1.8] would give $\varphi = 0$ on $M$) one has $\int_{\partial M} \langle D_2\varphi, \varphi\rangle v_g^{\partial M} < \beta \int_{\partial M} |\varphi|^2 v_g^{\partial M} \leq 0$, so that (3.30) with $H \geq 0$ (beware the different notations, in particular for the boundary Dirac operator $D_2$) implies

$$\int_M \left(\frac{S}{4}|\varphi|^2 - \frac{n-1}{n}|D\varphi|^2\right) v_g < 0,$$

which straightforward implies the inequality. In particular it cannot be sharp. □

## 4.2 Case of the CHI boundary condition

The chirality (CHI) boundary condition is defined by $B_{\mathrm{CHI}} := \frac{1}{2}(\mathrm{Id} - \nu \cdot \mathcal{G})$, where $\nu$ is the inner unit normal and $\mathcal{G}$ is an endomorphism-field of $\Sigma M$ (whose restriction on $\partial M$ is also denoted by $\mathcal{G}$) which is involutive, unitary, parallel and anti-commuting with the Clifford multiplication on $M$.

**Theorem 4.2.1 (O. Hijazi, S. Montiel and A. Roldán [143])**
*Let $(M^n, g)$ be an $n(\geq 2)$-dimensional compact Riemannian spin manifold with non-empty boundary $\partial M$. Assume that $\partial M$ has* non-negative mean curvature *w.r.t. the inner normal. Then any eigenvalue $\lambda$ of the fundamental Dirac operator $D$ of $(M^n, g)$ under the* CHI *boundary condition satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M(S), \tag{4.1}$$

*where $S$ is the scalar curvature of $(M^n, g)$. Moreover, (4.1) is an equality if and only if $(M^n, g)$ is isometric to the half sphere with radius $\frac{n}{2|\lambda|}$.*

*Proof*: There exists a non-zero smooth solution $\varphi \in \Gamma(\Sigma M)$ to the boundary value problem

$$\left| \begin{array}{lll} D\varphi & = \lambda\varphi & \text{on } M \\ (\mathrm{Id} - \nu \cdot \mathcal{G})(\varphi_{|\partial M}) & = 0 & \text{on } \partial M. \end{array} \right.$$

From (1.22) and the definition of $\mathcal{G}$ it can be easily proved that $D_2\mathcal{G} = \mathcal{G}D_2$, so that, if $\psi := \varphi_{|\partial M}$ then

$$\begin{aligned} \int_{\partial M} \langle D_2\psi, \psi\rangle v_g^{\partial M} &= \int_{\partial M} \langle D_2(\nu \cdot \mathcal{G}\psi), \psi\rangle v_g^{\partial M} \\ &\overset{(1.23)}{=} -\int_{\partial M} \langle \nu \cdot D_2(\mathcal{G}\psi), \psi\rangle v_g^{\partial M} \\ &= -\int_{\partial M} \langle \nu \cdot \mathcal{G}(D_2\psi), \psi\rangle v_g^{\partial M} \\ &= -\int_{\partial M} \langle D_2\psi, \nu \cdot \mathcal{G}\psi\rangle v_g^{\partial M} \\ &= -\int_{\partial M} \langle D_2\psi, \psi\rangle v_g^{\partial M}, \end{aligned}$$

that is, $\int_{\partial M}\langle D_2\psi,\psi\rangle v_g^{\partial M} = 0$. Formula (3.30) together with the assumption $H \geq 0$ imply

$$\int_M \left(\frac{S}{4}|\varphi|^2 - \frac{n-1}{n}|D\varphi|^2\right) v_g \leq 0,$$

which leads to (4.1).

If inequality (4.1) is an equality, then (3.30) implies on the one hand that $\varphi$ must be a Killing spinor to the real Killing constant $-\frac{\lambda}{n}$ (it is an eigenvector of $D$ lying in the kernel of the Penrose operator $P$ of $(M^n, g)$, see Appendix A) and on the other hand $H = 0$, i.e., the boundary $\partial M$ must be minimal in $M$. Moreover, $f := \langle \mathcal{G}\varphi, \varphi\rangle$ defines a smooth real function on $M$ whose differential is given on any $X \in M$ by

$$\begin{aligned} X(f) &= \langle \nabla_X(\mathcal{G}\varphi), \varphi\rangle + \langle \mathcal{G}\varphi, \nabla_X\varphi\rangle \\ &= \langle \mathcal{G}(-\frac{\lambda}{n}X\cdot\varphi), \varphi\rangle - \frac{\lambda}{n}\langle \mathcal{G}\varphi, X\cdot\varphi\rangle \\ &= -\frac{\lambda}{n}(\langle \mathcal{G}\varphi, X\cdot\varphi\rangle + \langle X\cdot\varphi, \mathcal{G}\varphi\rangle) \\ &= -\frac{2\lambda}{n}\Re e\left(\langle \mathcal{G}\varphi, X\cdot\varphi\rangle\right). \end{aligned}$$

Hence the Hessian of $f$ evaluated on any $X, Y \in TM$ is given by

$$\begin{aligned} \mathrm{Hess}(f)(X,Y) &= -\frac{2\lambda}{n}\Re e\left(\langle \nabla_X(\mathcal{G}\varphi), Y\cdot\varphi\rangle + \langle \mathcal{G}\varphi, Y\cdot\nabla_X\varphi\rangle\right) \\ &= \frac{2\lambda^2}{n^2}\Re e\left(\langle \mathcal{G}(X\cdot\varphi), Y\cdot\varphi\rangle + \langle \mathcal{G}\varphi, Y\cdot X\cdot\varphi\rangle\right) \\ &= \frac{2\lambda^2}{n^2}\Re e\left(\langle \mathcal{G}\varphi, X\cdot Y\cdot\varphi\rangle + \langle \mathcal{G}\varphi, Y\cdot X\cdot\varphi\rangle\right) \\ &= -\frac{4\lambda^2}{n^2}\langle \mathcal{G}\varphi, \varphi\rangle g(X,Y), \end{aligned}$$

i.e., $\mathrm{Hess}(f) = -\frac{4\lambda^2}{n^2}fg$. On the other hand neither $\lambda$ nor the function $f$ vanish, since (1.26) implies

$$\int_M \langle D\varphi, \mathcal{G}\varphi\rangle - \langle \varphi, D(\mathcal{G}\varphi)\rangle v_g = \int_{\partial M}\langle \psi, \nu\cdot\mathcal{G}\psi\rangle v_g^{\partial M},$$

i.e., $2\lambda f = \int_{\partial M}|\psi|^2 v_g^{\partial M}$, which does not vanish because of the unique continuation property mentioned above. Hence Reilly's characterization of the hemisphere (see reference in [143]) implies that $(M^n, g)$ is isometric to a hemisphere with radius $\frac{n}{2|\lambda|}$. Since the standard sphere for its canonical spin structure and metric with constant sectional curvature 1 carries non-zero $-\frac{1}{2}$ and $\frac{1}{2}$-Killing spinors (see e.g. Examples A.1.3.2) the other implication is trivial. □

In the context of manifolds with non-empty boundary the conformal covariance of the fundamental Dirac operator also improves the Friedrich-type lower bound (4.1). For the CHI boundary condition S. Raulot proved in [217, 218] a Hijazi-type inequality (3.18), which in dimension 2 is equivalent to a Bär-type inequality (3.17) and where the lower bound is the smallest eigenvalue of the Yamabe operator under a suitable boundary condition. Moreover, the lower bound is attained exactly for the round hemispheres in $\mathbb{R}^{n+1}$.

## 4.3 Case of the MIT bag boundary condition

Remember that the "MIT bag" (denoted by MIT) boundary condition is by definition the endomorphism-field of $\Sigma$ given by $B_{\mathrm{MIT}} := \frac{1}{2}(\mathrm{Id} - i\nu\cdot)$, where $\nu$ is the inner unit normal of the boundary.

**Theorem 4.3.1 (O. Hijazi, S. Montiel and A. Roldán [143])**
*Let $(M^n, g)$ be an $n(\geq 2)$-dimensional compact Riemannian spin manifold with non-empty boundary $\partial M$. Assume that $\partial M$ has* non-negative mean curvature *w.r.t. the inner normal. Then any eigenvalue $\lambda$ of the fundamental Dirac operator $D$ of $(M^n, g)$ under the* MIT bag *boundary condition satisfies*

$$|\lambda|^2 > \frac{n}{4(n-1)} \inf_M(S), \tag{4.2}$$

*where $S$ is the scalar curvature of $(M^n, g)$.*

*Proof*: There exists a non-zero smooth solution $\varphi \in \Gamma(\Sigma M)$ to the boundary value problem

$$\left| \begin{array}{lll} D\varphi & = \lambda\varphi & \text{on } M \\ (\mathrm{Id} - i\nu\cdot)(\varphi_{|_{\partial M}}) & = 0 & \text{on } \partial M. \end{array} \right.$$

Defining $\psi := \varphi_{|_{\partial M}}$, one has

$$\begin{aligned} \int_{\partial M} \langle D_2\psi, \psi\rangle v_g^{\partial M} &= \int_{\partial M} \langle D_2(i\nu\cdot\psi), \psi\rangle v_g^{\partial M} \\ &\overset{(1.23)}{=} -\int_{\partial M} \langle i\nu\cdot D_2(\psi), \psi\rangle v_g^{\partial M} \\ &= -\int_{\partial M} \langle D_2(\psi), i\nu\cdot\psi\rangle v_g^{\partial M} \\ &= -\int_{\partial M} \langle D_2\psi, \psi\rangle v_g^{\partial M}, \end{aligned}$$

that is, $\int_{\partial M} \langle D_2\psi, \psi\rangle v_g^{\partial M} = 0$. Formula (3.30) together with the assumption $H \geq 0$ imply

$$\int_M \left( \frac{S}{4}|\varphi|^2 - \frac{n-1}{n}|D\varphi|^2 \right) v_g \leq 0,$$

which leads to (4.2).
If (4.2) is an equality, then again $\varphi$ must be a $-\frac{\lambda}{n}$-Killing spinor on $(M^n, g)$. Since in that case $\Im m(\lambda) > 0$ (see Section 1.5), one deduces from Proposition A.4.1 that $\lambda \in i\mathbb{R}_+^*$ and hence $S = 4\frac{n-1}{n}\lambda^2 < 0$, contradiction. Therefore (4.2) is always a strict inequality. $\square$

Assuming the stronger condition $H > 0$ on $\partial M$, Theorem 4.3.1 can be improved. More precisely, using a suitable modified connection (or Penrose-like operator) S. Raulot showed the following:

**Theorem 4.3.2 (S. Raulot [216, 218])** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional compact Riemannian spin manifold with non-empty boundary $\partial M$. Assume* *that $\partial M$ has* positive mean curvature *w.r.t. the inner normal. Then any eigenvalue $\lambda$ of the fundamental Dirac operator $D$ of $(M^n, g)$ under the* MIT bag *boundary condition satisfies*

$$|\lambda|^2 \geq \frac{n}{4(n-1)} \inf_M(S) + n\Im m(\lambda) \inf_{\partial M}(H). \tag{4.3}$$

*Moreover,* (4.3) *is an equality if and only if the mean curvature of the boundary is constant and $(M^n, g)$ admits a non-zero imaginary Killing spinor.*

Besides, as for the CHI boundary condition, there exists a Hijazi-type conformal lower bound for the Dirac spectrum under MIT bag boundary condition, which was proved by S. Raulot in [217, 218] but which is never sharp.

## 4.4 Case of the mgAPS boundary condition

The modified generalized Atiyah Patodi Singer (mgAPS) boundary condition is defined by $B_{\mathrm{mgAPS}} := B_{\mathrm{gAPS}}(\mathrm{Id} + \nu\cdot)$, where $B_{\mathrm{gAPS}} = \pi_{\geq\beta}$ is the generalized Atiyah Patodi Singer boundary condition and $\nu$ is the inner unit normal to the boundary. It depends in particular on a parameter $\beta \leq 0$. The following theorem is a particular case of [78, Thm. 3.3] which itself generalizes [143, Thm. 5].

**Theorem 4.4.1** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional compact Riemannian spin manifold with non-empty boundary $\partial M$. Assume that $\partial M$ has* non-negative mean curvature *w.r.t. the inner normal. Then any eigenvalue $\lambda$ of the fundamental Dirac operator $D$ of $(M^n, g)$ under the* mgAPS *boundary condition to $\beta$ satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M(S), \tag{4.4}$$

*where $S$ is the scalar curvature of $(M^n, g)$. Furthermore* (4.4) *is an equality if and only if $(M^n, g)$ carries a non-zero $\alpha$-Killing spinor for real $\alpha < \frac{\beta}{n-1}$ and $\partial M$ is minimal in $M$.*

*Proof*: There exists a non-zero smooth solution $\varphi \in \Gamma(\Sigma M)$ to the boundary value problem

$$\left| \begin{array}{lll} D\varphi & = \lambda\varphi & \text{on } M \\ \pi_{\geq\beta}(\mathrm{Id} + \nu\cdot)(\varphi_{|\partial M}) & = 0 & \text{on } \partial M. \end{array} \right. \tag{4.5}$$

Again if we prove that $\int_{\partial M} \langle D_2\psi, \psi\rangle v_g^{\partial M} = 0$, where $\psi := \varphi_{|\partial M}$, then (3.30) together with the assumption $H \geq 0$ will provide the inequality. Denoting by $(\cdot\,,\cdot)_{\partial M} := \int_{\partial M} \langle\cdot\,,\cdot\rangle v_g^{\partial M}$ one has

$$\begin{aligned} 2(D_2\psi, \psi)_{\partial M} &= \Big(\{\mathrm{Id} + \nu\cdot\}D_2\psi, \{\mathrm{Id} + \nu\cdot\}\psi\Big)_{\partial M} \\ &= \Big(\pi_{<\beta}(\{\mathrm{Id} + \nu\cdot\}D_2\psi), \pi_{<\beta}(\{\mathrm{Id} + \nu\cdot\}\psi)\Big)_{\partial M} \\ &\overset{(1.23)}{=} \Big(D_2(\pi_{<\beta}(\{\mathrm{Id} - \nu\cdot\}\psi)), \pi_{<\beta}(\{\mathrm{Id} + \nu\cdot\}\psi)\Big)_{\partial M}, \end{aligned}$$

where we used the notations of Section 1.5. But $\pi_{\geq\beta}(\psi + \nu\cdot\psi) = 0$ implies

$$\begin{aligned}\pi_{<\beta}(\psi - \nu\cdot\psi) &\overset{(1.23)}{=} \pi_{<\beta}\psi - \nu\cdot\pi_{>-\beta}\psi \\ &= \pi_{\leq-\beta}\psi - \pi_{[\beta,-\beta]}\psi - \nu\cdot\pi_{>-\beta}\psi \\ &= \nu\cdot\pi_{\geq\beta}\psi - \pi_{[\beta,-\beta]}\psi - \nu\cdot\pi_{>-\beta}\psi \\ &= \nu\cdot\pi_{[\beta,-\beta]}\psi - \pi_{[\beta,-\beta]}\psi,\end{aligned}$$

in particular $\pi_{<\beta}(\psi - \nu\cdot\psi) = 0$, which implies $(D_2\psi, \psi)_{\partial M} = 0$ and (4.4).
If (4.4) is an equality, then $H = 0$ and $\varphi$ must be a $\alpha$-Killing spinor on $(M^n, g)$ with $\alpha := -\frac{\lambda}{n} \in \mathbb{R}$. Moreover (1.22) with $H = 0$ and for the restriction $\psi$ on $\partial M$ of an $\alpha$-Killing spinor $\varphi$ on $M$ implies

$$D_2(\psi + \nu\cdot\psi) = (n-1)\alpha(\psi + \nu\cdot\psi). \tag{4.6}$$

Since $\psi + \nu\cdot\psi \neq 0$ (the eigenvalues of the pointwise Clifford action by $\nu$ are $\pm i$), the mgAPS boundary condition then requires $(n-1)\alpha < \beta$. Conversely, if $(M^n, g)$ has minimal boundary and carries a non-zero $\alpha$-Killing spinor $\varphi$ with real $\alpha < \frac{\beta}{n-1}$, then on the one hand Proposition A.4.1 implies that $(M^n, g)$ is Einstein with scalar curvature $4n(n-1)\alpha^2 > 0$, on the other hand $D\varphi = -n\alpha\varphi$ on $M$. Furthermore the identity (4.6) holds on $\partial M$ so that $\varphi$ solves (4.5), hence $-n\alpha$ is an eigenvalue of $D$ w.r.t. the mgAPS boundary condition. This shows the characterization of the limiting-case of (4.4) and concludes the proof. $\square$

# Chapter 5

# Upper eigenvalue bounds on closed manifolds

In this chapter we turn to the very different game of looking for upper eigenvalue bounds for the fundamental Dirac operator. We concentrate on two methods for obtaining them. The first method, due to C. Vafa and E. Witten [236], consists in comparing $D$ to another Dirac-type operator $\mathcal{D}$ of which kernel is non trivial for index-theoretical reasons, then in estimating the zero-order difference $D - \mathcal{D}$ by geometric quantities. As a result, those bound from above a topologically determined number of eigenvalues of $D$. This method was applied to prove Theorems 5.1.1, 5.2.1 and 5.2.2. The second method relies on the min-max principle, which is a general variational principle characterizing eigenvalues of self-adjoint elliptic operators, see e.g. [77, pp.16-17]:

**Lemma 5.0.2 (min-max principle)** *Let $(M^n, g)$ be a closed Riemannian spin manifold. Order the eigenvalues of $D^2$ (which are exactly the squares of the eigenvalues of $D$) into a nondecreasing sequence $\lambda_1(D^2) \leq \ldots \leq \lambda_k(D^2) \leq \lambda_{k+1}(D^2) \leq \ldots$. Then, for every $k \geq 1$,*

$$\lambda_k(D^2) = \min_{E_k} \Big\{ \max_{\substack{\varphi \in E_k \\ \varphi \neq 0}} \frac{\int_M \langle D^2\varphi, \varphi\rangle v_g}{\int_M |\varphi|^2 v_g} \Big\},$$

*where the minimum runs over all $k$-dimensional vector subspaces $E_k$ of $\Gamma(\Sigma M)$. In particular one has, for every non-zero $\varphi \in \Gamma(\Sigma M)$,*

$$\lambda_1(D^2) \leq \frac{\int_M \langle D^2\varphi, \varphi\rangle v_g}{\int_M |\varphi|^2 v_g}, \qquad (5.1)$$

*with equality if and only if $D^2\varphi = \lambda_1(D^2)\varphi$.*

Applying this method means choosing $\varphi$ such that the r.h.s. of (5.1) can be computed in terms of geometric quantities. The quotient $\frac{\int_M \langle D^2\varphi, \varphi\rangle v_g}{\int_M |\varphi|^2 v_g}$ is called the Rayleigh quotient of $D^2$ evaluated at $\varphi$.

## 5.1 Intrinsic upper bounds

The following theorem, which can be formulated under weaker assumptions (see [54, Prop. 1]), is an application of Vafa-Witten's method. The upper bound which is derived only depends on the sectional curvature of $M$.

**Theorem 5.1.1 (H. Baum [54])** *Let* $(M^n, g)$ *be* even-dimensional*, closed* with positive sectional curvature $K^M$*. Then the first eigenvalue* $\lambda_1$ *of* $D$ *satisfies*

$$|\lambda_1| \leq 2^{\frac{n}{2}-1}\sqrt{\frac{n}{2}} \cdot \sqrt{\max_M(K^M)}.$$

The inequality in Theorem 5.1.1 is sharp for $M = \mathbb{S}^2$, however it is not clear if it can be sharp for any other manifold. If $n$ is odd one can apply Theorem 5.1.1 to the product of the manifold with a circle. In that case H. Baum proved in [54, Cor. 2] an analogous estimate, however under the supplementary assumptions that $K^M$ is pinched and $M$ is simply-connected. Moreover Theorem 5.1.1 improves a qualitative result by J. Lott [187, Prop. 4] valid on $\mathbb{S}^2$.

*Sketch of proof of* Theorem 5.1.1: Construct a suitable map $\iota$ of degree 1 from $M$ into a round sphere of suitable radius and such that the derivative of this map does not deviate too far from the identity. This goes as follows: fix a point $p \in M$ and take for $\iota$ the composition of the exponential map of the ($n$-dimensional) round sphere of radius $\frac{1}{\sqrt{K^M(p)}}$ with the inverse of the exponential map of $M$ at $p$. Of course the exponential map of $M$ at $p$ is only invertible on its injectivity domain; furthermore one has to control the behaviour of the exponential far from $p$ by introducing a smoothing function in the definition of $\iota$. Actually that smoothing function may be chosen such that $\iota$ can be extended by a constant to a smooth function on the whole of $M$. Then the norm of the derivative of $\iota$ can be estimated against $\sqrt{K^M}$, which together with Theorem 5.2.1 below leads to the result. □

Another way to obtain upper bounds consists in comparing the Dirac spectra for different metrics and applying the min-max principle. The remarkable property of conformal covariance of the Dirac operator allows this method to work. The first result is this direction is due to J. Lott [187, Prop. 3].

**Theorem 5.1.2 (J. Lott [187])** *Let* $(M^n, g)$ *be an* $n(\geq 2)$*-dimensional closed spin manifold. Then for any conformal class of Riemannian metrics* $[g]$ *on* $M^n$*, there exists a finite positive constant* $b([g])$ *such that*

$$\lambda_1(D_g^2) \leq b([g]) \sup_M(-S_g) \tag{5.2}$$

*for any* $g \in [g]$ *with scalar curvature* $S_g < 0$*.*

*Proof*: Fix $g_0 \in [g]$ with $S_{g_0} < 0$ on $M^n$. For some $u \in C^\infty(M^n, \mathbb{R})$, let $g = e^{2u}g_0 \in [g]$ with $S_g < 0$. As in Proposition 1.3.10, we denote by $\varphi \mapsto \overline{\varphi}$ the unitary isomorphism $\Sigma_{g_0}M \longrightarrow \Sigma_g M$. Choose $\psi_0$ to be a non-zero eigenvector for $D_{g_0}$ associated to the eigenvalue $\lambda_1(D_{g_0})$ and set $\psi := e^{-\frac{n-1}{2}u}\overline{\psi_0}$. The min-max principle together with (1.16) imply

$$\begin{aligned}\lambda_1(D_g^2) &\leq \frac{\int_M |D_g\psi|^2 v_g}{\int_M |\psi|^2 v_g}\\ &= \frac{\int_M e^{-(n+1)u}|D_{g_0}\psi_0|^2 e^{nu} v_{g_0}}{\int_M e^{-(n-1)u}|\psi_0|^2 e^{nu} v_{g_0}}\\ &\leq \lambda_1(D_{g_0}^2)\cdot\frac{\sup_M(|\psi_0|^2)}{\inf_M(|\psi_0|^2)}\cdot\frac{\int_M e^{-u}v_{g_0}}{\int_M e^u v_{g_0}}. \qquad (5.3)\end{aligned}$$

We now estimate the quotient of integrals on the r.h.s. with the help of (3.19). Namely (3.19) reads

$$S_g e^u = S_{g_0} e^{-u} + 2(n-1)e^{-u}\Delta_{g_0}u - (n-1)(n-2)e^{-u}|\mathrm{grad}_{g_0}(u)|^2_{g_0},$$

so that integrating one obtains

$$\begin{aligned}\int_M S_{g_0}e^{-u}v_{g_0} &= \int_M S_g e^u v_{g_0}\\ &\quad +(n-1)\int_M -2e^{-u}\Delta_{g_0}u + (n-2)e^{-u}|\mathrm{grad}_{g_0}(u)|^2_{g_0}v_{g_0}\\ &= \int_M S_g e^u v_{g_0} + n(n-1)\int_M e^{-u}|\mathrm{grad}_{g_0}(u)|^2_{g_0}v_{g_0},\end{aligned}$$

where we have used $\Delta_{g_0}(e^{-u}) = -e^{-u}\Delta_{g_0}u - e^{-u}|\mathrm{grad}_{g_0}(u)|^2_{g_0}$. We deduce that

$$\begin{aligned}\frac{\int_M e^{-u}v_{g_0}}{\int_M e^u v_{g_0}} &\leq \frac{\sup_M(-S_g)}{\inf_M(-S_{g_0})}\cdot\frac{\int_M -S_{g_0}e^{-u}v_{g_0}}{\int_M -S_g e^u v_{g_0}}\\ &= \frac{\sup_M(-S_g)}{\inf_M(-S_{g_0})}\Big(1 + n(n-1)\frac{\int_M e^{-u}|\mathrm{grad}_{g_0}(u)|^2_{g_0}v_{g_0}}{\int_M S_g e^u v_{g_0}}\Big)\\ &\leq \frac{\sup_M(-S_g)}{\inf_M(-S_{g_0})},\end{aligned}$$

which together with (5.3) gives the result. □

On surfaces, Lott's estimate (5.2) only applies if the genus is at least 2. For genus 0 or 1 the use of (3.19) can be avoided through the fact that eigenvectors associated to the lowest Dirac eigenvalue have constant length:

**Theorem 5.1.3 (I. Agricola and T. Friedrich [2])** *Let $M^2 := \mathbb{S}^2$ or $\mathbb{T}^2$ with arbitrary Riemannian metric $g$. Let $u \in C^\infty(M^2, \mathbb{R})$ be such that $g_0 := e^{-2u}g$ has constant curvature.*

*i) The smallest eigenvalue of $D_g^2$ satisfies*

$$\lambda_1(D_g^2)\mathrm{Area}(M^2, g) \leq \lambda_1(D_{g_0}^2)\mathrm{Area}(M^2, g_0) + \frac{1}{4}\int_{M^2}|\mathrm{grad}_{g_0}(u)|^2_{g_0}v_{g_0}. \qquad (5.4)$$

*ii) The smallest eigenvalue of $D_g^2$ satisfies*

$$\lambda_1(D_g^2) \leq \lambda_1(D_{g_0}^2)\frac{\int_{M^2} e^{-u}v_{g_0}}{\int_{M^2} e^u v_{g_0}}. \qquad (5.5)$$

*iii) The smallest positive eigenvalue of $D^2_{\mathbb{T}^2,g}$ on $\mathbb{T}^2$ endowed with trivial spin structure satisfies*

$$\lambda_2(D^2_{\mathbb{T}^2,g}) \leq \frac{\int_{\mathbb{T}^2} e^{-3u}\{\lambda_2(D^2_{\mathbb{T}^2,g_0}) + |\mathrm{grad}_{g_0}(u)|^2_{g_0}\}v_{g_0}}{\int_{\mathbb{T}^2} e^{-u}v_{g_0}}. \tag{5.6}$$

*Proof*: Let $\psi_0$ be a non-zero eigenvector for $D_{M,g_0}$ associated to the eigenvalue $\lambda_1(D_{M,g_0})$ in *i)*, *ii)* and to the eigenvalue $\lambda_2(D_{M,g_0})$ in *iii)* respectively. If $M = \mathbb{S}^2$ then the spinor $\psi_0$ is a real Killing spinor, hence has constant length on $\mathbb{S}^2$. If $M = \mathbb{T}^2$, the formula (2.1) implies that all eigenvectors of $D$ have constant length on $\mathbb{T}^2$. Therefore we may assume that $|\psi_0| = 1$ on $M^2$. As in Proposition 1.3.10, we denote by $\varphi \mapsto \overline{\varphi}$ the unitary isomorphism $\Sigma_{g_0}M \longrightarrow \Sigma_g M$.
*i)* Set $\psi := \overline{\psi_0}$. Identity (1.16) gives

$$\begin{aligned} D_{M,g}\psi &= e^{-u}(\overline{D_{M,g_0}\psi_0} + \frac{1}{2}\overline{\mathrm{grad}_{g_0}(u)\cdot\psi_0}) \\ &= e^{-u}(\lambda_1(D_{M,g_0})\psi + \frac{1}{2}\overline{\mathrm{grad}_{g_0}(u)\cdot\psi_0}), \end{aligned}$$

where we have denoted by "$\cdot$" the Clifford multiplication on $\Sigma_{g_0}M$. We deduce that

$$\begin{aligned} |D_{M,g}\psi|^2 &= e^{-2u}\Big(\lambda_1(D^2_{M,g_0})|\psi_0|^2 + \frac{1}{4}|\mathrm{grad}_{g_0}(u)\cdot\psi_0|^2 \\ &\qquad + \lambda_1(D_{M,g_0})\Re e(\langle\psi_0, \mathrm{grad}_{g_0}(u)\cdot\psi_0\rangle)\Big) \\ &= e^{-2u}\Big(\lambda_1(D^2_{M,g_0}) + \frac{1}{4}|\mathrm{grad}_{g_0}(u)|^2_{g_0}\Big). \end{aligned}$$

The min-max principle together with $v_g = e^{2u}v_{g_0}$ provides

$$\begin{aligned} \lambda_1(D^2_{M,g}) &\leq \frac{\int_{M^2}|D_{M,g}\psi|^2 v_g}{\int_{M^2}|\psi|^2 v_g} \\ &= \frac{\int_{M^2}\{\lambda_1(D^2_{M,g_0}) + \frac{1}{4}|\mathrm{grad}_{g_0}(u)|^2_{g_0}\}v_{g_0}}{\mathrm{Area}(M^2,g)}, \end{aligned}$$

which leads to (5.4).
*ii)* Set $\psi := e^{-\frac{u}{2}}\overline{\psi_0}$. Then identity (1.16) implies that

$$\begin{aligned} D_{M,g}\psi &= e^{-\frac{3u}{2}}\overline{D_{M,g_0}\psi_0} \\ &= \lambda_1(D_{M,g_0})e^{-\frac{3u}{2}}\overline{\psi_0}, \end{aligned}$$

so that $|D_{M,g}\psi|^2 = \lambda_1(D^2_{M,g_0})e^{-3u}$ and, by the min-max principle,

$$\lambda_1(D^2_{M,g}) \leq \frac{\int_{M^2}|D_{M,g}\psi|^2 v_g}{\int_{M^2}|\psi|^2 v_g} = \lambda_1(D^2_{M,g_0})\frac{\int_{M^2} e^{-u}v_{g_0}}{\int_{M^2} e^{u}v_{g_0}},$$

which is (5.5).
*iii)* Set $\psi := e^{-\frac{3u}{2}}\overline{\psi_0}$. On the one hand,

$$D_{\mathbb{T}^2,g}\psi \overset{(1.16)}{=} e^{-u}(\overline{D_{\mathbb{T}^2,g_0}(e^{-\frac{3u}{2}}\psi_0)} + \frac{e^{-\frac{3u}{2}}}{2}\overline{\mathrm{grad}_{g_0}(u)\cdot\psi_0})$$

$$\begin{aligned}
&\overset{(1.11)}{=} && e^{-u}\Big(-\frac{3}{2}e^{-\frac{3u}{2}}\overline{\mathrm{grad}_{g_0}(u)\cdot\psi_0}+e^{-\frac{3u}{2}}\overline{D_{\mathbb{T}^2,g_0}\psi_0}\\
& && +\frac{e^{-\frac{3u}{2}}}{2}\overline{\mathrm{grad}_{g_0}(u)\cdot\psi_0}\Big)\\
&= && e^{-u}\cdot e^{-\frac{3u}{2}}(\lambda_2(D_{\mathbb{T}^2,g_0})\overline{\psi_0}-\overline{\mathrm{grad}_{g_0}(u)\cdot\psi_0}).
\end{aligned}$$

On the other hand, $\psi$ is $\mathrm{L}^2$-orthogonal to the kernel of $D_{\mathbb{T}^2,g}$, which is spanned by $e^{-\frac{u}{2}}\overline{\sigma^\pm}$, where $\sigma^\pm$ are non-zero parallel spinors on $(\mathbb{T}^2,g_0)$ with $\sigma^\pm\in\Gamma(\Sigma^\pm\mathbb{T}^2)$: indeed $\psi_0$ is $\mathrm{L}^2$-orthogonal to $\sigma^\pm$, so that

$$\begin{aligned}
\int_{\mathbb{T}^2}\langle\psi,e^{-\frac{u}{2}}\overline{\sigma^\pm}\rangle v_g &= \int_{\mathbb{T}^2}e^{-2u}\langle\psi_0,\sigma^\pm\rangle v_g\\
&= \int_{\mathbb{T}^2}\langle\psi_0,\sigma^\pm\rangle v_{g_0}\\
&= 0.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\lambda_2(D^2_{\mathbb{T}^2,g}) &\leq \frac{\int_{\mathbb{T}^2}|D_{\mathbb{T}^2,g}\psi|^2v_g}{\int_{\mathbb{T}^2}|\psi|^2v_g}\\
&= \frac{\int_{\mathbb{T}^2}e^{-2u}\cdot e^{-3u}\{\lambda_2(D^2_{\mathbb{T}^2,g_0})+|\mathrm{grad}_{g_0}(u)|^2_{g_0}\}v_g}{\int_{\mathbb{T}^2}e^{-u}v_{g_0}}\\
&= \frac{\int_{\mathbb{T}^2}e^{-3u}\{\lambda_2(D^2_{\mathbb{T}^2,g_0})+|\mathrm{grad}_{g_0}(u)|^2_{g_0}\}v_{g_0}}{\int_{\mathbb{T}^2}e^{-u}v_{g_0}},
\end{aligned}$$

which shows (5.6) and concludes the proof. □

As a consequence of (5.4), if $M^2=\mathbb{S}^2$, then [2, Thm. 2]

$$\begin{aligned}
&\lambda_1(D^2_{\mathbb{S}^2,g})\mathrm{Area}(\mathbb{S}^2,g)-4\pi \leq\\
&\inf_{\Phi\in\mathrm{Conf}(\mathbb{S}^2,[g_0])}\Big\{\frac{1}{4}\int_{\mathbb{S}^2}|\mathrm{grad}_{g_0}(u_\Phi)|^2_{g_0}v_{g_0}\ ,\ \Phi^*g_0=e^{-2u_\Phi}g\Big\},
\end{aligned}\tag{5.7}$$

where $\mathrm{Conf}(\mathbb{S}^2,[g_0])$ denotes the group of conformal transformations of $(\mathbb{S}^2,g_0)$. Note that the l.h.s. of (5.7) is non-negative because of (3.17). Beware that the r.h.s. of (5.7) is not a conformal invariant, since the product $\lambda_1(D^2_{\mathbb{S}^2,g})\mathrm{Area}(\mathbb{S}^2,g)$ is not bounded in the conformal class of $g$ [27, Thm. 1.1]. As an application of (5.7), the smallest eigenvalue of $D^2$ of a one-parameter-family of ellipsoids can be asymptotically estimated: given $a>0$, let $M_a:=\{x\in\mathbb{R}^3\,|\,x_1^2+x_2^2+\frac{x_3^2}{a^2}=1\}$ carry the induced metric from $\mathbb{R}^3$, then [2, Thm. 5]

$$\overline{\lim_{a\to 0}}\ \lambda_1(D^2_{M_a})\in[2,\frac{3}{2}+\ln(2)]\qquad\text{and}\qquad\overline{\lim_{a\to\infty}}\ \lambda_1(D^2_{M_a})\leq\frac{1}{4}.\tag{5.8}$$

Both estimates provide much sharper upper bounds as C. Bär's one (5.19) in terms of the averaged total squared mean curvature.
In the case of the 2-torus and with the notations of Theorem 2.1.1, the smallest eigenvalue of $D^2_{\mathbb{T}^2,g}$ w.r.t. the $(\delta_1,\delta_2)$-spin structure and flat metric is not greater than $4\pi^2|\frac{1}{2}(\delta_1\gamma_1^*+\delta_2\gamma_2^*)|^2$, so that by (5.5) it satisfies [2, Thm. 4]

$$\lambda_1(D^2_{\mathbb{T}^2,g})\mathrm{Area}(\mathbb{T}^2,g)\leq\pi^2|\delta_1\gamma_1^*+\delta_2\gamma_2^*|^2\frac{\int_{\mathbb{T}^2}e^{-u}v_{g_0}}{\int_{\mathbb{T}^2}e^{u}v_{g_0}}.\tag{5.9}$$

Of course (5.9) is empty if $\delta_1 = \delta_2 = 0$, in which case only (5.6) gives information on the first positive eigenvalue. Besides, inequality (5.9) provides asymptotical estimates of the smallest eigenvalue of $D^2$ on round tori: given $0 < r < R$, let $\mathbb{T}^2_{r,R}$ denote the tube of radius $r$ about a circle of radius $R$. If $\mathbb{T}^2_{r,R}$ carries the induced metric and the $(1,0)$-spin structure, then [2, p.5]

$$\lim_{r\to 0} \lambda_1(D^2_{\mathbb{T}^2_{r,R}})\mathrm{Area}(\mathbb{T}^2_{r,R}) = \lim_{R\to\infty} \lambda_1(D^2_{\mathbb{T}^2_{r,R}})\mathrm{Area}(\mathbb{T}^2_{r,R}) = 0$$

and in case it carries the $(0,1)$-spin structure,

$$\overline{\lim_{\frac{r}{R}\to 1}}\ \lambda_1(D^2_{\mathbb{T}^2_{r,R}})\mathrm{Area}(\mathbb{T}^2_{r,R}) \leq \pi^2.$$

For both the $(1,0)$- and $(0,1)$-spin structures (however not for the $(1,1)$-one) these estimates enhance those obtained from (5.19) below.

In higher dimensions another general upper bound in terms of the sectional curvature can be obtained from the min-max principle. In the following theorem we denote by $B_r(p)$ the geodesic ball of radius $r > 0$ around some point $p \in M$ and, provided $r$ is smaller than the injectivity radius $\mathrm{rad_{inj}}(M^n, g)$ of the Riemannian manifold $(M^n, g)$, by $0 < \mu_1(B_r(p)) \leq \mu_2(B_r(p)) \leq \ldots$ the spectrum of the scalar Laplace operator with Dirichlet boundary condition on $B_r(p)$. For any $x \geq 0$ and $t > 0$ we also define

$$f_x(t) := \begin{cases} \frac{1-\cos(\sqrt{x}t)}{\sqrt{x}\sin(\sqrt{x}t)} & \text{if } x > 0 \\ \frac{t}{2} & \text{if } x = 0. \end{cases}$$

**Theorem 5.1.4 (C. Bär [39])** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional closed Riemannian spin manifold and assume that there exist nonnegative real constants $\rho_1, \rho_2, \kappa$ such that the sectional curvature of $(M^n, g)$ lies everywhere in $[\kappa - \rho_1, \kappa + \rho_2]$. Let $N := 2^{[\frac{n}{2}]}$. Then the $jN^{\mathrm{th}}$ eigenvalue in absolute value of the Dirac operator $D$ of $(M^n, g)$ satisfies*

$$\begin{aligned} |\lambda_{jN}(D)| \quad \leq \quad & \frac{n\sqrt{\kappa}}{2} + \inf_{p\in M} \inf_{r} \Big\{ \sqrt{\mu_j(B_r(p))} \\ & + (n-1)(1 + \frac{2}{3}\sqrt{[\frac{n-2}{2}]})(\rho_1 + \rho_2) f_{\kappa+\rho_2}(r) \Big\}, \quad (5.10) \end{aligned}$$

*where the second infimum ranges over $r \in ]0, \min\{\mathrm{rad_{inj}}(M^n, g), \frac{\pi}{\sqrt{\kappa+\rho_2}}\}[$. Moreover, (5.10) is an equality for $(M^n, g) = (\mathbb{S}^n, \mathrm{can})$, $j = 1$ and $\rho_1 = \rho_2 = 0$.*

*Sketch of proof of* Theorem 5.1.4: First consider a geodesic ball $B_r(p)$ of radius $0 < r < \mathrm{rad_{inj}}(M^n, g)$. Since $B_r(p)$ is convex its spinor bundle can be trivialized by a pointwise orthonormal family $\varphi_1, \ldots, \varphi_N$. Let $\{f_j\}_{j\geq 1}$ be a Hilbert basis of $\mathrm{L}^2(B_r(p), \mathbb{C})$ made out of eigenfunctions for the scalar Laplace operator $\Delta$ with Dirichlet boundary condition on $B_r(p)$, in particular $\Delta f_j = \mu_j(B_r(p)) f_j$ with $f_j|_{\partial B_r(p)} = 0$ holds. Then $\{f_j\varphi_k\}_{1\leq j, 1\leq k\leq N}$ is a Hilbert basis of $\mathrm{L}^2(\Sigma B_r(p))$. Since $D^2_{B_r(p)}$ is of Laplace type (see Schrödinger-Lichnerowicz' formula (3.2)) one may talk about its eigenvalues with respect to the Dirichlet boundary condition and the min-max principle also applies. Fix $1 \leq k \leq N$ and $j \geq 1$, then using the formal self-adjointness of the Dirac operator we have

$$\begin{aligned}
\frac{(D^2_{B_r(p)}(f_j\varphi_k), f_j\varphi_k)}{\|f_j\varphi_k\|^2} &= \frac{\|D_{B_r(p)}(f_j\varphi_k)\|^2}{\|f_j\|^2} \\
&\overset{(1.11)}{=} \|df_j\cdot\varphi_k + f_jD_{B_r(p)}\varphi_k\|^2 \\
&= \|df_j\|^2 + 2\Re e\left(df_j\cdot\varphi_k, f_jD_{B_r(p)}\varphi_k\right) \\
&\quad +\|f_jD_{B_r(p)}\varphi_k\|^2 \\
&\leq \mu_j(B_r(p)) + 2\|df_j\|\cdot\|f_jD_{B_r(p)}\varphi_k\| \\
&\quad + \sup_{B_r(p)}(|D_{B_r(p)}\varphi_k|^2) \\
&\leq \mu_j(B_r(p)) + 2\sqrt{\mu_j(B_r(p))}\cdot\sup_{B_r(p)}(|D_{B_r(p)}\varphi_k|) \\
&\quad + \sup_{B_r(p)}(|D_{B_r(p)}\varphi_k|^2) \\
&= \Big(\sqrt{\mu_j(B_r(p))} + \sup_{B_r(p)}(|D_{B_r(p)}\varphi_k|)\Big)^2,
\end{aligned}$$

therefore the min-max principle implies

$$\lambda_{jN}(D^2_{B_r(p)}) \leq \Big(\sqrt{\mu_j(B_r(p))} + \sup_{B_r(p)}(|D_{B_r(p)}\varphi_k|)\Big)^2.$$

The second step in the proof, which is the main and the most technical one, consists in estimating the supremum on the r.h.s. by geometric data. This can be done essentially by controlling the growth of the pointwise norm along geodesics and applying Rauch's comparison theorem, we refer to [39, Lemma 1] and [39, Sec. 4]. The third and last step consists in comparing the eigenvalues of $D^2$ with those of $D^2_{B_r(p)}$ subject to Dirichlet boundary condition: the monotonicity principle (see e.g. [77, Cor. 1 p.18]) implies that

$$\lambda_j(D^2) \leq \lambda_j(D_{B_r(p)})^2.$$

This proves the inequality (5.10).
If $(M^n, g) = (\mathbb{S}^n, \mathrm{can})$, then on the one hand $|\lambda_1(D_{\mathbb{S}^n})| = \ldots = |\lambda_N(D_{\mathbb{S}^n})| = \frac{n}{2}$ (see Theorem 2.1.3) and on the other hand the choice $\rho_1 = \rho_2 = 0$ provides $\inf_{p\in M}\inf_r\left\{\sqrt{\mu_1(B_r(p))}\right\}$ in the r.h.s. of (5.10). Since $\mu_1(B_r(p)) \underset{r\to\pi}{\longrightarrow} 0$ (see e.g. [77, Thm. 6 p.50]), we conclude that equality holds in (5.10). □

## 5.2 Extrinsic upper bounds

In this section we assume the existence of some map from the manifold $M$ to another manifold and want to derive upper eigenvalue estimates in terms of geometric invariants associated to this map. The first situation which has been studied is the case where there exists a map of sufficiently high degree from $M$ into the round sphere of same dimension.

**Theorem 5.2.1 (H. Baum [54])** *Let* $(M^n, g)$ *be an* even-dimensional *closed Riemannian spin manifold and assume the existence of* a smooth map $\iota : M^n \longrightarrow \mathbb{S}^n$ with degree

$$\deg(\iota) \geq 1 + 2^{\frac{n}{2}-1} \sum_{j=1}^{k-1} m_j$$

*for some positive integer* $k$*, where* $m_j$ *is the multiplicity of the* $j^{th}$ *eigenvalue of* $D^2$*. Then the* $k^{th}$ *eigenvalue* $\lambda_k$ *of* $D$ *satisfies*

$$|\lambda_k| \leq 2^{\frac{n}{2}-1} \sqrt{\frac{n}{2}} \cdot \max_{x \in M} \|d_x \iota\|.$$

The inequality in Theorem 5.2.1 is sharp for $M = \mathbb{S}^2$ and $\iota = \mathrm{Id}$ (but of course only for $k = 1$). It is unclear if it can be sharp in higher dimensions.

*Sketch of proof of* Theorem 5.2.1: The proof is based on Vafa-Witten's method. Consider the tensor-product bundle $\mathbb{S} := \Sigma M \otimes \iota^*(\Sigma\mathbb{S}^n)$. Define the tensor-product connection of $\nabla$ (on $\Sigma M$) with on the one hand the pull-back-connection $\iota^*(\nabla^{\Sigma\mathbb{S}^n})$ and on the other hand with a flat connection coming from a trivialization of $\Sigma\mathbb{S}^n$ through $\pm\frac{1}{2}$-Killing spinors, see Example A.1.3.2. One obtains two different covariant derivatives on $\mathbb{S}$ to which two different Dirac-type operators (called twisted Dirac operators) may be associated. The latter one (i.e., involving the flat connection on $\iota^*(\Sigma\mathbb{S}^n)$) is by construction just the direct sum of $2^{[\frac{n}{2}]} = \mathrm{rk}(\Sigma\mathbb{S}^n)$ copies of $D$. Applying the Atiyah-Singer index theorem for twisted Dirac operators and computing explicitly the Chern character of both positive and negative half-spinor bundles of $\mathbb{S}^n$ (remember that $n$ is assumed to be even) the dimension of the kernel of the other twisted Dirac operator can be bounded from below by a positive constant depending on the $\widehat{A}$-genus of $TM$ and the degree of $\iota$. A further observation shows that this lower bound may be made dependent of $\deg(\iota)$ only. The assumption on $\deg(\iota)$ plugged into Vafa-Witten's method then ensures that $|\lambda_k|$ is not greater than the norm of the difference of both Dirac operators, which can be itself easily estimated against the supremum norm of $d\iota$. This concludes the sketch of proof of Theorem 5.2.1. $\square$

Turning to the case where $M$ is isometrically immersed into some Euclidean space, upper eigenvalue bounds can be found in terms of the second fundamental form of the immersion. The first result in this direction is due to U. Bunke.

**Theorem 5.2.2 (U. Bunke [69])** *Assume that* $(M^n, g)$ *is* even-dimensional*, closed, and that* there exists an isometric immersion $\iota : M \longrightarrow \mathbb{R}^N$ *for some positive integer* $N$*. Then there is a topologically determined number of eigenvalues* $\lambda$ *of* $D$ *satisfying*

$$\lambda^2 \leq 2^{\frac{n}{2}} \max_{x \in M}(\|II_x\|^2),$$

*where* $II$ *is the second fundamental form of* $\iota$*.*

It is not clear whether the estimate in Theorem 5.2.2 can be sharp or not, since it is a strict inequality even for the standard immersion of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. It has been improved by C. Bär in [43] (see (5.19)). His idea consists in choosing the restriction of particular spinor fields (such as parallel or Killing spinors) onto the hypersurface as test-spinors in view of the min-max principle. We formulate a general statement, from which all known estimates *à la Reilly* will follow. The notion of twistor-spinor is explained in Appendix A.

**Theorem 5.2.3** *Let $(M^n, g)$ be closed, $n \geq 2$ and assume that there exists an isometric immersion $M \xrightarrow{\iota} \widetilde{M}^{n+1}$, where $(\widetilde{M}^{n+1}, g)$ admits a non-zero twistor-spinor $\psi$. If the spin structure on $M$ coincides with the one induced by $\iota$, then the smallest eigenvalue of $D^2$ satisfies*

$$\lambda_1(D^2) \leq \inf_{\substack{f\in C^\infty(M,\mathbb{R})\\ f\neq 0}} \Big(\frac{n^2\int_M (H^2+R(\iota))f^2|\psi|^2 v_g}{4\int_M f^2|\psi|^2 v_g} + \frac{\int_M |df|^2|\psi|^2 v_g}{\int_M f^2|\psi|^2 v_g}\Big), \quad (5.11)$$

*where $H := -\frac{1}{n}\mathrm{tr}(\widetilde{\nabla}\nu)$ is the mean curvature of $\iota$ and*

$$R(\iota) := \frac{1}{n(n-1)}\Big(\widetilde{S} - 2\widetilde{\mathrm{ric}}(\nu,\nu)\Big).$$

*Proof*: As in Section 1.4 we denote by "$\cdot$" the Clifford multiplication on $(\widetilde{M}^{n+1}, g)$ and by "$\underset{M}{\cdot}$" the one on $(M^n, g)$. In analogy with Proposition 1.4.1 we also denote by $D_2$ the operator

$$D_2 := \left| \begin{array}{ll} D & \text{if } n \text{ is even} \\ D\oplus -D & \text{if } n \text{ is odd} \end{array}\right.$$

where $D$ is the fundamental Dirac operator of $(M^n, g)$.
Since $M$ is a Riemannian hypersurface of $\widetilde{M}$, the operator $D_2^2$ can be related with the square of another Dirac operator, namely the Dirac-Witten operator introduced by E. Witten in his proof of the positive mass theorem [241] and defined by

$$\widehat{D} := \sum_{j=1}^n e_j \cdot \widetilde{\nabla}_{e_j}$$

in any local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $TM$: let $\varphi \in \Gamma(\Sigma\widetilde{M}_{|_M})$, then

$$\begin{aligned} D_2\varphi &\overset{(1.21)}{=} \sum_{j=1}^n e_j\cdot\nu\cdot\widetilde{\nabla}_{e_j}\varphi - \frac{1}{2}\sum_{j=1}^n e_j\cdot\nu\cdot A(e_j)\cdot\nu\cdot\varphi \\ &= -\nu\cdot\widehat{D}\varphi + \frac{nH}{2}\varphi, \end{aligned} \quad (5.12)$$

so that

$$\begin{aligned} D_2^2\varphi &\overset{(1.23)}{=} \nu\cdot D_2(\widehat{D}\varphi) + D_2(\frac{nH}{2}\varphi) \\ &\overset{(1.11)}{=} \nu\cdot D_2(\widehat{D}\varphi) + \frac{n}{2}(\mathrm{grad}(H)\cdot\nu\cdot\varphi + HD_2\varphi) \\ &\overset{(5.12)}{=} (\widehat{D} + \frac{nH}{2}\nu\cdot)\widehat{D}\varphi + \frac{n}{2}\mathrm{grad}(H)\cdot\nu\cdot\varphi + \frac{nH}{2}D_2\varphi \\ &= \widehat{D}^2\varphi + \frac{n^2H^2}{4}\varphi + \frac{n}{2}\mathrm{grad}(H)\cdot\nu\cdot\varphi. \end{aligned} \quad (5.13)$$

Since $\psi$ is a twistor-spinor on $(\widetilde{M}^{n+1}, g)$ one has, in any local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $TM$,

$$\begin{aligned}
\widehat{D}^2\psi &= \widehat{D}(\sum_{j=1}^{n} e_j\cdot\widetilde{\nabla}_{e_j}\psi)\\
&= -\frac{1}{n+1}\sum_{j=1}^{n}\widehat{D}(e_j\cdot e_j\cdot\widetilde{D}\psi)\\
&= \frac{n}{n+1}\widehat{D}(\widetilde{D}\psi)\\
&\overset{(A.4)}{=} \frac{n}{n-1}\sum_{j=1}^{n}(-\frac{1}{2}e_j\cdot\widetilde{\mathrm{Ric}}(e_j)\cdot\psi+\frac{\widetilde{S}}{4n}e_j\cdot e_j\cdot\psi)\\
&= \frac{n}{n-1}\Big(\frac{\widetilde{S}}{2}\psi+\frac{1}{2}\nu\cdot\widetilde{\mathrm{Ric}}(\nu)\cdot\psi-\frac{\widetilde{S}}{4}\psi\Big)\\
&= \frac{n}{n-1}\Big(\frac{1}{4}(\widetilde{S}-2\widetilde{\mathrm{ric}}(\nu,\nu))\psi+\frac{1}{2}\nu\cdot\widetilde{\mathrm{Ric}}(\nu)^{\mathrm{T}}\cdot\psi\Big)\\
&= \frac{n}{n-1}\Big(\frac{n(n-1)}{4}R(\iota)\psi+\frac{1}{2}\nu\cdot\widetilde{\mathrm{Ric}}(\nu)^{\mathrm{T}}\cdot\psi\Big)\\
&= \frac{n^2}{4}R(\iota)\psi+\frac{n}{2(n-1)}\nu\cdot\widetilde{\mathrm{Ric}}(\nu)^{\mathrm{T}}\cdot\psi, \qquad (5.14)
\end{aligned}$$

where we denoted by $\widetilde{\mathrm{Ric}}(\nu)^{\mathrm{T}} := \sum_{j=1}^{n}\widetilde{\mathrm{ric}}(\nu,e_j)e_j$ the tangential projection of $\widetilde{\mathrm{Ric}}(\nu)$. Combining (1.13) (which obviously holds for $D_2^2$ instead of $D^2$), (5.13) and (5.14) we obtain, for every $f\in C^\infty(M,\mathbb{R})$,

$$\begin{aligned}
D_2^2(f\psi) &= f(\widehat{D}^2\psi+\frac{n^2H^2}{4}\psi+\frac{n}{2}\mathrm{grad}(H)\cdot\nu\cdot\psi)\\
&\quad -2(\widetilde{\nabla}_{\mathrm{grad}(f)}\psi-\frac{A(\mathrm{grad}(f))}{2}\cdot\nu\cdot\psi)+(\Delta f)\psi\\
&= \frac{n^2}{4}(H^2+R(\iota))f\psi+\frac{nf}{2}\mathrm{grad}(H)\cdot\nu\cdot\psi+\frac{nf}{2(n-1)}\nu\cdot\widetilde{\mathrm{Ric}}(\nu)^{\mathrm{T}}\cdot\psi\\
&\quad +\frac{2}{n+1}\mathrm{grad}(f)\cdot D_{\widetilde{M}}\psi+A(\mathrm{grad}(f))\cdot\nu\cdot\psi+(\Delta f)\psi. \qquad (5.15)
\end{aligned}$$

We deduce that

$$\begin{aligned}
\Re e(\langle D_2^2(f\psi),f\psi\rangle) &= \frac{n^2}{4}(H^2+R(\iota))f^2|\psi|^2+\frac{2f}{n+1}\Re e(\langle\mathrm{grad}(f)\cdot D_{\widetilde{M}}\psi,\psi\rangle)\\
&\quad +f(\Delta f)|\psi|^2\\
&= \frac{n^2}{4}(H^2+R(\iota))f^2|\psi|^2-g(f\mathrm{grad}(f),\mathrm{grad}(|\psi|^2))\\
&\quad +f(\Delta f)|\psi|^2,
\end{aligned}$$

so that, integrating over $M$ and applying Green's formula,

$$\begin{aligned}
\int_M\langle D_2^2(f\psi),f\psi\rangle v_g &= \frac{n^2}{4}\int_M(H^2+R(\iota))f^2|\psi|^2v_g\\
&\quad -\int_M\delta(f\mathrm{grad}(f))|\psi|^2v_g+\int_M f(\Delta f)|\psi|^2v_g
\end{aligned}$$

$$= \frac{n^2}{4}\int_M (H^2 + R(\iota))f^2|\psi|^2 v_g + \int_M |\mathrm{grad}(f)|^2|\psi|^2 v_g. \qquad (5.16)$$

The result straightforward follows from the min-max principle. □

In particular, if $(\widetilde{M}^{n+1}, g)$ admits an $(m \geq 2)$-dimensional space of twistor-spinors then

$$\lambda_1(D^2) \leq \inf_{\substack{\psi \text{ twistor-spinor} \\ \psi\neq 0}} \{\text{r.h.s. of (5.11)}\}.$$

The case $n = 1$ - the "baby case" - should not be of interest since the spectrum of $\mathbb{S}^1$ for both spin structures is explicitly known (see Theorem 2.1.1). However similar results turn out to hold - at least for $\widetilde{M} = \mathbb{R}^2$ or $\mathbb{S}^2$ - and to follow from very elementary geometric properties of plane or space curves:

**Proposition 5.2.4** *Let $M$ be a closed regular curve in $(\widetilde{M}^2, g) := (\mathbb{R}^2, \mathrm{can})$ or $(\mathbb{S}^2, \mathrm{can})$. Then the smallest eigenvalue $\lambda_1(D^2)$ of the square of the Dirac operator on $M$ for the induced metric and spin structure satisfies*

$$\lambda_1(D^2) \leq \frac{1}{4L}\int_0^L (H(t)^2 + \kappa)dt$$

*where $H$ is the curvature of $M$ parametrized by arc-length in $\widetilde{M}$, $L := \mathrm{Length}(M)$ and $\kappa$ denotes the sectional curvature of $(\widetilde{M}^2, g)$.*
*Moreover this inequality is an equality if and only if $M$ is a simply-parametrized-circle in $\widetilde{M}$.*

*Proof*: First we may assume that $M := c([0, L])$, where $c : \mathbb{R} \longrightarrow \widetilde{M}$ is an $L$-periodic arc-length-parametrized curve in $\widetilde{M}$. In other words $M$ is isometric to $\mathbb{S}^1(L) := \{z \in \mathbb{C} \text{ s.t. } |z| = \frac{L}{2\pi}\}$. Therefore the smallest eigenvalue of $D^2$ w.r.t. the $\delta$-spin structure (where $\delta = 0$ for the trivial spin structure and 1 for the non-trivial one, see Example 1.4.3.1) is $\lambda_1(D^2) = \frac{\delta\pi^2}{L^2}$. We separate the two cases.
• *Case* $\kappa = 0$: Let $n_c \in \mathbb{Z}$ be the turning number of $c$. It is easy to show that the induced spin structure of $M$ in $\mathbb{R}^2$ is the trivial one in case $n_c$ is even and the non-trivial one otherwise. If $n_c$ is even then the inequality is trivial and cannot be an equality, so that we assume that $n_c$ is odd. From the elementary formula

$$n_c = \frac{1}{2\pi}\int_0^L H(t)dt$$

and the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned} \lambda_1(D^2) &= \frac{\pi^2}{L^2} \\ &\leq \frac{\pi^2}{L^2}n_c^2 \\ &= \frac{\pi^2}{L^2}\cdot\frac{1}{4\pi^2}\left(\int_0^L H(t)dt\right)^2 \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{4L^2}\cdot L\cdot\int_0^L H^2(t)dt \\ &= \frac{1}{4L}\int_0^L H^2(t)dt, \end{aligned}$$

which proves the inequality. The equality holds if and only if $H$ is constant and $|n_c| = 1$, hence we obtain the statement in that case.

• *Case* $\kappa = 1$: Let $\mu_c$ be the bridge number and $\widetilde{H}$ be the curvature of $c$ as space curve. From $\widetilde{H}^2 = H^2 + 1$ and

$$\mu_c \leq \frac{1}{2\pi}\int_0^L \widetilde{H}(t)dt$$

we obtain exactly as above

$$\begin{aligned} \lambda_1(D^2) &\leq \frac{\pi^2}{L^2}\mu_c^2 \\ &\leq \frac{1}{4L^2}(\int_0^L \widetilde{H}(t)dt)^2 \\ &\leq \frac{1}{4L}\int_0^L \widetilde{H}^2(t)dt \\ &= \frac{1}{4L}\int_0^L (H^2(t)+1)dt, \end{aligned}$$

which shows the inequality. As before the equality only holds if $H$ is constant and $\mu_c = 1$, hence we obtain the statement in that case and conclude the proof.□

**Corollary 5.2.5** *Under the assumptions of* Theorem 5.2.3*, the smallest eigenvalue* $\lambda_1$ *of* $D^2$ *satisfies:*

$$\lambda_1(D^2) \leq \frac{n^2\int_M (H^2 + R(\iota))|\psi|^2 v_g}{4\int_M |\psi|^2 v_g}. \qquad (5.17)$$

*If furthermore* $\psi_x \neq 0$ *for every* $x \in M$*, then* [100, Thm. 3.1 p.44]

$$\lambda_1(D^2) \leq \frac{n^2}{4\mathrm{Vol}(M)}\int_M (H^2 + R(\iota))v_g + \frac{1}{\mathrm{Vol}(M)}\int_M |d(\ln(|\psi|))|^2 v_g. \qquad (5.18)$$

*Proof*: Both results follow directly from Theorem 5.2.3: considering the expression inside the infimum of the r.h.s. of inequality (5.11) one just has to choose $f$ to be a non-zero constant in the first case and to be $\frac{1}{|\psi|}$ in the second one. □

Corollary 5.2.5 provides the following estimates which were proved by C. Bär [43, Cor. 4.2 & 4.3] for hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$ or of the round sphere $\mathbb{S}^{n+1}$ and by N. Ginoux [100, 102, 101] for hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$: let $\widetilde{M}^{n+1}(\kappa) := \mathbb{R}^{n+1}$ if $\kappa = 0$, $\mathbb{S}^{n+1}$ if $\kappa = 1$ and $\mathbb{H}^{n+1}$ if $\kappa = -1$, then for any closed hypersurface $M$ of $\widetilde{M}^{n+1}(\kappa)$ carrying the induced metric and spin structure,

$$\lambda_1(D^2) \leq \frac{n^2}{4\mathrm{Vol}(M)}\int_M (H^2 + \kappa)v_g \qquad (5.19)$$

if $\kappa \geq 0$,

$$\lambda_1(D^2) \leq \frac{n^2}{4}(\max_M (H^2) - 1) \tag{5.20}$$

and

$$\lambda_1(D^2) \leq \frac{n^2}{4\mathrm{Vol}(M)} \int_M (H^2 - 1)v_g + \frac{1}{\mathrm{Vol}(M)} \inf_{\substack{\psi \in \widetilde{\mathcal{K}}_{\pm \frac{i}{2}} \\ \psi \neq 0}} \Big( \int_M |d(\ln(|\psi|))|^2 v_g \Big) \tag{5.21}$$

if $\kappa = -1$ (the space $\widetilde{\mathcal{K}}_{\pm \frac{i}{2}}$ refers here to the space of $\pm\frac{i}{2}$-Killing spinors on $\mathbb{H}^{n+1}$). Indeed $\widetilde{M}^{n+1}(\kappa)$ carries at least one non-zero $\pm\frac{\sqrt{\kappa}}{2}$-Killing spinor $\psi$ (which is in particular a twistor-spinor and vanishes nowhere), see e.g. [59, 66] and Examples A.1.3. Moreover $R(\iota) = \kappa$ and, if $\kappa \geq 0$, then Proposition A.4.1 implies that $|\psi|$ is constant on $\widetilde{M}^{n+1}(\kappa)$.
The inequalities (5.19) and (5.20) actually hold for higher eigenvalues of $D^2$, since the space of $\pm\frac{\sqrt{\kappa}}{2}$-Killing spinors on $\widetilde{M}^{n+1}(\kappa)$ is $2^{[\frac{n+1}{2}]}$-dimensional and the upper bound in (5.19) or (5.20) does not depend on the Killing spinor $\psi$, see [43, 102].

The inequalities (5.19), (5.20) and (5.21) are equalities for all geodesic hyperspheres in $\widetilde{M}^{n+1}(\kappa)$, see [43], [102] and [101] respectively. For $\kappa \leq 0$ the question remains open whether those are the only hypersurfaces enjoying this property. For $\kappa = 1$, generalized Clifford tori in $\mathbb{S}^{n+1}$ as well as minimally embedded $\mathbb{S}^3/\mathrm{Q}_8$ (where $\mathrm{Q}_8$ denotes the finite group of quaternions) in $\mathbb{S}^4$ also satisfy the limiting-case in (5.19) and it is conjectured that this actually holds for every homogeneous hypersurface in the round sphere, see [103, 106].

For $n = 2$ the upper bound in (5.19) is nothing but the so-called Willmore functional of the immersion $M \hookrightarrow \widetilde{M}^3(\kappa)$. Combining the estimate (5.19) with lower bounds of the Dirac spectrum (see Section 3.6) B. Ammann [9] and C. Bär [43] proved the Willmore conjecture ("$\int_M H^2 v_g \geq 2\pi^2$ *for every embedded torus* $M$ *in* $\mathbb{R}^3$") for particular metrics.

**Note 5.2.6** Could (5.18), (5.20) and (5.21) be enhanced in

$$\lambda_1(D^2) \leq \frac{n^2}{4\mathrm{Vol}(M)} \int_M (H^2 + R(\iota))v_g? \tag{5.22}$$

They are already of that form as soon as there exists a twistor-spinor $\psi$ on $(\widetilde{M}^{n+1}, g)$ with constant norm on $M$ (in (5.18) or (5.21)) or if the mean curvature $H$ of $\iota$ is constant (in (5.20)) respectively. Both conditions are very strong; for example the level sets of the norm of a non-zero imaginary Killing spinor on $\mathbb{H}^{n+1}$ are either geodesic hyperspheres or horospheres. Since however (5.19) is already of the form (5.22) and since an analog of (5.22) for the smallest positive Laplace-eigenvalue holds in virtue of a work by A. El Soufi and S. Ilias (see reference in [101]), it is natural to ask if (5.22) could hold in full generality, i.e., for every $n \geq 2$, for any isometric immersion $M^n \to \widetilde{M}^{n+1}$ into any Riemannian spin manifold $(\widetilde{M}^{n+1}, g)$ admitting a non-zero twistor-spinor. This is unfortunately false, at least in dimension $n = 2$: indeed the integral $\int_{M^2}(H^2 + R(\iota))v_g$

- the Willmore functional - is a conformal invariant, i.e., it only depends on the conformal class of $g$. However, the product $\lambda_1(D^2_{M,g})\mathrm{Area}(M^2, g)$ is in general not bounded on a conformal class (take e.g. $M = \mathbb{S}^2$) [27].
In dimension $n \geq 3$ one could look for a sequence $\{f_n\}_{n\in\mathbb{N}}$ of smooth real-valued functions on $M$ which would converge in $\mathrm{L}^2$-norm towards $f := \frac{1}{|\psi|}$ (provided the twistor-spinor $\psi$ has no zero on $M$) and such that $df_n \stackrel{\mathrm{L}^2}{\longrightarrow} 0$ as $n \to +\infty$. In that case $\{f_n\}_{n\in\mathbb{N}}$ would be bounded in the Sobolev space $\mathrm{H}^{1,2}(M)$, so that there would exist a subsequence $\{f_{\varphi(n)}\}_{n\in\mathbb{N}}$ converging weakly in $\mathrm{H}^{1,2}(M)$ towards some $F \in \mathrm{H}^{1,2}(M)$. In particular $\{f_{\varphi(n)}\}_{n\in\mathbb{N}}$ would converge weakly towards $F$ in $\mathrm{L}^2(M)$ as well as $\{df_{\varphi(n)}\}_{n\in\mathbb{N}}$ towards $dF$. From the uniqueness of the weak limit one would conclude that $F = f$ and $dF = 0$ almost everywhere, in particular $df = 0$ almost everywhere, that is, $df = 0$, which does not hold in general as explained just above.
Nevertheless, it is not excluded in case $M$ is a closed hypersurface of the hyperbolic space $\mathbb{H}^{n+1}$ that

$$\lambda_1(D^2) \leq \frac{n^2}{4\mathrm{Vol}(M)} \int_M (H^2 - 1)v_g$$

holds.

# Chapter 6

# Prescription of eigenvalues on closed manifolds

From its definition the Dirac spectrum *a priori* depends on the metric, the spin structure and of course the underlying manifold. In a very formal manner, the Dirac spectrum can be thought of as a functor from the category of closed Riemannian spin manifolds to that of real discrete sequences with closed image and unbounded on both sides. In this chapter, we investigate this functor, in particular its injectivity and surjectivity. In other words, does its Dirac spectrum determine a given Riemannian spin manifold? For a given real sequence as above, is there a Riemannian spin manifold whose spectrum coincides with this sequence? If the answer to the former question is definitely negative (Section 6.1), only partial results have been found regarding the latter in the case where the whole spectrum is replaced by a finite set of eigenvalues. In this situation one has to distinguish between the eigenvalue 0 - whose associated eigenvectors are called harmonic spinors - and the other ones. We shall see in Section 6.2 that, in dimension $n \geq 3$ (the case of surfaces must be handled separately), the metric can in general be modified so as to make 0 a Dirac eigenvalue, whereas generic metrics just have as many harmonic spinors as the Atiyah-Singer-index theorem forces them to do. If the finite set of real numbers does not contain 0 (and is symmetric about 0 if $n \not\equiv 3$ (4)), then it is always the lower part of the Dirac spectrum of a given metric on a fixed spin manifold (Section 6.3).

## 6.1 Dirac isospectrality

The question we address in this section is: do closed Riemannian spin manifolds which are Dirac isospectral (i.e., which show the same Dirac spectrum) have to be isometric, and if not are they at least diffeomorphic or homeomorphic? Note here that we *a priori* have to require the isometry condition to take the spin structure into account, that is, isometries are supposed to preserve both the orientation and the spin-structure.

From Weyl's asymptotic formula (see e.g. [20, Thm. 2.6]), the Dirac spectrum determines the dimension and the volume of the underlying manifold. A sharper insight in the formula shows for example that, in dimension 4, the Euler characteristic of the manifold is determined by its Dirac spectrum as soon as the scalar curvature is assumed to be constant [57]. However, the Dirac spectrum in general determines neither the isometry class nor the topology of the manifold. To illustrate this, we describe different families of examples, evolving from the "simplest" to the most sophisticated ones where even the explicit knowledge of the Dirac spectrum is not needed.

The founding result for isospectrality issues is indisputably J. Milnor's famous one-page-long article (see reference in [17]), where the author describes two Laplace-isospectral but non-isometric 16-dimensional tori. The idea is the following. The spectrum of the scalar Laplace operator on a flat torus $\Gamma\backslash\mathbb{R}^n$ is $\{4\pi^2|\gamma^*|^2,\ \gamma^* \in \Gamma^*\}$, where we keep the notations of Theorem 2.1.1. For any $r \geq 0$, the multiplicity of the eigenvalue $4\pi^2 r^2$ is the number of $\gamma^* \in \Gamma^*$ with $|\gamma^*| = r$. Now it is a surprising fact that there exist two lattices $\Gamma_1, \Gamma_2$ in $\mathbb{R}^{16}$ which induce non-isometric metrics on $\mathbb{T}^{16}$ but such that, for every $r \geq 0$, the sets $\{\gamma^* \in \Gamma_1^*, |\gamma^*| = r\}$ and $\{\gamma^* \in \Gamma_2^*, |\gamma^*| = r\}$ have the same cardinality (see reference in Milnor's article). Therefore the flat tori $\Gamma_1\backslash\mathbb{R}^{16}$ and $\Gamma_2\backslash\mathbb{R}^{16}$ possess the same Laplace spectrum, however they are not isometric. From Theorem 2.1.1, the Dirac spectrum of a flat torus with trivial spin structure is $\{\pm 2\pi|\gamma^*|,\ \gamma^* \in \Gamma^*\}$, hence the same argument shows that the Dirac spectra of $\Gamma_1\backslash\mathbb{R}^{16}$ and $\Gamma_2\backslash\mathbb{R}^{16}$ also coincide, at least for the trivial spin structure.

On positively curved spaceforms, Theorem 2.1.4 straightforward results in the following criterion for producing isospectrality.

**Theorem 6.1.1 (C. Bär [41])** *For $n \geq 3$ odd let $\Gamma_1, \Gamma_2 \subset \mathrm{SO}_{n+1}$ be finite subgroups acting freely on $\mathbb{S}^n$. For $j = 1, 2$ let $\epsilon_j : \Gamma_j \longrightarrow \mathrm{Spin}_{n+1}$ be a group homomorphism such that $\xi \circ \epsilon_j$ is the inclusion map $\Gamma_j \subset \mathrm{SO}_{n+1}$ and consider the induced spin structure on $\Gamma_j\backslash\mathbb{S}^n$. Assume the existence of a bijective map $f : \Gamma_1 \longrightarrow \Gamma_2$ such that, for every $\gamma_1 \in \Gamma_1$, the elements $\epsilon_1(\gamma_1)$ and $\epsilon_2(f(\gamma_1))$ are conjugated in $\mathrm{Spin}_{n+1}$.*
*Then the spaceforms $\Gamma_1\backslash\mathbb{S}^n$ and $\Gamma_2\backslash\mathbb{S}^n$ are Dirac isospectral.*

*Proof*: Both the character as well as the determinant remain unchanged under conjugation, hence Theorem 2.1.4 implies that the corresponding series $F_\pm(z)$ giving the multiplicities of the Dirac eigenvalues $\pm(\frac{n}{2}+k)$ are the same for $\Gamma_1\backslash\mathbb{S}^n$ and $\Gamma_2\backslash\mathbb{S}^n$. □

For three positive integers $a, b, r$ with $a, b$ odd, we denote by $\Gamma(a, b, r)$ the abstract group generated by two elements $A, B$ satisfying the relations $A^a = B^b = 1$ and $BAB^{-1} = A^r$. It is an exercise to prove that, if $\Gamma(a, b, r)$ is embedded in $\mathrm{SO}_{n+1}$ so as to act freely on $\mathbb{S}^n$, then the corresponding spaceform has a unique spin structure [41, Lemma 7]. A more detailed study shows that, for a good choice of $a, b, r$ and $n$ (as in Corollary 6.1.2 below), the group $\Gamma(a, b, r)$ can be embedded in two different ways in $\mathrm{SO}_{n+1}$ so as to satisfy the assumptions of Theorem 6.1.1 but such that the corresponding spaceforms are not isometric [41, Sec. 5].

**Corollary 6.1.2 (C. Bär [41])** *Let $n \equiv 3\ (8)$, $n \geq 19$. Let $a$ be a prime number with $a \equiv 1\ (\frac{n+1}{4})$, let $b := (\frac{n+1}{4})^2$ and let $r$ be chosen such that its mod $a$ class* *is of order $\frac{n+1}{4}$ in $(\mathbb{Z}_a)^\times$. Then there exist two Dirac isospectral non-isometric spaceforms diffeomorphic to $\Gamma(a,b,r)\backslash\mathbb{S}^n$.*

Returning to the flat setting, Bieberbach manifolds (quotients of $\mathbb{R}^n$ through Bieberbach groups, i.e., discrete co-compact and freely acting subgroups of the isometry group of $\mathbb{R}^n$) provide a large family of manifolds where the Dirac spectrum can be theoretically computed - and hence the isospectrality question be answered. As already mentioned in Section 2.1, the complexity of Bieberbach groups in $n$ dimensions grows up abruptly with $n$, which makes the computation of the Dirac spectrum tedious. For particular holonomies it remains possible and whole families of Dirac isospectral examples can be obtained. As an interesting fact, even pairwise non-homeomorphic ones can be fished out. To state the result we need to introduce the following notations, which are those of [194]. Fix an integer $n \geq 3$. For non-negative integers $j, h$ with $n - 2j - h > 0$ and $j + h \in 2\mathbb{N} \setminus \{0\}$, define in the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ the orthogonal involution $B_{j,h}$ by

$$B_{j,h} := \mathrm{diag}\Big(\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{j}, \underbrace{-1, \ldots, -1}_{h}, 1, \ldots, 1\Big)$$

and set $\Gamma_{j,h} := \mathrm{Span}(B_{j,h} \cdot (\frac{e_n}{2} + \mathrm{Id}), e_1 + \mathrm{Id}, \ldots, e_n + \mathrm{Id})$. The subgroup $\Gamma_{j,h}$ of the isometry group of $\mathbb{R}^n$ is orientation-preserving and co-compact, therefore the quotient

$$M^n_{j,h} := \Gamma_{j,h}\backslash\mathbb{R}^n$$

is a compact orientable flat manifold with holonomy group $\mathbb{Z}_2$. Moreover it can be shown that $M^n_{j,h}$ is spin and that its first homology group with integer coefficients is $H_1(M^n_{j,h}) = \mathbb{Z}^{n-j-h} \oplus (\mathbb{Z}_2)^h$ [193, Prop. 4.1]. In particular, the quotients corresponding to different pairs $(j,h)$ as above are not homeomorphic to each other. Set, for fixed $k \in \{0,1\}$,

$$\mathcal{F}^+_k := \{M^n_{j,h} \,|\, \frac{j+h}{2} \equiv k\ (2)\}.$$

Among the $2^{n-j}$ spin structures on some fixed $M^n_{j,h} \in \mathcal{F}^+_k$, we shall only consider the two so-called $\varepsilon_k$-ones, which are defined in [193, Prop. 4.2] and [194, eq. (4.13)] and which can be roughly described as follows with the help of Proposition 1.4.2: lift the translations $e_1+\mathrm{Id}, \ldots, e_{n-1}+\mathrm{Id}$ to $1 \in \mathrm{Spin}_n$ and $B_{j,h} \in \mathrm{SO}_n$ to one of its both pre-images through the double covering $\xi$. This assignment fixes exactly two spin structures since with those assumptions the lift of $e_n + \mathrm{Id}$ must be $(-1)^k$, see [193, Prop. 4.2] for details.

**Theorem 6.1.3 (R. Miatello and R. Podestá [194])** *For a fixed integer $n \geq 3$ consider the $M^n_{j,h}$'s defined above.*

*i) All elements of the family $\{(M^n_{j,h}, \varepsilon_0) \,|\, M_{j,h} \in \mathcal{F}^+_0\}$ are pairwise Dirac isospectral closed flat Riemannian spin manifolds, which are pairwise non-homeomorphic as soon as the pairs $(j,h)$ are different.*

*ii) If $n \not\equiv 3\ (4)$ then all elements of the family $\{(M^n_{j,h}, \varepsilon_1) \,|\, M^n_{j,h} \in \mathcal{F}^+_1\}$ are pairwise Dirac isospectral closed flat Riemannian spin manifolds, which are pairwise non-homeomorphic as soon as the pairs $(j,h)$ are different.*

In [194] Theorem 6.1.3 comes as a corollary of a whole series of computations of spectra on the $M_{j,h}$'s, one of which motivation is to compare Dirac isospectrality with other isospectrality issues. The results are obtained in a much more general setting where one twists the spinor bundle by a vector bundle associated to some representation of the group $\Gamma_{j,h}$. We refer to [194] for further statements and the proof of Theorem 6.1.3.

In another direction, one can try to generalize Milnor's result, replacing flat tori by compact quotients of nilpotent Lie groups. Initiated for Laplace isospectrality issues (see references in [17]), this ansatz has turned out to provide very rich families of manifolds on which, among others, Dirac isospectrality can be tested. In that case there also exists a general criterion for isospectrality, see [17, Thm. 5.1] for a proof.

**Theorem 6.1.4 (B. Ammann and C. Bär [17])** *Let $M := {}_{\Gamma}\backslash^{G}$ and $M' := {}_{\Gamma'}\backslash^{G}$ be spin homogeneous spaces where $\Gamma, \Gamma'$ are co-compact lattices in the simply-connected Lie group $G$. If the right-representations of $G$ onto $\mathrm{L}^2(\Sigma M)$ and $\mathrm{L}^2(\Sigma M')$ are equivalent, then for any left-invariant metric on $G$, the manifolds $M$ and $M'$ are Dirac isospectral.*

The independence of the spectrum on the left-invariant metric is a very strong statement. It allows in particular to produce continuous families of isospectral metrics, the spin structures staying fixed. Of course the real difficulty consists in applying Theorem 6.1.4, i.e., in picking groups $G$ so that the equivalence of the $G$-representations is satisfied. For nilpotent Lie groups that are strictly non-singular (i.e., for every $z$ in the center of $G$ and $x$ in its complement, there exists a $y$ with $xyx^{-1}y^{-1} = z$), this condition simplifies in terms of group theoretical data [17, Thm. 5.3]. The nilpotent Lie groups chosen by R. Gornet provide concrete examples where this criterion is fulfilled, we refer to [17, Thm. 5.6] for details and references.

**Theorem 6.1.5 (B. Ammann and C. Bär [17])** *There exist in dimensions 7 and 8 a continuous family of Dirac isospectral non-isometric closed Riemannian spin manifolds. Each manifold inside a family is diffeomorphic to the same quotient of some nilpotent Lie group by a co-compact lattice.*

## 6.2 Harmonic spinors

In analogy with the differential-form-setting, a (smooth) spinor field is called *harmonic* if and only if it lies in the kernel of the Dirac operator. If the manifold is closed then this space has always finite dimension since each eigenspace does. For instance, if $(M^n, g)$ has positive scalar curvature, then Theorem 3.1.1 implies that there is no non-zero harmonic spinor on $M$, as we have seen in Section 3.1. The problem we investigate here is: does this dimension depend on the metric, and if so, how? Before going further on note that it clearly depends on the choice of spin structure as the elementary example $M = \mathbb{T}^n$ with flat metric already shows: the space of harmonic spinors w.r.t. the trivial spin structure is $2^{[\frac{n}{2}]}$-dimensional whereas it is reduced to 0 for all other ones, see Theorem 2.1.1.

The first fundamental remark on the dependence of the Dirac kernel in terms of the metric was formulated by N. Hitchin in [152]: the dimension $d$ of the kernel of $D$ stays constant under conformal changes. Indeed, using (1.16), if $\varphi$ is a harmonic spinor on $(M^n, g)$ then so is $e^{-\frac{n-1}{2}u}\overline{\varphi}$ on $(M^n, \overline{g} := e^{2u}g)$ for any $u \in C^\infty(M, \mathbb{R})$. It can for example be deduced from this fact combined with Theorem 2.1.3 that, whatever the metric chosen, the 2-sphere $\mathbb{S}^2$ does not carry any non-zero harmonic spinor (there exists only one conformal class as well as one spin structure on $\mathbb{S}^2$). Alternatively this straightforward follows from Bär's inequality (3.17) since the lower bound $2\pi\chi(M^2)$ is a topological invariant which is positive for $M^2 = \mathbb{S}^2$.

On closed surfaces the presence of a quaternionic structure on the spinor bundle, which commutes with the Dirac operator (see e.g. [104, Lemma 1]), forces the number $d$ to be even. On the other hand, $d$ is bounded from above independently of the metric and spin structure: N. Hitchin proved that $d \leq 2[\frac{g+1}{2}]$ for every closed Riemann surface $M^2$ of genus $g$ and, if $g \leq 2$ then $d$ does not even depend on the conformal class [152, Prop. 2.3]. The case of $\mathbb{S}^2$ has just been discussed. For the 2-dimensional torus $\mathbb{T}^2$ it can be seen as a consequence of the conformal property above and of Theorem 2.1.1, which implies that $\mathbb{T}^2$ has a 2-dimensional space of harmonic spinors for the trivial spin structure and no non-zero one otherwise. If $g = 2$ the independence on the conformal class follows from [152, Prop. 2.3] or, alternatively, from the following argument: the Atiyah-Singer-index theorem implies that $\frac{d}{2} \equiv \alpha(M)\,(2)$, where $\alpha(M) \in \mathbb{Z}_2$ is the $\alpha$-genus of $M^2$ (see e.g. [48, Sec. 3] for a definition); either $\alpha(M) = 1$ and it follows from Hitchin's upper bound that $d = 2$ as soon as $g \leq 4$, or $\alpha(M) = 0$ and, for $g \leq 2$, one has $d = 0$.

In higher genus the picture is more complex. As a consequence of [21, Thm. 1.1], for any given spin structure, there exists a conformal class for which $d$ coincides with the lower bound provided by the Atiyah-Singer-index theorem (see (6.1) below), hence for which $d \in \{0, 2\}$. Next one can ask whether $d$ can be made maximal. To answer this question, it is more convenient to study the variations of $d$ in terms of the spin structure, the conformal class being fixed. Recall first that, for any closed surface $M^2$ of genus $g$, the group $H^1(M, \mathbb{Z}_2)$ is isomorphic to $(\mathbb{Z}_2)^{2g}$, so that $M^2$ has exactly $2^{2g}$ spin structures; in fact, there are exactly $2^{2g-1} + 2^{g-1}$ spin structures with $\alpha(M) = 0$ (those spin structures are the induced ones on the boundary $M^2$ if we see $M^2$ as embedded in $\mathbb{R}^3$) and exactly $2^{2g-1} - 2^{g-1}$ ones with $\alpha(M) = 1$ (those do not bound). Moreover, two spin structures having the same $\alpha$-genus are spin diffeomorphic (i.e., there exists an orientation-preserving diffeomorphism of $M^2$ sending the first spin structure onto the second one), see [177, Sec. 2]. Assuming now the metric on $M^2$ to be hyperelliptic (i.e., the existence of an isometric involution of $M^2$ with exactly $2g+2$ fixed points), C. Bär and P. Schmutz Schaller [51] have computed $d$ explicitly for each spin structure and shown the following: if $g \geq 5$ or ($g \geq 3$ and $\alpha(M) = 0$) then for any spin structure on the surface $M^2$ of genus $g$, there exists a conformal class for which $d$ is “almost” maximal, that is, $d = 2[\frac{g+1}{2}]$ or $d = 2[\frac{g-1}{2}]$ according to the parity of $[\frac{g+1}{2}]$. More precisely, if $\alpha(M) = 0$ and $g \geq 3$, then there exists a conformal class for which $d = 2[\frac{g+1}{2}] > 2$ if $g \equiv 0, 3\ (4)$ and $d = 2[\frac{g-1}{2}] > 2$ if $g \equiv 1, 2\ (4)$; if $\alpha(M) = 1$ and $g \geq 5$, then there exists a conformal class for which $d = 2[\frac{g+1}{2}] > 2$ if $g \equiv 1,2$ (4) and $d = 2[\frac{g-1}{2}] > 2$ if $g \equiv 0,3$ (4). For $g \in \{3,4\}$ and $\alpha(M) = 1$, the number $d$ does not depend on the conformal class (hence $d = 2$). The case $g \in \{3,4\}$ and $\alpha(M) = 0$, where both possibilities $d = 0$ and $d = 4$ occur, is also illustrated in [15]. Independently H. Martens showed [192] that, if $d = 2[\frac{g+1}{2}]$, then either $M^2$ is hyperelliptic or $g = 4$ or $g = 6$. For both $g = 4$ and $g = 6$ and any non-hyperelliptic conformal class, there exists exactly one spin structure such that $d = 2[\frac{g+1}{2}]$, see [152, 51]. We summarise the main known results for orientable surfaces.

**Theorem 6.2.1** *Let $M^2$ be a closed oriented surface of genus $g$, of $\alpha$-genus $\alpha(M) \in \mathbb{Z}_2$ and let $d$ be the complex dimension of the space of harmonic spinors on $M^2$ for some conformal class.*

*i) The integer $d$ is even and satisfies $d \leq 2[\frac{g+1}{2}]$. In particular, for any metric, the 2-sphere $\mathbb{S}^2$ has no non-zero harmonic spinor.*

*ii) If $g \leq 2$ and $\alpha(M) = 0$, then $d = 0$.*

*iii) If $g \leq 4$ and $\alpha(M) = 1$, then $d = 2$.*

*iv) If $\alpha(M) = 0$ and $g \geq 3$, then there exists a conformal class for which $d = 0$ and a conformal class for which*

$$d = \left| \begin{array}{ll} 2[\frac{g+1}{2}] & \textit{in case } g \equiv 0,3\ (4) \\ 2[\frac{g-1}{2}] & \textit{in case } g \equiv 1,2\ (4). \end{array} \right.$$

*v) If $\alpha(M) = 1$ and $g \geq 5$, then there exists a conformal class for which $d = 2$ and a conformal class for which*

$$d = \left| \begin{array}{ll} 2[\frac{g+1}{2}] & \textit{in case } g \equiv 1,2\ (4) \\ 2[\frac{g-1}{2}] & \textit{in case } g \equiv 0,3\ (4). \end{array} \right.$$

*In particular, the only closed orientable surfaces which do not admit any metric with non-zero harmonic spinors are those of genus at most 2 and with $\alpha(M) = 0$.*

By contrast, in higher dimensions, if one fixes the manifold and the spin structure, a given metric admits in general few non-zero harmonic spinors whereas there exist particular metrics having lots of them. First note that the Atiyah-Singer-index theorem [31] provides an *a priori* lower bound for $d$, since the inequality $\dim(\mathrm{Ker}(D)) \geq |\mathrm{ind}(D^+)|$ in even dimensions combined with Theorem 1.3.9 implies that

$$d = \dim(\mathrm{Ker}(D)) \geq \begin{cases} |\widehat{A}(M)| & \text{if } n \equiv 0\ (4) \\ |\alpha(M)| & \text{if } n \equiv 1\ (8) \\ 2|\alpha(M)| & \text{if } n \equiv 2\ (8) \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

It was first conjectured by C. Bär [42] that (6.1) is an equality for generic metrics, i.e., metrics belonging to some subset which is open in the $C^1$- and dense in the

$C^\infty$-topology in the space of all Riemannian metrics. In dimensions $n = 3$ and 4 perturbation methods combined with the formula linking spinors for different metrics [65] suffice in order to prove the conjecture to hold true (S. Maier [191]). If the underlying manifold is assumed to be simply-connected, then C. Bär and M. Dahl [48] proved the conjecture to hold true for all dimensions $n \geq 5$. Based on bordism theory, their argument consists of the three following steps. First, the conjecture holds true for some generators of the spin bordism ring. Second, any closed spin manifold is spin bordant to a spin manifold where the conjecture holds true. By a theorem of Gromov and Lawson, any $n(\geq 5)$-dimensional closed simply-connected manifold which is spin bordant to a spin manifold can be obtained from it by surgeries of codimension at least 3. It remains in the last step to show that the conjecture survives to surgeries of codimension at least 3. A set of generators is given by spin manifolds admitting metrics of positive scalar curvature (for which we already know that $\mathrm{Ker}(D) = 0$) as well as products of some of the irreducible manifolds of Mc.K. Wang's classification (see Theorem A.4.2), on which it can be relatively easily shown that the conjecture holds true. The second step is a pure argument of bordism theory using a theorem by S. Stolz (see [48] for references) and the explicit set of generators described above. The crucial step is the last one, which can be deduced from the following theorem.

**Theorem 6.2.2 (C. Bär and M. Dahl [48])** *Let $(M^n, g)$ be a closed Riemannian spin manifold. Let $\widetilde{M}$ be obtained from $M$ by surgery of codimension at least 3. Let $\varepsilon > 0$ and $L > 0$ with $\pm L \notin \mathrm{Spec}(D)$.*
*Then there exists a Riemannian metric $\tilde{g}$ on $\widetilde{M}$ such that the Dirac eigenvalues of $(M^n, g)$ and $(\widetilde{M}^n, \tilde{g})$ in $]-L, L[$ differ at most by $\varepsilon$.*

Theorem 6.2.2 is a generalization of an earlier result by C. Bär about the convergence of Dirac spectra on a connected sum in odd dimensions [42, Thm. B]. Coming back to the conjecture, it has been proved by B. Ammann, M. Dahl and E. Humbert [21] to hold true in full generality. Their argument relies on a very fine generalization of the surgery theorem (Theorem 6.2.2) to codimension greater than 1, we refer to [21] for a detailed proof.

At what seems to be the opposite side there have appeared since N. Hitchin's pioneering article [152] several results showing the existence in certain dimensions $n \geq 3$ of metrics with lots of non-zero harmonic spinors. Already computing the Dirac spectrum on $\mathbb{S}^3$ with Berger metric (and canonical spin structure), N. Hitchin noticed [152] that, for every $N \in \mathbb{N}$, there exists a metric on $\mathbb{S}^3$ admitting at least $N$ linearly independent harmonic spinors. Furthermore he constructed with the help of differential topological methods metrics with non-zero harmonic spinors on all closed spin manifolds $M^n$ of dimension $n \equiv 0, 1, 7\ (8)$. Extending the computation of the Dirac spectrum to all Berger spheres C. Bär showed [42] the existence of such metrics on all closed spin manifolds $M^n$ with $n \equiv 3, 7\ (8)$. His proof is based on the following simple ideas: first, for any closed odd-dimensional Riemannian spin manifolds $M_1$ and $M_2$, their connected sum $M_1 \sharp M_2$ admits a Riemannian metric for which its Dirac spectrum gets close to the union of both Dirac spectra of $M_1$ and $M_2$; second, there exists, for any $n \equiv 3\ (4)$, a one-parameter-family of Riemannian metrics on $\mathbb{S}^n$ for which at least one eigenvalue crosses the zero line (put $t = 2(m+1)$, $a_1 = a_2 = 0$ and

$j = \frac{m-1}{2}$ in one of both eigenvalues given in Theorem 2.2.2.*iii*)). We refer to [42] for a detailed proof. As for the remaining dimensions $n \geq 3$, the question of existence of metrics with non-zero harmonic spinors is still open, although it has been conjectured by C. Bär [42] that such metrics always exist. For the sphere $\mathbb{S}^n$, $n \geq 4$, the conjecture has been proved to hold true by L. Seeger [227] in case $n = 2m \geq 4$ and by M. Dahl [83] for all $n \geq 5$. The latter work, which contains the results of C. Bär [42] and N. Hitchin [152] on the existence of metrics with harmonic spinors on all closed spin manifolds of dimension $n \equiv 0, 1, 3, 7$ (8) ($n \geq 3$), is proved in the following way: one shows that, on a given closed spin manifold, the space of Riemannian metrics for which the Dirac operator is invertible is disconnected, if non-empty. The argument involves special metrics with positive scalar curvature on $\mathbb{S}^n$ which do not bound any metric with positive scalar curvature on the unit ball $B^{n+1}$ of $\mathbb{R}^{n+1}$. We refer to [83, Sec. 3] for a detailed proof.

We summarise the main known results for closed $n(\geq 3)$-dimensional spin manifolds.

**Theorem 6.2.3** *Let $M^n$ be a closed $n(\geq 2)$-dimensional spin manifold and $d(g)$ be the complex dimension of the space of harmonic spinors on $M^n$ for some metric $g$. Then the following holds.*

*i) There exists a subset in the space of all Riemannian metrics on $M^n$, which is open in the $C^1$- and dense in the $C^\infty$-topology, such that $d(g)$ coincides with the lower bound given by* (6.1) *for every metric $g$ in this subset* (B. Ammann, M. Dahl and E. Humbert [21])*.*

*ii) If $n \equiv 0, 1, 3, 7$ (8) and $n \geq 3$ then there exists a Riemannian metric $g$ on $M^n$ such that $d(g) \geq 1$.*

It would be interesting to know whether the following stronger conjecture by C. Bär [42, p. 41] holds true: on every closed spin manifold of dimension $\geq 3$ there exists a sequence $\{g_m\}_m$ of Riemannian metrics for which $d(g_m) \geq m$ for every $m \in \mathbb{N}$.

## 6.3 Prescribing the lower part of the spectrum

In the last section we have seen that, if $n \geq 3$ and $n \equiv 0, 1, 3, 7$ (8), then any $n$-dimensional closed spin manifold admits a metric for which the kernel of the corresponding Dirac operator is non-trivial. In other words, the eigenvalue 0 can be always prescribed in those dimensions. What about prescribing the rest of the spectrum? Although this question remains open, at least the lower part of the spectrum can be fixed. Note that, in dimension $n \not\equiv 3$ (4), Theorem 1.3.7.*iv*) imposes *a priori* this part to be symmetric about the origin.

**Theorem 6.3.1 (M. Dahl [82])** *For $n \geq 3$ let $M^n$ be any closed spin manifold. Let $L > 0$ (resp. $m \geq 1$) be real (resp. integral) and $l_1, \ldots, l_m$ be non-zero real numbers. Then the following holds:*

*i) If* $n \equiv 3$ (4) *and* $-L < l_1 < \ldots < l_m < L$*, then there exists a Riemannian metric on* $M^n$ *such that* $\mathrm{Spec}(D, g) \cap (]-L, L[\setminus\{0\}) = \{l_1, \ldots, l_m\}$*, where each of those eigenvalues is simple.*

*ii) If* $n \not\equiv 3$ (4) *and* $0 < l_1 < \ldots < l_m < L$*, then there exists a Riemannian metric on* $M^n$ *such that* $\mathrm{Spec}(D, g) \cap (]-L, L[\setminus\{0\}) = \{\pm l_1, \ldots, \pm l_m\}$*, where each of those eigenvalues is simple.*

The proof of Theorem 6.3.1 relies on techniques similar to those used for the construction of metrics with harmonic spinors. Roughly speaking, after possibly rescaling a fixed given metric on $M$, one has to add sufficiently many spheres of different sizes, each having one of the $l_i$'s as single and simple eigenvalue in $]-L, L[$. The surgery theorem (Theorem 6.2.2) ensures that the resulting manifold has the desired eigenvalues modulo a small error. We refer to [82, Sec. 4] for a detailed proof.

# Chapter 7

# The Dirac spectrum on non-compact manifolds

In this chapter we investigate the less familiar situation where the underlying manifold is non-compact. A good but somewhat not up-to-date reference is [45, Sec. 8.2]. The Dirac operator of a non-compact Riemannian spin manifold has in general no well-defined spectrum, even if its square does as a non-negative operator (Section 7.1). If the Riemannian manifold is complete then its Dirac spectrum - which is well-defined - is composed of a discrete part and of a disjoint union of intervals. After giving examples where the Dirac spectrum can be explicitly computed (Section 7.2), we survey the different situations where the geometry or particular analytical properties of the underlying manifold produce either gaps in their Dirac spectrum (Section 7.3) or the non-existence of one of both spectral components (Section 7.4).

## 7.1 Essential and point spectrum

We have seen in Proposition 1.3.5 that the Dirac operator is essentially self-adjoint as soon as the underlying Riemannian spin manifold is complete. This provides the existence of a canonical self-adjoint extension of $D$, which makes its spectral theory somewhat easier, see below. In case the manifold is not complete, there exists no canonical such extension as we have already seen in Note 1.3.6. However, since the square of $D$ is symmetric and non-negative on its domain $\Gamma_c(\Sigma M) = \{\varphi \in \Gamma(\Sigma M) \,|\, \mathrm{supp}(\varphi) \text{ compact }\}$, it admits a canonical non-negative self-adjoint extension, as was noticed by C. Bär in [47, Sec. 2]:

**Theorem 7.1.1 (Friedrichs' extension)** *Let $(M^n, g)$ be any Riemannian spin manifold. Then the operator $D^2$ has a unique non-negative self-adjoint extension in $\mathrm{H}^1_D(\Sigma M) \subset \mathrm{L}^2(\Sigma M)$, where $\mathrm{H}^1_D(\Sigma M)$ is the completion of $\Gamma_c(\Sigma M)$ w.r.t. the Hermitian inner product $(\varphi, \psi) \mapsto (\varphi, \psi) + (D\varphi, D\psi)$. It is called the Friedrichs' extension of $D$.*

For the general construction of the Friedrichs' extension we refer to [239, Thm. VII.2.11]. Thus one may consider the spectrum of the Friedrichs' extension of $D^2$, which we also denote by $\sigma(D^2)$. In general the spectrum of an operator may be decomposed as follows:

**Definition 7.1.2** *Let $T$ be a densely defined (unbounded) operator in a Hilbert space $H$.*

*i) The* point spectrum *of $T$ is the set $\sigma_p(T) := \{\lambda \in \mathbb{C} \,|\, \mathrm{Ker}(\lambda \mathrm{Id} - T) \neq \{0\}\}$.*

*ii) The* continuous spectrum *of $T$ is the set $\sigma_c(T) := \{\lambda \in \mathbb{C} \,|\, \mathrm{Ker}(\lambda \mathrm{Id} - T) = \{0\}$, $\overline{\mathrm{Im}(\lambda \mathrm{Id} - T)} = H$ and $(\lambda \mathrm{Id} - T)^{-1}$ unbounded$\}$.*

*iii) The* residual spectrum *of $T$ is the set $\sigma_r(T) := \{\lambda \in \mathbb{C} \,|\, \mathrm{Ker}(\lambda \mathrm{Id} - T) = \{0\}$ and $\overline{\mathrm{Im}(\lambda \mathrm{Id} - T)} \neq H\}$.*

*iv) The* essential spectrum *of $T$ is the set $\sigma_e(T) := \{\lambda \in \mathbb{C} \,|\, \lambda \mathrm{Id} - T$ not Fredholm$\}$.*

*v) The* discrete spectrum *of $T$ is the set $\sigma_d(T) := \sigma_p(T) \setminus \sigma_e(T)$.*

In particular $\sigma(T) = \sigma_p(T) \bigsqcup \sigma_c(T) \bigsqcup \sigma_r(T)$. The point spectrum is the set of eigenvalues and the discrete spectrum the subset of eigenvalues with finite multiplicity. Any self-adjoint operator $T$ has real spectrum and no residual spectrum. Furthermore, $\sigma_c(T) = \sigma_e(T) \setminus \sigma_p(T)$ and $\sigma_e(T)$ may be characterized as the set of $\lambda$'s for which an orthonormal sequence $(\varphi_k)_k$ exists satisfying $\|T\varphi_k - \lambda\varphi_k\| \underset{k\to\infty}{\longrightarrow} 0$. The essential spectrum of a self-adjoint elliptic differential operator remains unchanged after modifying any compact part of the underlying manifold [44, Prop. 1]:

**Proposition 7.1.3 (decomposition principle)** *Let $(M_1, g_1)$ and $(M_2, g_2)$ be Riemannian spin manifolds. Assume the existence of a spin-structure-preserving isometry $M_1 \setminus K_1 \longrightarrow M_2 \setminus K_2$ for some compact subsets $K_j \subset M_j$, $j = 1, 2$. Then $D_{M_1,g_1}$ and $D_{M_2,g_2}$ have the same essential spectrum.*

Note that, as a consequence of Theorem 1.3.7, the Dirac operator has no essential spectrum as soon as the underlying manifold is compact.

## 7.2 Explicit computations of spectra

We first determine the spectrum of the Euclidean space explicitly.

**Theorem 7.2.1** *The Dirac operator on $(\mathbb{R}^n, \mathrm{can})$ has no point spectrum and its continuous spectrum is $\mathbb{R}$.*

*Proof*: We adapt [45, Sec. 8.2.2] to the case $n \geq 1$. If $\varphi \in C^\infty(\mathbb{R}^n, \Sigma_n)$ were an eigenvector of $D$ associated to the eigenvalue $\lambda$, then it would be in particular an eigenvector of $D^2 = \Delta$ (acting on $\Sigma_n \cong \mathbb{C}^N$, where $N = 2^{[\frac{n}{2}]}$) associated to the eigenvalue $\lambda^2$. But there is no non-zero square-integrable eigenfunction for the scalar Laplacian on $\mathbb{R}^n$ as a consequence of a result by F. Rellich, see reference in [157]. Therefore $\sigma_p(D) = \varnothing$.
Let now $\lambda \in \mathbb{R}^n$ be an arbitrary vector. For $n = 1$ and $\lambda \neq 0$ let $\varphi_{\lambda,\epsilon} := 1 \in \mathbb{C} = \Sigma_1$ and $\epsilon := -\mathrm{sgn}(\lambda)$. For $\lambda = 0$ let $\varphi_{\lambda,\epsilon}$ be any unit-length-element in $\Sigma_n$ and put $\epsilon := 0$. For $n \geq 2$ and $\lambda \neq 0$ let $\varphi_{\lambda,\epsilon}$ be any unit-length-elements in $\mathrm{Ker}(i\frac{\lambda}{|\lambda|} \cdot -\epsilon \mathrm{Id})$ for $\epsilon = \pm 1$ (see proof of Theorem 2.1.1). Pick $\chi \in C^\infty(\mathbb{R}^n, [0, 1])$ with $\chi(x) = 1$ whenever $|x| \leq 1$ and $\chi(x) = 0$ whenever $|x| \geq 2$. For $j \in \mathbb{N} \setminus \{0\}$ set

$$\phi^\lambda_{j,\epsilon}(x) := \chi(\frac{x}{j} + 3je_1) \cdot e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}$$

for all $x \in \mathbb{R}^n$, where $e_1$ denotes the first canonical basis vector of $\mathbb{R}^n$. By construction $\mathrm{supp}(\phi^\lambda_{j,\epsilon}) \cap \mathrm{supp}(\phi^\lambda_{j',\epsilon}) = \varnothing$ whenever $j \neq j'$, so that $(\phi^\lambda_{j,\epsilon})_{j\geq 1}$ forms an orthogonal system in $\mathrm{L}^2(\mathbb{R}^n, \Sigma_n)$. In particular setting $\psi^\lambda_{j,\epsilon} := \frac{\phi^\lambda_{j,\epsilon}}{\|\phi^\lambda_{j,\epsilon}\|}$ one obtains an orthonormal system $(\psi^\lambda_{j,\epsilon})_{j\geq 1}$ in $\mathrm{L}^2(\mathbb{R}^n, \Sigma_n)$. Next we show that $\|(D - \epsilon|\lambda|\mathrm{Id})\psi^\lambda_{j,\epsilon}\| \underset{j\to\infty}{\longrightarrow} 0$, which will show $\epsilon|\lambda| \in \sigma_e(D)$ and conclude the proof. It follows from (1.11) that

$$
\begin{aligned}
(D - \epsilon|\lambda|\mathrm{Id})\psi^\lambda_{j,\epsilon} \quad = \quad & \frac{1}{\|\phi^\lambda_{j,\epsilon}\|}\Big(\chi(\frac{x}{j} + 3je_1)\cdot(D - \epsilon|\lambda|)(e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}) \\
& + \frac{1}{j}\mathrm{grad}(\chi)(\frac{x}{j} + ke_1)\cdot e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}\Big),
\end{aligned}
$$

with

$$
\begin{aligned}
D(e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}) &= \sum_{l=1}^n e_l \cdot \frac{\partial}{\partial x_l}(e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}) \\
&= \sum_{l=1}^n i\lambda_l e_l \cdot (e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}) \\
&= i\lambda\cdot(e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}) \\
&= \epsilon|\lambda|(e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}),
\end{aligned}
$$

so that

$$
(D - \epsilon|\lambda|\mathrm{Id})\psi^\lambda_{j,\epsilon} = \frac{1}{j\|\phi^\lambda_{j,\epsilon}\|}\mathrm{grad}(\chi)(\frac{x}{j} + ke_1)\cdot e^{i\langle\lambda,x\rangle}\varphi_{\lambda,\epsilon}.
$$

A simple transformation formula provides $\|\phi^\lambda_{j,\epsilon}\| = j^{\frac{n}{2}}\|\chi\|_{\mathrm{L}^2}$, from which

$$
\begin{aligned}
\|(D - \epsilon|\lambda|\mathrm{Id})\psi^\lambda_{j,\epsilon}\| &= \frac{1}{j\cdot j^{\frac{n}{2}}\|\chi\|_{\mathrm{L}^2}}\Big(\int_{\mathbb{R}^n}|\mathrm{grad}(\chi)(\frac{x}{j} + ke_1)|^2 dx_1\ldots dx_n\Big)^{\frac{1}{2}} \\
&= \frac{1}{j\cdot j^{\frac{n}{2}}\|\chi\|_{\mathrm{L}^2}}\cdot j^{\frac{n}{2}}\|\mathrm{grad}(\chi)\|_{\mathrm{L}^2} \\
&= \frac{\|\mathrm{grad}(\chi)\|_{\mathrm{L}^2}}{j\|\chi\|_{\mathrm{L}^2}}
\end{aligned}
$$

and the result follow. □

The second family where the spectrum can be explicitly described is that of hyperbolic spaces, whose Dirac spectra are given in the following table (note however that 0 is never a Dirac eigenvalue on $\mathbb{R}\mathrm{H}^n$ as claimed in [68, Cor. 4.6] for $n$ even):

| $M$ | $\sigma_p(D)$ | $\sigma_c(D)$ | references |
|---|---|---|---|
| $\mathbb{R}\mathrm{H}^n$ | $\varnothing$ | $\mathbb{R}$ | [68], [32] |
| $\mathbb{C}\mathrm{H}^n$, $n$ odd | $\varnothing$ | $\mathbb{R}$ | [76] |
| $\mathbb{C}\mathrm{H}^n$, $n$ even | $\{0\}$ | $]-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty[$ | [76] |
| $\mathbb{H}\mathrm{H}^n$ | $\varnothing$ | $\mathbb{R}$ | [76] |
| $\mathbb{O}\mathrm{H}^2$ | $\varnothing$ | $\mathbb{R}$ | [76] |

As a generalization, S. Goette and U. Semmelmann have shown [114] that the point spectrum of $D$ on a Riemannian symmetric space of non-compact type is either empty or $\{0\}$, and in the latter case each irreducible factor of $M$ is of the form $\mathbb{U}(p+q)/\mathbb{U}(p)\times\mathbb{U}(q)$ with $p+q$ odd. This explains in particular why the even complex dimensional hyperbolic space stands out as the only one in the list above having non-empty point spectrum.

## 7.3 Lower bounds on the spectrum

In general not much can be said on both components of the Dirac spectrum. However, as in the compact case, a spectral gap about 0 occurs as soon as the scalar curvature is bounded below by a positive constant, even if the underlying manifold is not complete:

**Theorem 7.3.1 (C. Bär [47])** *Let $(M^n, g)$ be any $n(\geq 2)$-dimensional Riemannian spin manifold, then*

$$\min(\sigma(D^2)) \geq \frac{n}{4(n-1)} \inf_M(S), \tag{7.1}$$

*where $S$ is the scalar curvature of $(M^n, g)$.*

*Proof*: Recall that $\sigma(D^2)$ denotes the spectrum of the Friedrichs' extension of $D^2$. As in the compact setting, $\min(\sigma(D^2))$ can be characterized as follows, see e.g. [239]:

$$\min(\sigma(D^2)) = \inf_{\varphi\in\Gamma_c(\Sigma M)\setminus\{0\}} \Big\{\frac{(D^2\varphi,\varphi)}{\|\varphi\|^2}\Big\}.$$

For every $\varphi \in \Gamma_c(\Sigma M)$ the identity $(D^2\varphi, \varphi) = \|D\varphi\|^2$ holds (Proposition 1.3.4), moreover (3.3) is valid on every Riemannian spin manifold provided $\varphi$ is smooth and has compact support, therefore inequality (7.1) is satisfied. □

As a consequence, if furthermore $(M^n, g)$ is complete with scalar curvature uniformly bounded below by a positive constant $S_0$, then the spectrum of $D$ satisfies $\sigma(D) \subset\, ]-\infty, -\sqrt{\frac{nS_0}{4(n-1)}}] \cup [\sqrt{\frac{nS_0}{4(n-1)}}, \infty[$, see [47, Cor. 3.2]. In analogy with the compact setting, the equality $\lambda^2 = \frac{n}{4(n-1)}\inf_M(S) > 0$ for some eigenvalue $\lambda$ of $D$ on the complete Riemannian manifold $(M^n, g)$ implies the existence of a non-zero real non-parallel Killing spinor on $(M^n, g)$ [47, Thm. 3.4], in particular the manifold must be Einstein and closed (Proposition A.4.1). Here one should beware that it is *a priori* not clear whether $\|\nabla\varphi\|$ is finite or not for a non-zero eigenvector $\varphi$ of $D$. Nevertheless this can be proved with the help of standard functional analytical techniques [47, Lemma 3.3]. The finiteness of $\|\nabla\varphi\|$ implies in turn that of $\|P\varphi\|$ (see (A.11)), therefore (3.3) makes sense for $\varphi$ being an eigenvector of $D$ and the statement follows as in the compact setting.

Combining Theorem 7.3.1 with the decomposition principle (Proposition 7.1.3) provides a positive lower bound on the essential spectrum of $D$ as soon as the scalar curvature is bounded below "at infinity": if $(M^n, g)$ is again arbitrary and contains a compact subset $K$ for which there exists a positive constant $S_0$ with $\inf_{M\setminus K}(S) \geq S_0$, then $\min(\sigma_e(D^2)) \geq \frac{n}{4(n-1)}S_0$ [47, Thm. 4.1]. In particular, the Dirac operator on $(M^n, g)$ has no essential spectrum as soon as the scalar curvature explodes at infinity [47, Cor. 4.3]. This is a particular situation where one of both components of the Dirac spectrum does not appear, see Section 7.4 for further results.

A natural question arising from Theorem 7.3.1 is whether conformal lower bounds for the Dirac spectrum can be obtained as in the compact setting. There is no complete answer to that question. In the case of surfaces an analog of Bär's inequality (3.17) holds, at least when the surface has finite area and can be embedded into $\mathbb{S}^2$ so as to inherit its spin structure:

**Theorem 7.3.2 (C. Bär [47])** *Let $(M^2, g)$ be a connected surface of finite area embedded into $\mathbb{S}^2$ and carrying the induced spin structure. Then*

$$\min(\sigma(D^2)) \geq \frac{4\pi}{\mathrm{Area}(M^2, g)}.$$

Theorem 7.3.2 follows from (3.17) using the above-mentioned characterization of $\min(\sigma(D^2))$ and a suitable sequence of metrics so as to make the area of $M^2$ close to that of $\mathbb{S}^2$.
In higher dimensions there exists no general analog of Hijazi's inequality (3.18), since for example the spectrum of the conformal Laplace operator on the real hyperbolic space $\mathbb{R}\mathrm{H}^n$ for $n \geq 3$ is $[\frac{n-1}{n-2}, \infty[$. However the corresponding inequality can be obtained under additional geometric assumptions:

**Theorem 7.3.3 (N. Große [118])** *Let $(M^n, g)$ be any $n(\geq 3)$-dimensional complete Riemannian spin manifold with finite volume and let $\lambda \in \sigma(D)$. Assume that either $\lambda \in \sigma_d(D)$ or $\lambda \in \sigma_e(D)$ and $n \geq 5$ as well as $S \geq S_0$ for some $S_0 \in \mathbb{R}$. Then*

$$\lambda^2 \geq \frac{n}{4(n-1)} \min(\sigma(L)), \tag{7.2}$$

*where $L := \frac{4(n-1)}{n-2}\Delta + S$ is the conformal Laplace operator and $S$ is the scalar curvature of $(M^n, g)$.*

The two situations according to $\lambda \in \sigma_d(D)$ or $\lambda \in \sigma_e(D)$ require different approaches. In the case where $\lambda$ is a eigenvalue of $D$, the proof consists in adapting that of Corollary 3.3.2 using a conformal factor given by the length of the corresponding eigenvector and cutting off near its zero-set and at infinity. The second case, where the supplementary assumptions $n \geq 5$ and $S \geq S_0$ are needed for technical reasons, relies on a Kato-type inequality [118, Lemma 2.1], we refer to [118, Sec. 4] for details. Moreover, as for inequality (7.1), equality in (7.2) for some non-zero eigenvalue $\lambda$ implies the existence of a non-zero real non-parallel Killing spinor on $(M^n, g)$, in particular $(M^n, g)$ must be Einstein and closed [118, Thm. 1.1].
Interestingly enough, the analog of the so-called conformal Hijazi inequality (3.20) also turns out to hold under suitable geometric assumptions. Given an $n$-dimensional manifold $M$ with Riemannian metric $\overline{g}$ and spin structure $\epsilon$, set

$$\lambda_1^+(M, \overline{g}, \epsilon) := \inf_{\substack{\varphi \in \Gamma_c(\Sigma M) \\ (\overline{D}\varphi, \varphi) > 0}} \Big\{ \frac{\|\overline{D}\varphi\|^2}{(\overline{D}\varphi, \varphi)} \Big\},$$

where $\overline{D} := D_{\overline{g}}$. The corresponding conformal invariant is defined by

$$\lambda^+_{\min}(M^n, [g], \epsilon) := \inf_{\substack{\overline{g}\in[g] \\ \mathrm{Vol}(M,\overline{g})<\infty}} \lambda_1^+(M, \overline{g}, \epsilon) \cdot \mathrm{Vol}(M, \overline{g})^{\frac{1}{n}},$$

where $[g]$ denotes a Riemannian conformal class on $M$.

**Theorem 7.3.4 (N. Große [118])** *Let $M^n$ be any $n(\geq 3)$-dimensional manifold with Riemannian conformal class $[g]$ and spin structure $\epsilon$. Assume the existence of a complete metric with finite volume in $[g]$ as well as $\lambda^+_{\min}(M^n, [g], \epsilon) > 0$. Then*

$$\lambda^+_{\min}(M^n, [g], \epsilon)^2 \geq \frac{n}{4(n-1)} \inf_{\substack{\overline{g}\in[g] \\ \mathrm{Vol}(M,\overline{g})<\infty}} \min(\sigma(L_{\overline{g}})) \cdot \mathrm{Vol}(M, \overline{g})^{\frac{2}{n}}, \tag{7.3}$$

*where $L_{\overline{g}}$ denotes the conformal Laplace operator on $(M^n, \overline{g})$.*

The assumption $\lambda^+_{\min}(M^n, [g], \epsilon) > 0$ actually implies $\sigma_e(D_{\overline{g}}) \subset ]-\infty, 0]$ for every complete $\overline{g} \in [g]$ with finite volume [118, Lemma 3.3]. As an example, if $(N^{n-1}, h)$ is any closed Riemannian spin manifold with positive scalar curvature and $n \geq 5$, then the Riemannian product $(N \times \mathbb{R}, h \oplus dt^2)$ endowed with the product spin structure satisfies the assumptions of Theorem 7.3.4, therefore (7.3) holds [118, Ex. 4.1], where the r.h.s. can be shown to be positive (see reference in [118]). However the cases $n = 3, 4$ remain open.

## 7.4 Absence of a spectral component

There are particular situations where one of both components of the Dirac spectrum can be excluded out of geometric considerations. This kind of question has attracted a lot of attention in the last years. We choose to present here five different settings with sometimes non-empty mutual intersection. Two of them deal with manifolds with cusps. A manifold with cusps can be written as the disjoint union of a compact manifold with non-empty boundary together with cusps, which are Riemannian manifolds of the form $(]0, \infty[\times N, dt^2 \oplus g_t)$ for some smooth 1-parameter-family of Riemannian metrics on the manifold $N$.

First consider an oriented complete $n$-dimensional hyperbolic manifold (recall that a metric is called hyperbolic if it has constant sectional curvature $-1$). Assume it to have finite volume. Then $M$ can be shown to possess a finite number of cusps of the form $(]0, \infty[\times N^{n-1}, dt^2 \oplus e^{-2t} g_{\mathrm{flat}})$ for some flat metric on some closed manifold $N^{n-1}$ (see reference in [44]). If $M$ is spin, then any spin structure on $M$ induces a spin structure on each cusp and hence on each $N^{n-1}$-factor. We call the spin structure trivial along a cusp if the Dirac operator of the corresponding $N$ has non-zero kernel and non-trivial otherwise. Since by the decomposition principle (Proposition 7.1.3) the essential spectrum of $D$ is unaffected by perturbations on a compact subset, it can be only influenced by the geometry of the cusps. As a striking fact, it turns out to depend only on the spin structure on $M$, where one obtains the following dichotomy as shown by C. Bär [44, Thm. 1]:

**Theorem 7.4.1 (C. Bär [44])** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional complete hyperbolic spin manifold of finite volume.*

*i) If the spin structure of $M$ is trivial along at least one cusp, then $\sigma_e(D) = \mathbb{R}$, in particular $\sigma(D) = \sigma_e(D) = \mathbb{R}$.*

*ii) If the spin structure of $M$ is non-trivial along all cusps, then $\sigma_e(D) = \varnothing$, in particular $\sigma(D) = \sigma_d(D)$.*

Explicit 3-dimensional examples constructed out of complements of knots are given in [44] where both possibilities occur according to the parity of the linking numbers, see [44, Thm. 4].

Theorem 7.4.1 has been generalized in several ways. W. Ballmann and J. Brüning [33, Thm. E] have proved that the conclusion of Theorem 7.4.1.*ii*) holds as soon as the sectional curvature is assumed to be pinched in $]-\infty, 0[$ and the cusp metric is of general warped product type:

**Theorem 7.4.2 (W. Ballmann and J. Brüning [33, 34])** *Let $(M^n, g)$ be any $n(\geq 2)$-dimensional complete Riemannian spin manifold with finitely many cusps. Assume that, on each cusp, the metric $g$ has sectional curvature in $[-b^2, -a^2]$ with $0 < a < b < \infty$ and that w.r.t. the induced spin structure the Dirac operator of $(N, g_t)$ has trivial kernel for large enough $t$.*
*Then $\sigma_e(D) = \varnothing$, in particular $\sigma(D) = \sigma_d(D)$.*

The case of a non-complete underlying manifold $(M^n, g)$ is somewhat more delicate to handle since there are no cusps in general. A geometric situation where Theorem 7.4.1 can be generalized has been discovered by A. Moroianu and S. Moroianu [207, Thm. 2.1]. Their setting appears as natural when considering the existence of Poincaré-Einstein metrics, where the metric is required to be conformal to a product metric at infinity.

**Theorem 7.4.3 (A. Moroianu and S. Moroianu [207])** *Let $M^n$ be any connected $n$-dimensional spin manifold with non-empty boundary $\partial M$. Consider a Riemannian metric $g$ on $M$ of the form $g = dx^2 \oplus g_{\partial M}$ in a neighbourhood of $\partial M$, where $x : M \longrightarrow [0, \infty[$ is the distance function to $\partial M$ and $g_{\partial M}$ is a Riemannian metric on $\partial M$. For $f \in C^\infty(M \setminus \partial M, ]0, \infty[)$ which only depends on $x$ in a neighbourhood of $\partial M$ set $\overline{g} := f^2 g$ on $M \setminus \partial M$.*
*If $\int_0^r f(x)dx = \infty$ for some $r \in ]0, \infty[$ and the Dirac operator of $(\partial M, g_{\partial M})$ is essentially self-adjoint in $\mathrm{L}^2(\Sigma\partial M)$, then the Dirac operator of $(M \setminus \partial M, \overline{g})$ has no point spectrum.*

The condition $\int_0^r f(x)dx = \infty$ imposes the boundary $\partial M$ to be at infinite distance from its complement in $M$ w.r.t. $\overline{g}$. Theorem 7.4.3 enhances an earlier result by J. Lott [188, Thm. 1]. As an application of Theorem 7.4.3, A. Moroianu and S. Moroianu showed the following: if $(M^n, g)$ is a complete Riemannian manifold carrying an incomplete vector field which, outside a compact subset, vanishes nowhere, is conformal and at the same time the gradient of a function, then $\sigma(D) = \sigma_e(D)$ (see [207, Thm. 4.1]). Note that the incompleteness assumption on the vector field is essential because of Theorem 7.4.1.*ii*) (take e.g. $X := e^{-t}\frac{\partial}{\partial t}$ along a cusp) [207, Rem. 4.4]). Further applications of Theorem

7.4.3 as well as references are discussed in [207, Sec. 5].

Theorem 7.4.3 was also inspired by an earlier work of S. Moroianu [210, Thm. 1 & Thm. 2], where the geometric condition $\int_0^r f(x)dx = \infty$ is replaced by the invertibility of the Dirac operator on the boundary as in the spirit of Theorem 7.4.1.*ii*):

**Theorem 7.4.4 (S. Moroianu [210])** *Let $M^n$ be any compact connected $n$-dimensional spin manifold with non-empty boundary $\partial M$. Consider a Riemannian metric $g$ on $M$ which in some local coordinates near $\partial M$ is of the form*

$$g = a_{00}(x,y)\frac{dx^2}{x^4} + \sum_{j=1}^{n-1} a_{0j}(x,y)\frac{dxdy_j}{x^2} + \sum_{i,j=1}^{n-1} a_{ij}(x,y)dy_i dy_j,$$

*where $x : M \longrightarrow [0,\infty[$ is the distance function to $\partial M$, $a_{\alpha\beta} \in C^\infty(M \times M, \mathbb{R})$ with $\big(a_{ij}(0,y)\big)_{1\leq i,j\leq n-1}$ positive definite. Set $g_{\partial M} := \sum_{i,j=1}^{n-1} a_{ij}(0,y)dy_i dy_j$ and, for $p > 0$, $\overline{g} := x^{2p}g$ on $M \setminus \partial M$.*
*If $a_{00}(0,y) = a_{0j}(0,y) = 1$ for every $1 \leq j \leq n$ and the Dirac operator of $(\partial M, g_{\partial M})$ is invertible, then the Dirac operator of $(M \setminus \partial M, \overline{g})$ is essentially self-adjoint and has no essential spectrum.*

The particular form of the metric near the boundary in Theorem 7.4.4 allows furthermore the existence of a nice Weyl's asymptotic estimate for the eigenvalue counting function [210, Thm. 3].

In the radically different situation where the curvature is non-negative, one still may guarantee the point spectrum to be empty or almost empty. However the hypotheses needed are much stronger. Let $(M^n, g)$ be geodesically starshaped w.r.t. some point $x_0$. Assume the scalar curvature of $(M^n, g)$ to be non-negative and that particular parts of the sectional curvature remain pointwise pinched in $[0, \frac{c(n)}{r^2}]$, where $r$ is the distance function from $x_0$ and $c(n)$ is a positive constant depending explicitly on $n$. Then $\sigma_d(D) = \varnothing$ or $\{0\}$ as shown by S. Kawai [157, Thm. 3].

# Chapter 8

# Other topics related with the Dirac spectrum

We outline the main topics in relation with the spectrum of Dirac operators that have been left aside in this overview.

## 8.1 Other eigenvalue estimates

As we have seen in Section 2.2, the Dirac operator on homogeneous spaces can be described as a family of matrices using the decomposition of the space of $L^2$-sections of $\Sigma M$ into irreducible components. What happens if the homogeneity assumption is slightly weakened? This question has first been addressed by M. Kraus in the cases of isometric $SO_n$-actions and warped products over $\mathbb{S}^1$ respectively. Although the explicit knowledge of the Dirac spectrum becomes out of reach, the eigenvalues can still be approximated in a reasonable way.

**Theorem 8.1.1 (M. Kraus [169, 170])** *For $n \geq 2$, let $g$ be any Riemannian metric on $\mathbb{S}^n$ such that $SO_n$ acts isometrically on $(S^n, g)$. Write $f_{\max}^{n-1} \cdot \mathrm{Vol}(S^{n-1}, \mathrm{can})$ for the maximal volume of the orbits of the $SO_n$-action. Then the Dirac spectrum of $(S^n, g)$ is symmetric about the origin,*

$$\lambda_1(D^2_{\mathbb{S}^n,g}) \geq \frac{(n-1)^2}{4f_{\max}^2}$$

*and there are at most $2^{[\frac{n}{2}]} \cdot \begin{pmatrix} n-1+k \\ k \end{pmatrix}$ eigenvalues of $D^2_{\mathbb{S}^n,g}$ in the interval $[\frac{(\frac{n-1}{2}+k)^2}{f_{\max}^2}, \frac{(\frac{n-1}{2}+k+1)^2}{f_{\max}^2}[$, for every nonnegative integer $k$.*

The proof of Theorem 8.1.1 relies on the following arguments: the $SO_n$-action allows a dense part of $(\mathbb{S}^n, g)$ to be written as a warped product of $\mathbb{S}^{n-1}$ with an interval. On this dense part the eigenvalue problem on $(\mathbb{S}^n, g)$ translates into a singular nonlinear differential equation of first order with boundary conditions at both ends. The rest of the proof involves Sturm-Liouville theory, we refer to [170] for details.
Note that the inequality in Theorem 8.1.1 is not sharp for the standard metric

on $\mathbb{S}^n$ since $\lambda_1(D^2_{\mathbb{S}^n,\mathrm{can}}) = \frac{n^2}{4}$ (see Theorem 2.1.3). However Theorem 8.1.1 provides sharp asymptotical eigenvalue estimates in the two following situations. First consider the cylinder $C^n(L) := ]0, L[\times\mathbb{S}^{n-1}$ with half $n$-dimensional spheres glued at both ends. Obviously $C^n(L)$ admits an isometric $\mathrm{SO}_n$-action for which $f_{\max} = 1$, in particular Theorem 8.1.1 implies that $\lambda_1(D^2_{C^n(L)}) \geq \frac{(n-1)^2}{4}$. On the other hand, $C^n(L)$ sits in $\mathbb{R}^{n+1}$ by construction; now C. Bär's upper bound (5.19) in terms of the averaged total squared mean curvature is not greater than

$$\frac{(n-1)^2 L\mathrm{Vol}(\mathbb{S}^{n-1},\mathrm{can}) + n^2\mathrm{Vol}(\mathbb{S}^n,\mathrm{can})}{4(L\mathrm{Vol}(\mathbb{S}^{n-1},\mathrm{can}) + \mathrm{Vol}(\mathbb{S}^n,\mathrm{can}))},$$

so that [170]

$$\lim_{L\to\infty} \lambda_1(D^2_{C^n(L)}) = \frac{(n-1)^2}{4}.$$

For the 2-dimensional ellipsoid $M_a := \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + \frac{x_3^2}{a^2} = 1\}$ (where $a > 0$) the maximal length of $\mathbb{S}^1$-orbits is $2\pi$, so that by Theorem 8.1.1 the inequality $\lambda_1(D^2_{M_a}) \geq \frac{1}{4}$ holds. Combining this with the upper bound (5.8) provides [169]

$$\lim_{a\to\infty} \lambda_1(D^2_{M_a}) = \frac{1}{4}.$$

The technique of separation of variables used in the proof of Theorem 8.1.1 also provides a lower eigenvalue bound on warped product fibrations over $\mathbb{S}^1$ in terms of the Dirac eigenvalues of the fibres, see [171, Thm. 2]. As for the case of higher dimensional fibres over arbitrary base manifolds, the only family which has been considered so far is that of warped products with fibre $\mathbb{S}^k$ with $k \geq 2$, where decomposing the Dirac operator into block operator matrices provides similar results to those of Theorem 8.1.1, see [173].

Another natural but completely different way to study the Dirac eigenvalues consists in comparing them with those of other geometric operators. Hijazi's inequality (3.18) is already of that kind since $\mu_1$ is the smallest eigenvalue of the conformal Laplace operator. As for spectral comparison results between the Dirac and the scalar Laplace operators, the first ones were proved by M. Bordoni. They rely on a very nice general comparison principle between two operators satisfying some kind of Kato-type inequality. The estimate which can be deduced reads as follows.

**Theorem 8.1.2 (M. Bordoni [63])** *Let* $0 = \lambda_0(\Delta) < \lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \ldots$ *be the spectrum of the scalar Laplace operator* $\Delta$ *on a closed* $n(\geq 2)$*-dimensional Riemannian spin manifold* $(M^n, g)$*. Then for any positive integer* $N$ [63, Prop. 4.20]

$$\lambda_{2N}(D^2) \geq \frac{n}{4(n-1)}\Big(\inf_M(S) + \frac{\lambda_k(\Delta)}{2(2^{[\frac{n}{2}]}+1)^2}\Big), \tag{8.1}$$

*where* $k = [\frac{N}{2^{[\frac{n}{2}]}+1}]$.

In particular Bordoni's inequality (8.1) implies Friedrich's inequality (3.1) as well as the presence of at most $2^{[\frac{n}{2}]+1}$ eigenvalues of $D^2$ in the interval

$$[\frac{n}{4(n-1)}\inf_M(S), \frac{n}{4(n-1)}\Big(\inf_M(S) + \frac{\lambda_1(\Delta)}{2(2^{[\frac{n}{2}]}+1)^2}\Big)[,$$

see Section 8.2 for further results on the spectral gap.
Bordoni's results were generalized by M. Bordoni and O. Hijazi in the Kähler setting [64], where essentially the Friedrich-like term in the lower bound must be replaced by the Kirchberg-type one of inequality (3.10) in odd complex dimension.
Comparisons between Dirac and Laplace eigenvalues which go the other way round can be obtained in particular situations. In the case of surfaces, J.-F. Grosjean and E. Humbert proved the following (see also [22]).

**Theorem 8.1.3 (J.-F. Grosjean and E. Humbert [116])** *Let $[g]$ be a conformal class on a closed orientable surface $M^2$ with fixed spin structure, then* [116, Cor. 1.2]

$$\inf_{\overline{g}\in[g]}\Big(\frac{\lambda_1(D_{\overline{g}}^2)}{\lambda_1(\Delta_{\overline{g}})}\Big)\leq\frac{1}{2},\tag{8.2}$$

*where here $\lambda_1(D_{\overline{g}}^2)$ denotes the smallest* positive *eigenvalue of $D_{\overline{g}}^2$.*

Inequality (8.2) is optimal and sharp for $M^2=\mathbb{S}^2$: indeed for any Riemannian metric $g$ one has $\lambda_1(D^2_{\mathbb{S}^2,g})\geq\frac{\lambda_1(\Delta_{\mathbb{S}^2,g})}{2}$ as a straightforward consequence of Bär's inequality (3.17) and Hersch's inequality (3.22). Moreover, (8.2) completes [1] where I. Agricola, B. Ammann and T. Friedrich prove the existence of a 1-parameter family $(g_t)_{t\geq0}$ of $\mathbb{S}^1$-invariant Riemannian metrics on $\mathbb{T}^2$ for which, in the same notations as just above, $\lambda_1(\Delta_{\mathbb{T}^2,g_t})<\lambda_1(D^2_{\mathbb{T}^2,g_t})$ for any $t\geq0$, where $\mathbb{T}^2$ is endowed with its trivial spin structure. The inequality $\lambda_k(\Delta_{\mathbb{T}^2,g})\geq\lambda_k(D^2_{\mathbb{T}^2,g})$ for $k$ large enough and for particular metrics $g$ on $\mathbb{T}^2$ with trivial spin structure has been proved independently by M. Kraus [172].
In the case where the manifold sits as a hypersurface in some spaceform, the best known result is the following.

**Theorem 8.1.4 (C. Bär [43])** *Let $(M^n,g)$ be isometrically immersed into $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$ and carry the induced spin structure, then* [43, Thm. 5.1]

$$\lambda_N(D^2)\leq\frac{n^2}{4}\Big(\sup_M(H^2)+\kappa\Big)+\lambda_{[\frac{N-1}{2^\mu}]}(\Delta)\tag{8.3}$$

*for every positive $N\in\mathbb{N}$, where $\kappa\in\{0,1\}$ denotes the sectional curvature of the ambient space, $H$ denotes the mean curvature of $M^n$ and $\mu$ is the integer defined by $\mu:=[\frac{n+1}{2}]-n\,\mathrm{mod}\,2$.*

Inequality (8.3) follows from the min-max principle and from (5.16) where one chooses $f$ to be an eigenfunction of $\Delta$ and $\psi$ to be the restriction of a non-zero Killing spinor.

## 8.2 Spectral gap

Another method to obtain information on the eigenvalues consists in estimating their difference, which is called the *spectral gap*. Initiated by H.C. Yang (see reference in [79]) for the scalar Laplacian, this approach turns out to provide similar results for the Dirac operator. The proof of the following theorem relies on the min-max principle and a clever input of coordinate functions of the immersion into the Rayleigh quotient, see [79] for details.

**Theorem 8.2.1 (D. Chen [79])** *Let $(M^n, g)$ be any $n$-dimensional closed immersed Riemannian spin submanifold of $\mathbb{R}^N$ for some $N \geq n+1$. Denote the spectrum of $D^2$ by $\{\lambda_j(D^2)\}_{j\geq 1}$ and set, for every $j \geq 1$,*

$$\mu_j := \lambda_j(D^2) + \frac{1}{4}\big(n^2 \sup_M(H^2) - \inf_M(S)\big),$$

*where $H$ and $S$ are the mean and the scalar curvature of $M$ respectively. Then for any $k \geq 1$*

$$\sum_{j=1}^{k}(\mu_{k+1} - \mu_j)(\mu_{k+1} - (1 + \frac{4}{n})\mu_j) \leq 0. \tag{8.4}$$

Note that the codimension of $M$ is arbitrary and that no compatibility condition between the spin structure of $M$ and that of $\mathbb{R}^N$ is required. Elementary computations show that inequality (8.4) implies

$$\mu_{k+1} \leq \frac{1}{k}(1 + \frac{4}{n})\sum_{j=1}^{k}\mu_j,$$

which itself provides

$$\mu_{k+1} - \mu_k \leq \frac{4}{nk}\sum_{j=1}^{k}\mu_j,$$

which had been shown independently by N. Anghel [29]. In particular precise estimates on the growth rate of the Dirac eigenvalues can be deduced. Theorem 8.2.1 has been extended by D. Chen and H. Sun to holomorphically immersed submanifolds of the complex projective space [80, Thm. 3.2].

## 8.3 Pinching Dirac eigenvalues

If Friedrich's inequality (3.1) is an equality for the smallest eigenvalue $\lambda_1(D^2)$, then from Theorem 3.1.1 and Proposition A.4.1 the underlying Riemannian manifold must be Einstein, which is a quite rigid geometric condition. Does the manifold remain "near to" Einstein if $\lambda_1(D^2)$ - or at least some lower eigenvalue - is close enough to Friedrich's lower bound? This kind of issue is designed under the name *eigenvalue pinching*. It addresses the continuous dependence of the geometry on the spectrum, in a sense that must be precised. We denote in the rest of this section by $K_{\mathrm{sec}}$, diam and $S$ the sectional curvature, diameter and scalar curvature of a given Riemannian manifold respectively. We also call two spin manifolds *spin diffeomorphic* if there exists a spin-structure-preserving diffeomorphism between them.

The first pinching result for Dirac eigenvalues is due to B. Ammann and C. Sprouse. It deals with the case where the scalar curvature almost vanishes. Tori with flat metric and trivial spin structure carry a maximal number of linearly independent parallel (hence harmonic) spinors. Theorem 8.3.1 below states that, under boundedness assumptions for the diameter and the sectional and scalar curvatures, one stays near to a flat torus in case some lower Dirac eigenvalue is not too far away from 0. Recall that a nilmanifold is the (left or right) quotient of a nilpotent Lie group by a cocompact lattice. If a (left or right) invariant metric is fixed on the nilmanifold, then the trivial lift of the lattice to the spin group provides a spin structure called the trivial one, see Proposition 1.4.2 for spin structures on coverings.

**Theorem 8.3.1 (B. Ammann and C. Sprouse [28])** *Let $K, d$ be positive real constants, $n \geq 2$ be an integer, $r := 1$ if $n = 2, 3$ and $r := 2^{[\frac{n}{2}]-1} + 1$ if $n \geq 4$. Then there exists an $\varepsilon = \varepsilon(n, K, d) > 0$ such that every $n$-dimensional closed Riemannian spin manifold $(M^n, g)$ with*

$$|K_{\mathrm{sec}}(M^n, g)| < K, \quad \mathrm{diam}(M^n, g) < d, \quad S(M^n, g) > -\varepsilon \text{ and } \lambda_r(D^2_{M^n,g}) < \varepsilon$$

*is spin diffeomorphic to a nilmanifold with trivial spin structure.*

Theorem 8.3.1 implies the existence of a uniform lower eigenvalue bound for the Dirac operator in the following family: there exists an $\varepsilon = \varepsilon(n, K, d) > 0$ such that on every $n$-dimensional closed Riemannian spin manifold $(M^n, g)$ with $|K_{\mathrm{sec}}(M^n, g)| < K$, $\mathrm{diam}(M^n, g) < d$, $S(M^n, g) > -\varepsilon$ and which is not spin diffeomorphic to a nilmanifold with trivial spin structure the $r^{\mathrm{th}}$ eigenvalue of $D^2$ satisfies

$$\lambda_r(D^2) \geq \varepsilon.$$

The choice for $r$, which looks *a priori* curious, is actually optimal since the product of a so-called K3-surface with a torus carries exactly $r - 1$ linearly independent parallel spinors, see [28, Ex. (2) p.411]. The proof of Theorem 8.3.1 makes use of an approximation result by U. Abresch (see reference in [28]) in an essential way, we refer to [28, Sec. 7] for details.
Under the supplementary assumption of a lower bound on the volume, the metric can even be shown to stay near to some with parallel spinors.

**Theorem 8.3.2 (B. Ammann and C. Sprouse [28])** *Let $K, d, V, \delta$ be positive real constants, $n \geq 2$ be an integer, $r := 1$ if $n = 2, 3$ and $r := 2^{[\frac{n}{2}]-1} + 1$ if $n \geq 4$. Then there exists an $\varepsilon = \varepsilon(n, K, d, V, \delta) > 0$ such that for every $n$-dimensional closed Riemannian spin manifold $(M^n, g)$ with*

$$|K_{\mathrm{sec}}(M^n, g)| < K, \quad \mathrm{diam}(M^n, g) < d, \quad S(M^n, g) > -\varepsilon, \quad \mathrm{Vol}(M^n, g) > V$$

*and $\lambda_r(D^2_{M^n,g}) < \varepsilon$, the metric $g$ is at $C^{1,\alpha}$-distance at most $\delta$ to a metric admitting a non-zero parallel spinor.*

The proof of Theorem 8.3.2 relies on a similar general eigenvalue pinching valid for arbitrary rough Laplacians on arbitrary vector bundles due to P. Petersen (see reference in [28]) and on the Schrödinger-Lichnerowicz formula (3.2). Petersen's method can also be applied to the rough Laplacian associated to the deformed covariant derivative $X \mapsto \nabla_X + \rho X \cdot$ and in this case it provides the following:

**Theorem 8.3.3 (B. Ammann and C. Sprouse [28])** *Let $K, d, V, \rho, \delta$ be positive real constants, $n \geq 2$ be an integer, $r := 1$ if $n = 2, 3$ and $r := 2^{[\frac{n}{2}]-1} + 1$ if $n \geq 4$. Then there exists an $\varepsilon = \varepsilon(n, K, d, V, \rho, \delta) > 0$ such that for every $n$-dimensional closed Riemannian spin manifold $(M^n, g)$ with*

$$|K_{\mathrm{sec}}(M^n, g)| < K, \ \mathrm{diam}(M^n, g) < d, \ \mathrm{Vol}(M^n, g) > V, \ S(M^n, g) \geq n(n-1)\rho^2$$

*and* $\lambda_r(D^2_{M^n,g}) < \frac{n^2\rho^2}{4} + \varepsilon$, *the metric* $g$ *is at* $C^{1,\alpha}$*-distance at most* $\delta$ *to a metric with constant sectional curvature* $\rho^2$.

Note that the bound on the sectional curvature is necessary because of Bär-Dahl's result [49] discussed in Section 3.2. However the minimal number $r$ necessary for the result to hold can be enhanced.

**Theorem 8.3.4 (A. Vargas [237])** *The conclusion of* Theorem 8.3.3 *holds with*

$$r := \begin{cases} 3 & \text{if } n = 6 \text{ or } n \equiv 1\ (4) \\ \frac{n+9}{4} & \text{if } n \equiv 3\ (4). \end{cases}$$

## 8.4 Spectrum of other Dirac-type operators

Up to now we have concentrated onto the fundamental (or spin) Dirac operator on a spin manifold. As already mentioned at the beginning of Chapter 1, Dirac-type operators may be defined in the more general context where a so-called Clifford bundle [178, Sec. II.3] is at hand. Roughly speaking, a Clifford bundle is given by a Hermitian vector bundle together with a covariant derivative and on which the tangent bundle acts by Clifford multiplication such that all three objects (Hermitian metric, covariant derivative and Clifford multiplication) are compatible with each other in the sense of Definition 1.2.2 and Proposition 1.2.3. The associated Dirac operator is defined as the Clifford multiplication applied to the covariant derivative. One may add a zero-order term and obtain a so-called Dirac-Schrödinger operator. In this section we discuss spectral results in relation with the spin$^c$ Dirac operator, with twisted Dirac-Schrödinger operators, with Dirac operators associated to particular geometrically relevant connections, with the basic Dirac operator and in the pseudo-Riemannian setting.

First, the concept of spin structure may be weakened to that of spin$^c$ structure, whose structure group is the spin$^c$ group $\mathrm{Spin}_n^c := {}^{\mathrm{Spin}_n \times \mathbb{S}^1}\!/_{\mathbb{Z}_2}$. Such a structure comes along with a $\mathbb{S}^1$-principal bundle, or equivalently with a complex line bundle $\mathcal{L}$. We do not want to define spin$^c$ structures more precisely but mention that all spin manifolds are spin$^c$ and that all almost-Hermitian manifolds have a canonical spin$^c$ structure [178, App. D]. Moreover the choice of a covariant derivative on the line bundle induces a covariant derivative and hence a Dirac operator on the associated spinor bundle over the underlying manifold. In that case it can be expected that most of the results valid for the spin Dirac operator remain valid for the spin$^c$ one, except that the curvature of the line bundle must in some situations be taken into account. For example, M. Herzlich and A. Moroianu proved the analog of Hijazi's inequality (3.18) in the spin$^c$ context: denote by $\omega$ the curvature form of the line bundle $\mathcal{L}$ and by $\mu_1$ the smallest eigenvalue of the scalar operator $L_\omega := 4\frac{n-1}{n-2}\Delta + S - 2[\frac{n}{2}]^{\frac{1}{2}}|\omega|$, then any eigenvalue $\lambda$ of the spin$^c$ Dirac operator satisfies [130, Thm. 1.2]

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1.$$

We note however that little has been done in the spin$^c$ context in comparison with the spin one.

If the underlying space is again our familiar spin manifold $(M^n, g)$ and if we choose an arbitrary Riemannian or Hermitian vector bundle $E$ over $M$, then the tensor product bundle $\Sigma M \otimes E$ carries a canonical Clifford multiplication (extend the Clifford multiplication by the identity on the second factor). If we endow $E$ with a metric covariant derivative, then we obtain a structure of Clifford bundle and an associated Dirac operator called Dirac operator of $M$ twisted with $E$. This operator is usually denoted by $D_M^E$. For example, the Euler operator $d+\delta$ can be seen as the Dirac operator of $M$ twisted with $\Sigma M$: this follows essentially from (1.2) and may actually be stated without any spin structure on $M$ [178, Sec. II.6]. Another prominent example is the Dirac operator of a spin submanifold twisted with the spinor bundle of its normal bundle (where the latter is assumed to be spin). Various studies have been devoted to the spectrum of twisted Dirac operators, therefore we restrict ourselves to a few ones which we hope to be representative. We include all that concerns Dirac-Schrödinger operators, since in that case the zero order term mainly translates the upper or lower bounds by a constant.

Let first $E$ be as above, $M$ be closed and $f$ be a smooth real function on $M$. Denote by $\kappa_1$ the smallest eigenvalue on $M$ of the pointwise linear operator $-2\sum_{k,l=1}^n e_k \cdot e_l \cdot R^E_{e_k,e_l}$, where $R^E$ is the curvature tensor of the chosen covariant derivative on $E$ (and $(e_k)_{1\leq k\leq n}$ is a local o.n.b. of $TM$). If the inequalities $n(S+\kappa_1) > (n-1)f^2 > 0$ hold on $M$, then any eigenvalue $\lambda$ of the Dirac-Schrödinger operator $D_M^E - f$ acting on $\Gamma(\Sigma M \otimes E)$ satisfies [108, Prop. 4.1]

$$\lambda^2 \geq \frac{1}{4}\inf_M \Big(\sqrt{\frac{n}{n-1}(S+\kappa_1)} - |f|\Big)^2. \tag{8.5}$$

Inequality (8.5), which can be deduced from a clever choice of modified covariant derivative, stands for the analog of Friedrich's inequality in this context, see [108] for other kinds of estimates and references to earlier works on that topic (such as [202]). In the particular case where $n = 4$, $f = 0$, $E$ is arbitrary and carries a selfdual covariant derivative, the estimate (8.5) can be enhanced using the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ and the vanishing of one half of the auxiliary curvature term computed from $R^E$: H. Baum proved [55, Thm. 2] that

$$\lambda^2 \geq \frac{1}{3}\inf_M(S)$$

for any eigenvalue $\lambda$ of $D_M^E$, which is exactly Friedrich's inequality (3.1) for the eigenvalues of the spin Dirac operator.

Staying in dimension 4, if the spin manifold $(M^4, g)$ carries a Hermitian structure $J$ (i.e., an orthogonal complex structure on $TM$) then one is led to the Dolbeault operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ twisted with $E = K_M^{\frac{1}{2}}$, where $K_M^{\frac{1}{2}}$ is the square-root of the canonical line bundle $K_M := \Lambda^2_{\mathbb{C}} T^*M$ which determines the spin structure. Although Kirchberg's inequality (3.10) does not apply, sharp lower bounds for the eigenvalues of the Dolbeault operator are still available: B. Alexandrov, G. Grantcharov and S. Ivanov proved [6, Thm. 2] that

$$\lambda^2 \geq \frac{1}{6}\inf_M(S)$$

for any eigenvalue $\lambda$ of $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$. Beware that equality cannot occur for a non-flat Kähler metric because of (3.10). The proof of that inequality relies on

Weitzenböck formulas and the clever choice of twistor operators associated to a canonical one-parameter-family of Hermitian connections, we refer to [6] for details. Besides, we mention that upper eigenvalue bounds for particular twisted Dirac operators have been obtained in [55], [43] and [104].

From the point of view of geometers investigating the integrability of particular $G$-structures, there exists another interesting family of Dirac-type operators which are usually denoted by $D^{\frac{1}{3}}$ and defined by $D^{\frac{1}{3}} := D_g + \frac{T}{4}\cdot$, where $T$ is some given 3-form and $D_g$ is the spin Dirac operator on the Riemannian spin manifold $(M^n, g)$. For example if $(M^n, g)$ is a so-called reductive homogeneous space then $D^{\frac{1}{3}}$ is the so-called Kostant Dirac operator (see reference in [3]); if $(M^n, g)$ is a Hermitian manifold then $D^{\frac{1}{3}}$ coincides with the Dolbeault-operator defined just above. In case $T$ is the characteristic torsion of a 5-dimensional closed spin Sasaki manifold with scalar curvature bounded from below, the use of suitable deformations of the connection by polynomials of the torsion form allowed I. Agricola, T. Friedrich and M. Kassuba to prove the following estimates of any eigenvalue $\lambda$ of $(D^{\frac{1}{3}})^2$ [3, Thm. 4.1]:

$$\lambda \geq \left| \begin{array}{ll} \frac{1}{16}(1 + \frac{1}{4}\inf_M(S))^2 & \text{if } -4 < S \leq 4(9+4\sqrt{5}) \\ \\ \frac{5}{16}\inf_M(S) & \text{if } S \geq 4(9+4\sqrt{5}). \end{array} \right.$$

Equality holds if $(M^5, g)$ is $\eta$-Einstein (see [3] for a definition). Surprisingly enough the first lower bound depends quadratically on the scalar curvature, which makes the estimate better for small $S$. We refer to [3] for the proof. We also note that in the context of contact metric manifolds (which have a canonical spin$^c$ structure) Weitzenböck formulas for the Dirac operator associated to the so-called Tanaka-Webster connection have also been produced in order to prove vanishing theorems [211], however no study of the spectrum is still available.

Sasaki manifolds can also be viewed as particular foliated Riemannian manifolds. Spin structures can be defined on Riemannian foliations in much the same way as on the tangent bundle and an associated covariant derivative and Dirac operator may be defined which are called the transversal covariant derivative and transversal Dirac operator respectively. The transversal Dirac operator, which acts on the space of basic spinors (spinors whose transversal covariant derivatives vanish along all directions normal to the leaves), is in general not formally self-adjoint, therefore one considers the symmetrized operator called *basic* Dirac operator of the foliation and denoted by $D_b$. It is a not-so-straightforward adaptation of the proof of Friedrich's inequality by G. Habib and K. Richardson to show that any eigenvalue $\lambda$ of $D_b$ on a closed underlying manifold $(M^n, g)$ satisfies [126, Eq. (1.1)]

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M(S^{\mathrm{tr}}),$$

where $q \geq 2$ stands for the codimension of the foliation and $S^{\mathrm{tr}}$ for its transversal scalar curvature. In case the normal bundle of the foliation carries a Kähler or a quaternionic Kähler structure, analogs of Kirchberg's inequality (3.10) and of (3.15) can also be derived [155, 156, 124, 123, 122].

To close this section we mention the only result known to us about the spectrum of the Dirac operator in the *pseudo-Riemannian* (non-Riemannian) setting. First spin structures require the pseudo-Riemannian manifold to be simultaneously space- and time-oriented in order to be well-defined, see [52] or [56, Sec. 2]. In that case the choice of a maximal timelike subbundle induces an $\mathrm{L}^2$-Hermitian inner product on the space of spinors. Unlike its Riemannian version the associated (spin) Dirac operator is neither formally self-adjoint w.r.t. that inner product nor elliptic. However H. Baum could show with the help of suitable endomorphisms of the spinor bundle commuting or anti-commuting with the Dirac operator that the point spectrum, the continuous spectrum and the residual spectrum of the Dirac operator on any even-dimensional pseudo-Riemannian manifold are symmetric w.r.t. the real and imaginary axes. We refer to [56] for further statements and the proof.

## 8.5 Conformal spectral invariants

In this section we are interested in two invariants associated to the Dirac spectrum. A good reference for the whole section is [153]. Given a closed spin manifold $M^n$ with fixed conformal class $[g]$ and spin structure denoted by $\epsilon$, let $\lambda_1(D^2_{M,g})$ be the smallest eigenvalue of the square of the Dirac operator of $(M^n, g)$. The *Bär-Hijazi-Lott invariant* [13, eq. (2.4.1) p.12] of $(M^n, [g], \epsilon)$ is the nonnegative real number $\lambda_{\min}(M^n, [g], \epsilon)$ defined by

$$\lambda_{\min}(M^n, [g], \epsilon) := \inf_{\overline{g}\in[g]} \left( \sqrt{\lambda_1(D^2_{M,\overline{g}})} \cdot \mathrm{Vol}(M, \overline{g})^{\frac{1}{n}} \right).$$

Of course the expression on the r.h.s. is chosen so as to remain scaling-invariant. By definition $\lambda_{\min}(M^n, [g], \epsilon)$ is a conformal invariant. The Bär-Hijazi-Lott invariant is tightly connected to and behaves much like the Yamabe invariant. Indeed, it already follows from Bär's inequality (3.17) and from Hijazi's inequality (3.20) that

$$\lambda_{\min}(M^2, [g], \epsilon)^2 \geq 2\pi\chi(M^2) \quad \text{and} \quad \lambda_{\min}(M^n, [g], \epsilon)^2 \geq \frac{n}{4(n-1)} Y(M, [g]) \tag{8.6}$$

for every $n \geq 3$, where $\chi(M^2)$ and $Y(M, [g])$ are the Euler characteristic and the Yamabe invariant respectively. For $M^2 = \mathbb{S}^2$ this implies that the Bär-Hijazi-Lott invariant is positive. More generally, as a consequence of J. Lott's estimate (3.21), the Bär-Hijazi-Lott invariant is positive as soon as the Dirac operator is invertible for some - hence any - metric in the conformal class. In particular $\lambda_{\min}(M^n, [g], \epsilon)$ vanishes if and only if $(M^n, g)$ admits non-zero harmonic spinors. Generalizing J. Lott's Sobolev-embedding techniques [187] to the case where the Dirac kernel is possibly non-trivial, B. Ammann showed the positivity of $\inf_{\overline{g}\in[g]} \left( \sqrt{\lambda^+(D^2_{M,\overline{g}})} \cdot \mathrm{Vol}(M, \overline{g})^{\frac{1}{n}} \right)$ to hold true in general [12, Thm. 2.3], where $\lambda^+(D^2_{M,\overline{g}})$ denotes the smallest positive eigenvalue of $D^2_{M,\overline{g}}$. As an example, the Bär-Hijazi-Lott invariant of $\mathbb{S}^n$ ($n \geq 2$) with standard conformal class [can] and canonical spin structure is given by $\frac{n}{2}\omega_n^{\frac{1}{n}}$, where $\omega_n$ is the volume of $\mathbb{S}^n$ carrying the metric of sectional curvature 1 (denoted by "can"): this follows from Corollaries 3.3.2 and 3.3.3 together with $\lambda_1(D^2_{\mathbb{S}^n,\mathrm{can}}) = \frac{n^2}{4}$ and

$Y(\mathbb{S}^n, [\mathrm{can}]) = n(n-1)\omega_n^{\frac{2}{n}}$ if $n \geq 3$.
In a similar way as for the Yamabe invariant, the Bär-Hijazi-Lott invariant cannot be greater than that of the sphere: if $n \geq 2$ then
$\inf_{\overline{g}\in[g]} \left(\sqrt{\lambda^+(D^2_{M,\overline{g}})} \cdot \mathrm{Vol}(M,\overline{g})^{\frac{1}{n}}\right) \leq \lambda_{\min}(\mathbb{S}^n, [\mathrm{can}])$, in particular

$$\lambda_{\min}(M^n, [g], \epsilon) \leq \lambda_{\min}(\mathbb{S}^n, [\mathrm{can}]). \tag{8.7}$$

This was proved by B. Ammann [12, Thm. 3.1 & 3.2] for $n \geq 3$ or $M^2 = \mathbb{S}^2$ and by B. Ammann, J.-F. Grosjean, E. Humbert and B. Morel [22, Thm. 1.1] in general. The proof relies on a suitable cut-off argument performed on Dirac eigenvectors on the gluing of a sphere with large radius to the manifold, see [12, Sec. 3] and [22] respectively for the details.
The next step would consist in showing that (8.7) is a strict inequality if $(M^n, [g])$ is not conformally equivalent to $(\mathbb{S}^n, [\mathrm{can}])$. This has been done by B. Ammann, E. Humbert and B. Morel in the conformally flat setting where one introduces a further datum, namely the so-called *mass endomorphism*. The mass endomorphism of a locally conformally flat Riemannian spin manifold is a self-adjoint endomorphism field of its spinor bundle and can be locally defined out of the difference between the Green's operators for the Dirac operators associated to the original metric and to the Euclidean one in suitable coordinates, see [26, Def. 2.10] for a precise definition. The name comes from the corresponding term for the Yamabe operator and which is known to provide the mass of an asymptotically flat Riemannian spin manifold. Moreover, the mass endomorphism is "well-behaved" regarding conformal changes of metric [26, Prop. 2.9]. In case the locally conformally flat manifold $(M^n, [g])$ has an invertible Dirac operator (for some hence any metric in the conformal class) and if its mass endomorphism has a non-zero eigenvalue somewhere on $M^n$, then [26, Thm. 1.2]

$$\lambda_{\min}(M^n, [g], \epsilon) < \lambda_{\min}(\mathbb{S}^n, [\mathrm{can}]). \tag{8.8}$$

At this point one should beware that the mass endomorphism of $(\mathbb{S}^n, [\mathrm{can}])$ vanishes and that this does not characterize the round sphere since flat tori also have vanishing mass endomorphism. We refer to [26] for the details. Interestingly enough, generic metrics on 3-dimensional manifolds have a non-zero mass endomorphism [128, Thm. 1.1], in particular (8.8) holds for those metrics. For a generalization of the Bär-Hijazi-Lott invariant to manifolds with non-empty boundary we refer to [218, 220].
We also mention that the Bär-Hijazi-Lott invariant has been generalized to the noncompact setting, where it provides an obstruction to the existence of conformal spin compactifications of the manifold [117]. More precisely, let $M^n$ be any $n$-dimensional manifold with conformal class $[g]$ and spin structure $\epsilon$ and define $\lambda^+_{\min}(M^n, [g], \epsilon)$ as in Section 7.3. If

$$\lim_{r\to\infty} \lambda^+_{\min}(M^n \setminus \overline{B}_r(p), [g], \epsilon) < \lambda_{\min}(\mathbb{S}^n, [\mathrm{can}]),$$

where $p \in M$ is arbitrary, then $(M^n, [g])$ is not conformal to a subdomain with induced spin structure of a closed spin manifold [119, Thm. 3.0.1] (see also [117, Thm. 1.4]). The vanishing of $\lambda^+_{\min}(M^n, [g], \epsilon)$ also prevents the existence of conformal spin compactifications of $M$, since $\lambda^+_{\min}(M^n, [g], \epsilon) > 0$ on closed manifolds [12, Thm. 2.3] and a monotonicity principle holds for $\lambda^+_{\min}$ [119, Lemma 2.0.3], see [119, Rem. 3.0.4].

The Green's operators for the Dirac operator have also revealed as a powerful tool in general problems from geometric analysis such as the classical Yamabe conjecture (*"find a metric with constant scalar curvature in a fixed conformal class"*). As shown by R. Schoen, the Yamabe conjecture is implied by the positive mass theorem through the fact that the constant term in the asymptotic expansion in inverted normal coordinates of the Green's operator for the conformal Laplace operator is proportional to the mass of the conformal blow-up. Furthermore but independently, E. Witten [241] showed that in the spin setting the positive mass theorem is in turn implied by the existence of spinor field which is harmonic and asymptotically constant on the conformal blow-up. Now it is a striking result by B. Ammann and E. Humbert [23] that the Green's operators for the Dirac operator provide such a spinor. More precisely, if $(M^n, g)$ is closed Riemannian spin manifold with positive Yamabe invariant and which is locally conformally flat if $n \geq 6$, then its conformal blow-up has positive mass. For positive mass theorems we refer to Section 8.8.

If one lets the conformal class vary on the closed manifold $M^n$, then one is led to the so-called $\tau$-invariant of $M^n$ with spin structure $\epsilon$ and which is defined by

$$\tau(M^n, \epsilon) := \sup_{[g]} \big(\lambda_{\min}(M^n, [g], \epsilon)\big).$$

The introduction of the spinorial invariant $\tau$ is inspired from that of R. Schoen's $\sigma$-invariant which is defined in dimension 2 by $\sigma(M^2) := 4\pi\chi(M^2)$ and in dimension $n \geq 3$ by

$$\sigma(M^n) := \sup_{[g]} \big(Y(M^n, [g])\big),$$

where $Y(M^n, [g])$ denotes the Yamabe invariant on $(M^n, [g])$. There are at least two motivations for the study of the $\tau$-invariant. First, the $\tau$-invariant bounds the $\sigma$-invariant from above since it follows from (8.6) that, in every dimension $n \geq 2$,

$$\tau(M^n, \epsilon)^2 \geq \frac{n}{4(n-1)}\sigma(M^n),$$

with equality for $\mathbb{S}^n$. Therefore upper bounds for $\tau(M^n, \epsilon)$ provide upper bounds for $\sigma(M^n)$, on which little is known. In an independent context, the inequality $\lambda_{\min}(M^2, [g], \epsilon) < 2\sqrt{\pi} = \tau(\mathbb{S}^2)$ guarantees the existence of a metric $\overline{g} \in [g]$ for which any simply-connected open subset of $(M^2, \overline{g})$ can be isometrically embedded with constant mean curvature into $\mathbb{R}^3$ [13, Sec. 5.4] (see also [16]). Hence it is of geometric interest to know when the inequality $\tau(M^2, \epsilon) < 2\sqrt{\pi}$ holds. In case $M^2 = \mathbb{T}^2$ B. Ammann and E. Humbert have shown [24, Thm. 1.1] that $\tau(\mathbb{T}^2, \epsilon) = 2\sqrt{\pi}$ for any of its non-trivial spin structures $\epsilon$ (obviously $\tau(\mathbb{T}^2, \epsilon_0) = 0$ for the trivial spin structure $\epsilon_0$). Note that this neither proves nor contradicts the existence of immersed constant mean curvature tori in $\mathbb{R}^3$. As a generalization, the $\tau$-invariant of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ is equal to zero if $n = 2$ and $\mathbb{S}^1$ carries the trivial spin structure and to $\tau(\mathbb{S}^n)$ otherwise [24, Thm. 1.2]. More recently, B. Ammann and E. Humbert have shown that the $\tau$-invariant does not decrease when adding a handle to the manifold [25, Cor. 1.2]. Interestingly enough, it can be deduced from that fact that the $\tau$-invariant of a closed oriented surface is given by [25, Thm. 1.3]

$$\tau(M^2, \epsilon) = (1 - \alpha(M)) \cdot 2\sqrt{\pi},$$

where $\alpha(M) \in \mathbb{Z}_2$ is the $\alpha$-genus of $(M^2, \epsilon)$. In particular $\tau(M^2, \epsilon)$ is always either equal to 0 or to that of $\mathbb{S}^2$ according to the spin structure.

## 8.6 Convergence of eigenvalues

Given a converging sequence of closed Riemannian spin manifolds, does their Dirac spectrum have to converge to that of the limit? Three very different contexts have up to now been considered where this question can be given sense and answered. The simplest and historically the first one deals with the behaviour of the Dirac spectrum of $\mathbb{S}^1$-bundles under collapse. In that case the behaviour depends sensitively of the spin structure as shown by B. Ammann and C. Bär [17]. Let $M$ denote the total space of an $\mathbb{S}^1$-bundle which is simultaneously a Riemannian submersion with totally geodesic fibres over a base manifold $B$. Two kinds of spin structures can be defined on $M$ according to whether the $\mathbb{S}^1$-action can be lifted to the spin level or not; in the former case the spin structure is called projectable and in the latter it is called non-projectable. Projectable spin structures on $M$ stand in one-to-one correspondence with spin structures on $B$. The main result of [17] states the following about the convergence of the Dirac spectrum of $M$ as the fibre-length goes to 0: either the spin structure of $M$ is projectable and there exist Dirac eigenvalues of $M$ converging to those of $B$ or it is non-projectable and all Dirac eigenvalues of $M$ tend to $\infty$ or $-\infty$ [17, Thm. 4.1 & 4.5]. As an interesting application, the Dirac spectrum of all complex odd-dimensional complex projective spaces can be deduced from that of the Berger spheres (Theorem 2.2.2). Parts of those results have been generalized by B. Ammann to $\mathbb{S}^1$-bundles with non-geodesic fibres [7, 8].

The second natural context deals with hyperbolic degenerations, i.e., with sequences of closed hyperbolic spin manifolds $(M_j)_{j\in\mathbb{N}}$ converging to a non-compact complete hyperbolic spin manifold $M$ (here a hyperbolic metric is a metric with constant sectional curvature $-1$). Those sequences only exist in dimensions 2 and 3 and, provided the convergence respects the spin structures in some sense, the limit manifold must have discrete Dirac spectrum in dimension 3 whereas it may have continuous spectrum in dimension 2, see references in [214] where a precise description of hyperbolic degenerations is recalled. In case the limit manifold $M$ is assumed to have discrete Dirac spectrum, F. Pfäffle proved the convergence of the Dirac spectrum of $(M_j)_{j\in\mathbb{N}}$ in the following sense [214, Thm. 1.2] (see also [213]): For all $\varepsilon > 0$ and $\Lambda \geq 0$, there exists an $N \in \mathbb{N}$ such that for all $j \geq N$ the real number $\Lambda$ lies neither in the spectrum of $D$ nor in that of $D_{M_j}$, both Dirac operators $D_{M_j}$ and $D$ have only discrete eigenvalues and no other spectrum in $[-\Lambda, \Lambda]$, they have the same number $m$ of eigenvalues in $[-\Lambda, \Lambda]$ which can be ordered so that $|\lambda_k^{(j)} - \lambda_k| \leq \varepsilon$ holds for all $1 \leq k \leq m$.

The diameter of the converging sequence of degenerating hyperbolic manifolds cannot be controlled since the limit-manifold must have a finite number of so-called cusps, which by definition are unbounded. The third context to have been considered precisely deals with the situation where both the diameter and the sectional curvature of the converging sequence are assumed to remain bounded. In that case J. Lott proved the following very general result [189]. Consider a sequence $(g_j)_{j\in\mathbb{N}}$ of bundle metrics on the total space of a spin fibre bundle $M$ over a base spin manifold $B$. Assume the fibre length to go to 0 as $j$ tends to $\infty$ while both the diameter and the sectional curvature of $(M, g_j)$ remain bounded. Then the Dirac spectrum of $(M, g_j)$ converges in the sense just above to that of some differential operator of first order on $B$ which can be explicitly constructed. Since a precise formulation and the discussion of the results would require too many details we recommend the introduction of [189].

## 8.7 Eta-invariants

As we have seen in Theorem 1.3.7, the Dirac spectrum of any closed $n$-dimensional Riemannian spin manifold is symmetric w.r.t. the origin in dimension $n \not\equiv 3$ (4). To measure the asymmetry of the Dirac spectrum in case $n \equiv 3$ (4), Atiyah, Patodi and Singer introduced [30] the so-called $\eta$-invariant of $D$ which is defined by $\eta(D) := \eta(0, D)$, where, for every $s \in \mathbb{C}$ with $\Re e(s) > n$,

$$\eta(s, D) := \sum_{\lambda_j \neq 0} \frac{\mathrm{sgn}(\lambda_j)}{|\lambda_j|^s}.$$

The $\lambda_j$'s denote the eigenvalues of $D$. It is already a non-trivial statement that $s \mapsto \eta(s, D)$ can be meromorphically extended onto $\mathbb{C}$ and is regular at $s = 0$, see [30]. Originally the $\eta$-invariant was introduced to describe some boundary term in the Atiyah-Patodi-Singer index theorem [30]. In a simple-minded way, the $\eta$-invariant of $D$ can be thought of as the difference between the number of positive and that of negative Dirac eigenvalues (of course this has no sense since both numbers are infinite). In particular the $\eta$-invariant of $D$ vanishes as soon as the Dirac spectrum is symmetric.

Few $\eta$-invariants are known explicitly. One of the first computations of $\eta$-invariant goes back to Hitchin [152], where the explicit knowledge of the Dirac spectrum on the Berger sphere $\mathbb{S}^3$ allows the $\eta$-invariant to be explicited. This was generalized onto all Berger spheres by D. Koh [166]. In the flat setting, the $\eta$-invariant can also be deduced from the Dirac spectrum in dimension $n = 3$ [212] and for particular holonomies in dimension $n \geq 4$ [194]. Theorem 2.2.3 provides the $\eta$-invariant on particular closed 3-dimensional hyperbolic manifolds [224]. The most general formula allowing the determination of the $\eta$-invariant has been proved by S. Goette [111, Thm. 2.33] on homogeneous spaces, where $\eta(D)$ arises as the sum of three terms: a representation-theoretical expression, an index-theoretical one and so-called equivariant $\eta$-invariants, which can themselves be deduced from finer representation-theoretical data [109, 110].

Though unknown in most cases, the $\eta$-invariant behaves nicely under connected sums: roughly speaking, if a closed Riemannian spin manifold is separated in two pieces $M_1, M_2$ by a closed hypersurface $N$, about which both the metric and the Dirac operator split as on a Riemannian product, then the $\eta$-invariant of $D$ consists of the sum of the $\eta$-invariants of $D_{M_1}$ and $D_{M_2}$ plus the so-called Maslov-index of a pair of Lagrangian subspaces of $\mathrm{Ker}(D_N)$ making $D_{M_j}$ self-adjoint, plus some index-theoretical integers (U. Bunke [70, Thm 1.9]). We refer to [70] for an overview of $\eta$-invariants of general Dirac-type operators and numerous useful references.

We also mention that some kind of $\eta$-invariant can be defined in the non-compact setting, see [121] and references therein.

## 8.8 Positive mass theorems

Although this section has more to do with physics as with the Dirac spectrum, we include it because on the one hand the proofs of the results presented involve simple spinorial techniques as already used above, and on the other hand positive mass theorems nowadays play a central role in many other topics of global analysis such as the Yamabe problem. A good but not up-to-date reference for that topic is [129].
A positive mass theorem (sometimes called positive energy theorem) is a twofold statement reading roughly as follows: Let $(M^n, g)$ be a Riemannian manifold which is asymptotic to a model manifold (in a sense that must be precised) and some of which curvature invariant satisfies a pointwise inequality, then some asymptotic geometric invariant called its mass also satisfies a similar inequality and, if this latter inequality is an equality, then the whole manifold is globally isometric to the original model manifold. To fix the ideas we concentrate from now on onto the original positive mass theorem as proved by R. Schoen and S.-T. Yau [221, 222] and independently by E. Witten [241] in the spinorial setting, in particular we leave aside all recent developments in what has become a whole field of research at the intersection between mathematics and general relativity, see e.g. [242] for references.

Let $(M^n, g)$ be a Riemannian manifold of dimension $n \geq 3$. Call it asymptotically flat of order $\tau \in \mathbb{R}$ if there exists a compact subset $K \subset M$, a positive real number $R$ and a diffeomorphism $M \setminus K \longrightarrow \{x \in \mathbb{R}^n,\ |x| > R\}$ such that the pushed-out metric fulfills: $g_{ij} - \delta_{ij} = O(|x|^{-\tau})$, $\frac{\partial g_{ij}}{\partial x_k} = O(|x|^{-\tau-1})$ and $\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} = O(|x|^{-\tau-2})$ as $|x| \to \infty$, for all $1 \leq i, j, k, l \leq n$. Given such a manifold $(M^n, g)$, set

$$m(g) := \frac{1}{16\pi} \cdot \lim_{r \to \infty} \int_{S_r} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j}\right) \nu_j dA,$$

where $S_r$ denotes the Euclidean sphere of radius $r$ about $0 \in \mathbb{R}^n$ with outside unit normal $\nu$ and $dA$ its canonical measure. Beware here that in general $m(g)$ does not make any sense: the integral need not converge, and even if it converges it depends on the choice of asymptotic coordinates. If however $\tau > \frac{n-2}{2}$ and the scalar curvature of $(M^n, g)$ is integrable, then a highly non-trivial theorem of R. Bartnik (see reference in [129]) ensures $m(g)$ to be well-defined. In that case it is called the ADM-mass of $(M^n, g)$. The canonical example of asymptotically flat manifold (of any order) is $(\mathbb{R}^n, \mathrm{can})$, whose ADM-mass vanishes. The positive mass theorem states that, with the assumptions above and if the scalar curvature $S$ of $(M^n, g)$ is non-negative, then $m(g) \geq 0$ with equality if and only if $(M^n, g) = (\mathbb{R}^n, \mathrm{can})$. This is a very deep statement since it establishes a direct relationship between the geometry at infinity and the global geometry of $M$.

For example, as a consequence, any Riemannian metric on $\mathbb{R}^n$ with $S \geq 0$ and which is flat outside a compact subset must be flat. Surprisingly enough, the positive mass theorem follows from relatively simple considerations involving some kind of boundary value problem for the Dirac operator, at least in case $M$ is spin, as shown by E. Witten [241]. Let us sketch his idea.

The first and main step in Witten's proof consists in choosing any non-zero "constant" spinor field $\psi_0$ at infinity and exhibiting a sufficiently regular non-zero spinor field $\psi$ lying in the kernel of $D^2$ and being asymptotic to $\psi_0$. This can be done by showing the invertibility of $D^2$ between suitable Hölder spaces. Applying Schrödinger-Lichnerowicz' formula (1.15), integrating on a Euclidean ball of (sufficiently large) radius $r$ and using (3.29) together with Green's formula one obtains

$$0 = \int_{B_r} \langle D^2\psi, \psi\rangle v_g = \int_{B_r} (|\nabla\psi|^2 + \frac{S}{4}|\psi|^2) v_g - \int_{S_r} \langle \nabla_\nu \psi, \psi\rangle dA,$$

where $\nu$ denotes here the outer unit normal to $S_r = \partial B_r$. The miracle in Witten's proof happens here: it can be easily shown that the boundary term $\int_{S_r} \langle \nabla_\nu \psi, \psi\rangle dA$ is asymptotic to $m(g)$ times some finite positive constant $c$ as $r$ goes to $\infty$. After passing to the limit one is left with $m(g) = c(\int_M (|\nabla\psi|^2 + \frac{S}{4}|\psi|^2) v_g$, which implies $m(g) \geq 0$. The equality $m(g) = 0$ requires $\psi$ to be parallel for any $\psi$ constructed this way, in particular the spinor bundle of $(M^n, g)$ must be trivialized by parallel spinors, from which the identity $(M^n, g) = (\mathbb{R}^n, \mathrm{can})$ can be deduced. An alternative spinorial proof but with supplementary assumptions on the dimension or the Weyl tensor has been given by B. Ammann and E. Humbert [23] using the Green's operators associated to the Dirac operator, see Section 8.5.

# Appendix A

# The twistor and Killing spinor equations

In this section we recall basic facts as well as classification results concerning twistor and Killing spinors on Riemannian spin manifolds. The reader is invited to refer to [59, 73, 58, 66] for further statements and references.

## A.1 Definitions and examples

**Definition A.1.1** *Let $(M^n, g)$ be a Riemannian spin manifold.*

*i) A* twistor-spinor *on $(M^n, g)$ is a section $\psi$ of $\Sigma M$ solving*

$$P\psi = 0, \tag{A.1}$$

*where $P_X\psi := \nabla_X\psi + \frac{1}{n}X \cdot D\psi$ for every $X \in TM$.*

*ii) Given a complex number $\alpha$, an $\alpha$-*Killing spinor *on $(M^n, g)$ is a section $\psi$ of $\Sigma M$ solving*

$$\nabla_X\psi = \alpha X \cdot \psi \tag{A.2}$$

*for every $X \in TM$. In case $\alpha \in \mathbb{R}$ (resp. $\alpha \in i\mathbb{R}^*$) an $\alpha$-Killing spinor is called real Killing (resp. imaginary Killing) spinor.*

The operator $P : \Gamma(\Sigma M) \longrightarrow \Gamma(T^*M \otimes \Sigma M)$ is called the *Penrose* or *twistor* operator. It is obtained as the orthogonal projection of the covariant derivative onto the kernel of the Clifford multiplication $\mu : T^*M \otimes \Sigma M \longrightarrow \Sigma M$. Obviously a section of $\Sigma M$ is a Killing spinor if and only if it is a twistor-spinor which is an eigenvector of $D$. Beware that our definition of real Killing spinor contains that of parallel spinor, compare [59].

**Notes A.1.2**

1. The name "Killing spinor" originates from the fact that, if $\alpha$ is real, then the vector field $V$ defined by $g(V, X) := i\langle\psi, X \cdot \psi\rangle$ for all $X \in TM$, is a Killing vector field on $(M^n, g)$.

2. A spinor field $\psi$ is an $\alpha$-Killing spinor on $(M^n, g)$ if and only if it is an $\frac{\alpha}{\lambda}$-Killing spinor on $(M^n, \lambda^2 g)$, for any positive real constant $\lambda$.

3. One could give a slightly more general definition of Killing spinor, requiring (A.2) to hold for a given smooth complex-valued function $\alpha$ on $M$. If $\alpha$ is real-valued and $n \geq 2$, then O. Hijazi has shown [131] that it must be constant on $M$, see [138, Prop. 5.12]. On the other hand, H.-B. Rademacher has proved [215] the existence of (and actually completely classified) manifolds carrying non-zero $\alpha$-Killing spinors with non-constant $\alpha \in C^\infty(M, i\mathbb{R})$, see Theorem A.4.5 below.

On 1-dimensional manifolds $M$ it is a simple exercise to show the following: every section of $\Sigma M$ is a twistor-spinor, therefore this space is infinite-dimensional. For any $\alpha \in \mathbb{C}$ the space of $\alpha$-Killing spinors on $(M, g) := (\mathbb{R}, \mathrm{can})$ is $\mathbb{C} \cdot e^{-i\alpha t}$, thus it is 1-dimensional. As for the circle $\mathbb{S}^1(L)$ of length $L > 0$ and carrying the $\delta$-spin structure with $\delta \in \{0, 1\}$ (see Example 1.4.3.1), it admits a non-zero - and hence 1-dimensional - space of $\alpha$-Killing spinors if and only if $\alpha \in \frac{\pi\delta}{L} + \frac{2\pi}{L}\mathbb{Z}$ (in particular $\alpha$ must be real).

Before we proceed to general properties of twistor-spinors in dimension $n \geq 2$, we discuss a few examples.

**Examples A.1.3** We describe the twistor spinors on simply-connected space-forms.

1. Let $(M^n, g) := (\mathbb{R}^n, \mathrm{can})$, $n \geq 2$, be endowed with its canonical spin structure. It obviously admits a $2^{[\frac{n}{2}]}$-dimensional space of parallel spinors, which are the constant sections of $\Sigma(\mathbb{R}^n) \cong \mathbb{R}^n \times \Sigma_n$. There exists moreover a $2^{[\frac{n}{2}]}$-dimensional space of non-parallel twistor-spinors, which are of the form $\varphi_x := x \cdot \psi$ for every $x \in \mathbb{R}^n$, where $\psi$ is some parallel spinor on $(\mathbb{R}^n, \mathrm{can})$. Since this space stands in direct sum with that of parallel spinors, we deduce that the space of twistor-spinors on $(\mathbb{R}^n, \mathrm{can})$ is at least $2^{[\frac{n}{2}]+1}$-dimensional. We shall show that for $n \geq 3$ the space $\mathrm{Ker}(P)$ is actually at most (hence here exactly) $2^{[\frac{n}{2}]+1}$-dimensional (see Proposition A.2.1.3.b)), however in dimension $n = 2$ there are many more twistor-spinors on $\mathbb{R}^n$ (see Proposition A.2.3).

2. Let $(M^n, g) := (\mathbb{S}^n, \mathrm{can})$, $n \geq 2$, be endowed with its canonical metric (with sectional curvature 1) and spin structure. Since it is a hypersurface of $(\mathbb{R}^{n+1}, \mathrm{can})$ with Weingarten-map $-\mathrm{Id}_{TM}$ w.r.t. the normal vector field $\nu_x := x$ we deduce from the Gauss-type formula (1.21) that the restriction of any parallel spinor (resp. positive half spinor for $n$ odd, see Proposition 1.4.1) onto $\mathbb{S}^n$ is a $\frac{1}{2}$-Killing spinor. Therefore the space of $\frac{1}{2}$-Killing spinors on $(\mathbb{S}^n, \mathrm{can})$ is at least $2^{[\frac{n}{2}]}$-dimensional. Using again Proposition 1.4.1, it is easy to show that the restriction of any spinor of the form $x \mapsto x \cdot \psi$, where $\psi$ is parallel on $(\mathbb{R}^{n+1}, \mathrm{can})$ (and positive if $n$ is odd), gives a $-\frac{1}{2}$-Killing spinor. Therefore the space of $-\frac{1}{2}$-Killing spinors on $(\mathbb{S}^n, \mathrm{can})$ is also at least $2^{[\frac{n}{2}]}$-dimensional. From Propositions A.2.1.4 and A.4.1.2 below we deduce that both spaces have exactly dimension $2^{[\frac{n}{2}]}$ and that the space of twistor-spinors is exactly the direct sum of them.

3. Let $(M^n, g) := (H^n, \mathrm{can})$, $n \geq 2$, be endowed with its canonical metric (with sectional curvature $-1$) and spin structure. It is a hypersurface of the Minkowski-space $(\mathbb{R}^{n+1}, \langle\!\langle \cdot\,,\cdot \rangle\!\rangle)$ with Weingarten-map $-\mathrm{Id}_{TM}$ w.r.t. the normal vector field $\nu_x := x$. In the Lorentzian setting Proposition 1.4.1 has an analog: there exists an isomorphism $\Sigma\widetilde{M}_{|_M} \longrightarrow \Sigma M$ (or to a double copy of it if $n$ is odd) for which quite the same Gauss-type-formula as (1.21) holds but for which the relation (1.20) becomes $X \cdot i\nu \cdot \varphi = -X \underset{M}{\cdot} \varphi$ (or $-(X \underset{M}{\cdot} \oplus - X \underset{M}{\cdot})\varphi$ if $n$ is odd) for all $X \in TM$ and $\varphi \in \Sigma\widetilde{M}_{|_M}$. As a consequence the restriction of any parallel spinor (resp. positive half spinor for $n$ odd) onto $H^n$ is a $\frac{i}{2}$-Killing spinor. Analogously the restriction of any spinor of the form $x \mapsto x \cdot \psi$, where $\psi$ is parallel on $(\mathbb{R}^{n+1}, \langle\!\langle \cdot\,,\cdot \rangle\!\rangle)$ (and positive if $n$ is odd), gives a $-\frac{i}{2}$-Killing spinor. We deduce from Proposition A.4.1.2 below that there are no other Killing spinors on $(H^n, \mathrm{can})$ than those constructed and from Proposition A.2.1.3.b) that, if $n \geq 3$, then the space of twistor-spinors is exactly the direct sum of both spaces (for $n = 2$ see Proposition A.2.3).

Non simply-connected spaceforms may also admit non-zero twistor-spinors. This is the case if and only if the (or at least some) twistor-spinor on the corresponding model space is preserved by the $\pi_1$-action, see Proposition 1.4.2. This condition is not always fulfilled: for example there does not exist any non-zero imaginary Killing spinor on closed Riemannian spin manifolds (otherwise the Dirac operator would have a purely imaginary eigenvalue, which cannot be on closed manifolds). In dimension 2 flat tori together with their trivial spin structure are the only closed non simply-connected spaceforms admitting twistor-spinors, which are then parallel. In dimension 3 flat tori together with their trivial spin structure are also the only closed flat manifolds admitting twistor-spinors [212], however the quotient of $\mathbb{S}^3$ through any of its finite subgroups carries Killing spinors [13] (for lens spaces it has been proved independently in [87]). The real projective space $\mathbb{R}\mathrm{P}^n$ admits for every $n \equiv 3\ (4)$ non-zero real Killing spinors: in the notations of Corollary 2.1.5, if the spin structure is fixed by $\delta = 0$, then there exists a $2^{\frac{n-1}{2}}$-dimensional space of $-\frac{1}{2}$-Killing spinors if $n \equiv 3\ (8)$ and of $\frac{1}{2}$-ones if $n \equiv 7\ (8)$ respectively (vice-versa for $\delta = 1$). More generally there exists a formula for the dimension of the space of Killing spinors on every (closed) spaceform with positive curvature [41, Thm. 3]. However, up to the knowledge of the author, there does not exist any full classification of complete (even closed) flat manifolds admitting parallel spinors in dimension $n \geq 4$.

Non conformally flat examples are much more involved, see for instance [174] where the authors construct on every $\mathbb{C}^n$ a half conformally flat (non conformally flat) metric carrying a non-zero space of twistor-spinors. For the reader interested in twistor-spinors on singular spaces such as orbifolds we suggest [60].

## A.2 Elementary properties of twistor-spinors

The following fundamental results on the twistor-spinor-equation are due to H. Baum, T. Friedrich and A. Lichnerowicz (see [59] for precise references):

**Proposition A.2.1 (see [59])** *Let $\psi$ be any twistor-spinor on an $n(\geq 2)$-dimensional Riemannian spin manifold $(M^n, g)$. Then the following holds:*

1. *For any conformal change $\overline{g} := e^{2u}g$ of metric on $M^n$,*
$$\overline{P} = e^{\frac{u}{2}} \circ P \circ e^{-\frac{u}{2}},$$
*where $\overline{P} := P_{\overline{g}}$. In particular $e^{\frac{u}{2}}\overline{\psi}$ is a twistor-spinor on $(M^n, \overline{g})$.*

2. *If $S$ denotes the scalar curvature of $(M^n, g)$ then*
$$D^2\psi = \frac{nS}{4(n-1)}\psi. \tag{A.3}$$

3. *If $n \geq 3$ then:*
   a) *for every $X \in TM$,*
$$\nabla_X(D\psi) = \frac{n}{n-2}\Big(-\frac{1}{2}\mathrm{Ric}(X)\cdot\psi + \frac{S}{4(n-1)}X\cdot\psi\Big). \tag{A.4}$$
   b) $\dim(\mathrm{Ker}(P)) \leq 2^{[\frac{n}{2}]+1}$.
   c) *The zero-set of $\psi$ is either discrete in $M^n$ or $M^n$ itself.*
   d) *If $(M^n, g)$ is Einstein with $S \neq 0$ then $\psi$ is the sum of two non-parallel Killing spinors.*
   e) *If $|\psi|$ is a non-zero constant then $(M^n, g)$ is Einstein. Moreover either $S = 0$ and $\psi$ is parallel or $S > 0$ and $\psi$ is the sum of two real non-parallel Killing spinors.*

4. *If $M^n$ is closed then $\mathrm{Ker}(P)$ is finite dimensional. In the case $\psi \neq 0$ if furthermore $S$ is constant then either $S = 0$ and $\psi$ is parallel or $S > 0$ and $\psi$ is the sum of two real non-parallel Killing spinors.*

*Proof*:

1. For any $\varphi \in \Gamma(\Sigma M)$, $f \in C^\infty(M)$ and $X \in TM$ one has
$$\begin{aligned} P_X(f\varphi) &= \nabla_X(f\varphi) + \frac{1}{n}X\cdot D(f\varphi) \\ &\overset{(1.11)}{=} X(f)\varphi + f\nabla_X\varphi \\ &\quad +\frac{1}{n}X\cdot\mathrm{grad}(f)\cdot\varphi + \frac{f}{n}X\cdot D\varphi \\ &= X(f)\varphi + \frac{1}{n}X\cdot\mathrm{grad}(f)\cdot\varphi + fP_X\varphi. \end{aligned} \tag{A.5}$$

   We deduce from (1.17) and (1.18) that
$$\begin{aligned} \overline{P}_X\overline{\varphi} &= \overline{\nabla}_X\overline{\varphi} + \frac{1}{n}X\overline{\cdot}\overline{D}\overline{\varphi} \\ &= \overline{\nabla_X\varphi} - \frac{1}{2}\overline{X\cdot\mathrm{grad}_g(u)\cdot\varphi} - \frac{X(u)}{2}\overline{\varphi} \\ &\quad +\frac{e^{-u}}{n}X\overline{\cdot}(\overline{D\varphi} + \frac{n-1}{2}\overline{\mathrm{grad}_g(u)\cdot\varphi}) \\ &= \overline{P_X\varphi} - \frac{1}{2n}\overline{X\cdot\mathrm{grad}_g(u)\cdot\varphi} - \frac{X(u)}{2}\overline{\varphi}, \end{aligned} \tag{A.6}$$

so that

$$\begin{aligned}
\overline{P}_X(e^{\frac{u}{2}}\overline{\varphi}) &\overset{(A.5)}{=} \frac{e^{\frac{u}{2}}}{2}X(u)\overline{\varphi} + \frac{e^{\frac{u}{2}}}{2n}X\bar{\cdot}\mathrm{grad}_{\overline{g}}(u)\bar{\cdot}\,\overline{\varphi} + e^{\frac{u}{2}}\overline{P}_X\overline{\varphi} \\
&\overset{(A.6)}{=} \frac{e^{\frac{u}{2}}}{2}X(u)\overline{\varphi} + \frac{e^{\frac{u}{2}}}{2n}X\bar{\cdot}\mathrm{grad}_{\overline{g}}(u)\bar{\cdot}\,\overline{\varphi} \\
&\quad + e^{\frac{u}{2}}(\overline{P_X\varphi} - \frac{1}{2n}\overline{X\cdot\mathrm{grad}_g(u)\cdot\varphi} - \frac{X(u)}{2}\overline{\varphi}) \\
&= e^{\frac{u}{2}}\overline{P_X\varphi},
\end{aligned}$$

which shows 1.

2. Let $X, Y \in \Gamma(TM)$, then

$$\nabla_X\nabla_Y\psi = -\frac{1}{n}\nabla_X Y\cdot D\psi - \frac{1}{n}Y\cdot\nabla_X D\psi,$$

from which we deduce

$$\begin{aligned}
R^{\nabla}_{X,Y}\psi &= \nabla_{[X,Y]}\psi - [\nabla_X,\nabla_Y]\psi \\
&= \frac{1}{n}(Y\cdot\nabla_X D\psi - X\cdot\nabla_Y D\psi). \qquad (A.7)
\end{aligned}$$

Let $\{e_j\}_{1\leq j\leq n}$ be a local orthonormal basis of $TM$. From (1.9) we obtain

$$\begin{aligned}
\frac{1}{2}\mathrm{Ric}(X)\cdot\psi &= \frac{1}{n}\sum_{j=1}^{n}(e_j\cdot e_j\cdot\nabla_X D\psi - e_j\cdot X\cdot\nabla_{e_j}D\psi) \\
&= \frac{1}{n}(-(n-2)\nabla_X D\psi + X\cdot D^2\psi). \qquad (A.8)
\end{aligned}$$

Hence

$$\begin{aligned}
-\frac{1}{2}S\psi &= \frac{1}{2}\sum_{j=1}^{n}e_j\cdot\mathrm{Ric}(e_j)\cdot\psi \\
&\overset{(A.8)}{=} \frac{1}{n}\sum_{j=1}^{n}(-(n-2)e_j\cdot\nabla_{e_j}D\psi + e_j\cdot e_j\cdot D^2\psi) \\
&= -\frac{2(n-1)}{n}D^2\psi
\end{aligned}$$

which shows 2.

3. Assume $n \geq 3$.

   a) Coming back to (A.8) using (A.3) we obtain

$$\frac{1}{2}\mathrm{Ric}(X)\cdot\psi = -\frac{n-2}{n}\nabla_X D\psi + \frac{S}{4(n-1)}X\cdot\psi$$

   which is the result.

b) It follows from (A.4) that $\psi$ is a twistor-spinor if and only if the section $\psi \oplus D\psi$ of $\Sigma M \oplus \Sigma M$ is parallel w.r.t. the covariant derivative

$$\begin{aligned}\nabla^T_X(\varphi_1 \oplus \varphi_2) &:= \Big(\nabla_X \varphi_1 + \frac{1}{n} X \cdot \varphi_2\Big)\\ &\oplus \Big(\frac{n}{n-2}(\frac{1}{2}\mathrm{Ric}(X)\cdot \varphi_1 - \frac{S}{4(n-1)} X \cdot \varphi_1) + \nabla_X \varphi_2\Big)\end{aligned}$$

for all $\varphi_1, \varphi_2 \in \Gamma(\Sigma M)$. From $\mathrm{rk}_{\mathbb{C}}(\Sigma M) = 2^{[\frac{n}{2}]}$ we conclude.

c) We compute the Hessian of $|\psi|^2$. Let $X, Y \in \Gamma(TM)$. From

$$\begin{aligned}X(|\psi|^2) &= 2\Re e\left(\langle \nabla_X \psi, \psi\rangle\right)\\ &= -\frac{2}{n}\Re e\left(\langle X \cdot D\psi, \psi\rangle\right)\end{aligned}$$

one has

$$\begin{aligned}\mathrm{Hess}(|\psi|^2)(X,Y) &= -\frac{2}{n}\Re e\left(\langle Y \cdot \nabla_X D\psi, \psi\rangle + \langle Y \cdot D\psi, \nabla_X \psi\rangle\right)\\ &\overset{(A.4)}{=} \frac{1}{n-2}\Re e(\langle Y \cdot \mathrm{Ric}(X)\cdot\psi, \psi\rangle)\\ &\quad -\frac{S}{2(n-1)(n-2)}\Re e(\langle Y \cdot X \cdot \psi, \psi\rangle)\\ &\quad +\frac{2}{n^2}\Re e(\langle Y \cdot D\psi, X \cdot D\psi\rangle)\\ &= -\frac{|\psi|^2}{n-2}\mathrm{ric}(X,Y)\\ &\quad +\Big(\frac{S|\psi|^2}{2(n-1)(n-2)} + \frac{2|D\psi|^2}{n^2}\Big) g(X,Y). \qquad (A.9)\end{aligned}$$

If $\psi_p = 0$ then $\mathrm{Hess}(|\psi|^2)_p = \frac{2|D\psi|_p^2}{n^2} g_p$. In the case $\psi \neq 0$ one must have $(D\psi)_p \neq 0$ (otherwise the $\nabla^T$-parallel section $\psi \oplus D\psi$ would vanish at $p$ and hence identically), therefore the Hessian of $|\psi|^2$ is positive definite at $p$ and the result follows.

d) If $(M^n, g)$ is Einstein then (A.4) becomes

$$\begin{aligned}\nabla_X D\psi &= \frac{n}{n-2}(-\frac{S}{2n} + \frac{S}{4(n-1)}) X \cdot \psi\\ &= -\frac{S}{4(n-1)} X \cdot \psi\end{aligned}$$

for every $X \in TM$. In case $S \neq 0$ the spinor $\psi$ can be written as $\psi = \psi_1 + \psi_{-1}$ where

$$\psi_{\pm 1} := \frac{1}{2}\left(\psi \pm \frac{1}{\lambda} D\psi\right)$$

and with $\lambda := \sqrt{\frac{nS}{4(n-1)}}$ (if $S < 0$ the square root may be chosen arbitrarily since changing its sign just exchanges the roles of $\psi_1$ and

$\psi_{-1}$). We show that $\psi_{\pm 1}$ is a $\mp\frac{\lambda}{n}$-Killing spinor on $(M^n, g)$: for every $X \in TM$,

$$\begin{aligned} \nabla_X \psi_{\pm 1} &= \frac{1}{2}(\nabla_X \psi \pm \frac{1}{\lambda} \nabla_X D\psi) \\ &= \frac{1}{2}\Big( -\frac{1}{n} X \cdot D\psi \pm \frac{1}{\lambda}(-\frac{\lambda^2}{n} X \cdot \psi)\Big) \\ &= \mp \frac{\lambda}{2n} X \cdot (\psi \pm \frac{1}{\lambda} D\psi) \\ &= \mp \frac{\lambda}{n} X \cdot \psi_{\pm 1}, \end{aligned}$$

which shows d).

e) If $|\psi|$ is a non-zero constant then (A.9) implies

$$\mathrm{ric} = \Big(\frac{S}{2(n-1)} + \frac{2(n-2)|D\psi|^2}{n^2|\psi|^2}\Big) g,$$

that is, $(M^n, g)$ is Einstein. Moreover from the latter equation the scalar curvature of $(M^n, g)$ is then given by

$$S = \frac{4(n-1)}{n} \cdot \frac{|D\psi|^2}{|\psi|^2} \geq 0. \tag{A.10}$$

If $S > 0$ then using d) we deduce that $\psi = \psi_1 + \psi_{-1}$ where $\psi_{\pm 1}$ is a $\mp\sqrt{\frac{S}{4n(n-1)}}$-Killing spinor with $\sqrt{\frac{S}{4n(n-1)}} \in \mathbb{R}$. In case $S = 0$ the identity (A.10) requires $D\psi = 0$ and hence $\nabla\psi = 0$, i.e., $\psi$ is parallel on $(M^n, g)$. This proves e).

4. Assume $(M^n, g)$ to be closed and $n \geq 2$. From

$$|\nabla\varphi|^2 = |P\varphi|^2 + \frac{1}{n}|D\varphi|^2 \tag{A.11}$$

and

$$\begin{aligned} P^*P &\overset{(A.11)}{=} \nabla^*\nabla - \frac{1}{n} D^2 \\ &\overset{(1.15)}{=} \frac{n-1}{n}\nabla^*\nabla - \frac{S}{4n}\mathrm{Id} \end{aligned} \tag{A.12}$$

the operator $P^*P$ is elliptic, hence its kernel is finite-dimensional. If furthermore $\psi \neq 0$ and $S$ is constant then integrating the Hermitian product of (A.3) with $\psi$ one obtains

$$\begin{aligned} \frac{nS}{4(n-1)} \int_M |\psi|^2 v_g &= \int_M \langle D^2\psi, \psi\rangle v_g \\ &= \int_M |D\psi|^2 v_g, \end{aligned}$$

which shows $S \geq 0$. On the other hand (A.3) already stands for the limiting-case in T. Friedrich's inequality (3.1), so that $\psi$ must either be parallel (in case $S = 0$) or the sum of two real Killing spinors (in case $S > 0$). This shows 4. and concludes the proof. □

**Note A.2.2** Actually Proposition A.2.1.4 implies that Proposition A.2.1.3.b) and c) hold on closed $M$ in dimension $n = 2$ as well: on the one hand we deduce that the only compact orientable surfaces admitting twistor-spinors are $\mathbb{S}^2$ and $\mathbb{T}^2$ carrying any conformal class, the latter one being endowed with its trivial spin structure. For $\mathbb{S}^2$ (resp. $\mathbb{T}^2$) that space is 4-dimensional (resp. 2-dimensional), corresponding to the direct sum of the space of $\frac{1}{2}$-Killing spinors with that of $-\frac{1}{2}$-ones for the canonical metric (resp. to the space of parallel spinors for any flat metric in the conformal class). On the other hand the sum of two Killing spinors on $\mathbb{S}^2$ has at most one zero.

For $\mathbb{R}^2$ and $\mathbb{H}^2$ (see Note A.2.2 for closed $M^2$) one can again make the space of twistor-spinors explicit, however that space turns out to be infinite-dimensional:

**Proposition A.2.3** *Let $M$ be any non-empty connected open subset of $\mathbb{R}^2$ carrying its canonical conformal class and spin structure. Then the space of twistor-spinors of $M$ for any metric in this conformal class is isomorphic to the direct sum of the space of holomorphic with that of anti-holomorphic functions on $M$. In particular the space of twistor-spinors on $\mathbb{R}^2$ and $\mathbb{H}^2$ respectively is infinite-dimensional.*

*Proof*: Since the twistor-spinor-equation is conformally invariant (see Proposition A.2.1.1) we may assume that $g$ is the canonical flat metric on $M$. Let $\{\varphi_+, \varphi_-\}$ be a basis of parallel spinors on $M$ w.r.t. $g$ such that $ie_1 \cdot e_2 \cdot \varphi_\pm = \pm\varphi_\pm$ where $\{e_1, e_2\}$ denotes the canonical basis of $\mathbb{R}^2$. Then there exist functions $f_+, f_- : M \longrightarrow \mathbb{C}$ such that $\psi = f_+\varphi_+ + f_-\varphi_-$. We compute $P\psi$: for every $X \in TM$,

$$P_X\psi = X(f_+)\varphi_+ + X(f_-)\varphi_- + \frac{1}{2}X \cdot (df_+ \cdot \varphi_+ + df_- \cdot \varphi_-).$$

For the Kähler structure $J$ associated to $g$ and the orientation of $M$ one has however

$$\begin{aligned} X \cdot Y \cdot \varphi_\pm &= \sum_{j,k=1}^{2} g(X, e_j)g(Y, e_k)e_j \cdot e_k \cdot \varphi \\ &= \sum_{j=1}^{2} g(X, e_j)g(Y, e_j)e_j \cdot e_j \cdot \varphi \\ &\quad +(g(X, e_1)g(Y, e_2) - g(X, e_2)g(Y, e_1))e_1 \cdot e_2 \cdot \varphi_\pm \\ &= -g(X, Y)\varphi - g(X, J(Y))e_1 \cdot e_2 \cdot \varphi_\pm \\ &= (-g(X, Y) \pm ig(X, J(Y)))\varphi_\pm \\ &= -2g(X, p_\pm(Y))\varphi_\pm, \end{aligned}$$

where $p_\pm(X) := \frac{1}{2}(X \mp iJ(X))$. We deduce that

$$\begin{aligned} P_X\psi &= X(f_+)\varphi_+ - g(X, p_+(df_+))\varphi_+ \\ &\quad +X(f_-)\varphi_- - g(X, p_-(df_-))\varphi_- \\ &= g(X, p_-(df_+))\varphi_+ + g(X, p_+(df_-))\varphi_-. \end{aligned}$$

Therefore $P\psi = 0$ if and only if $p_\pm(df_\mp) = 0$, that is, if and only if $f_+$ is anti-holomorphic and $f_-$ is holomorphic. From $(\mathbb{H}^2, \mathrm{can}_{\mathbb{H}^2}) = (\{z \in \mathbb{C}$ s.t. $|z| <$

$1\}, \frac{4}{(1-|z|^2)^2}\mathrm{can}_{\mathbb{R}^2})$ we conclude the proof. □

Note that Proposition A.2.3 together with Note A.2.2 imply in particular that Proposition A.2.1.3.c) still holds in dimension $n = 2$, since holomorphic and anti-holomorphic functions on a surface vanish either on a discrete subset or identically.

**Corollary A.2.4 ([59])** *Let $(M^n, g)$ be an $n(\geq 3)$-dimensional Riemannian spin manifold carrying a non-zero twistor-spinor $\psi$. Then $Z_\psi := \{x \in M \mid \psi(x) = 0\}$ is discrete in $M$ and $(\overline{M}^n := M^n \setminus Z_\psi, \overline{g} := \frac{1}{|\psi|^4} g)$ admits a real Killing spinor, which is parallel if $Z_\psi \neq \varnothing$.*

*Proof*: The statement on the zero set of $\psi$ has been proved in Proposition A.2.1.3.c). From Proposition A.2.1.1 the spinor $\overline{\phi} := \frac{\overline{\psi}}{|\psi|}$ is a twistor-spinor on $(\overline{M}^n, \overline{g})$. In dimension $n \geq 3$ since it has constant norm it is the sum of two real Killing spinors (Proposition A.2.1.3.e)); furthermore

$$\begin{aligned}
\overline{D}\,\overline{\phi} &\overset{(1.18)}{=} |\psi|^2(\overline{D\phi} - \frac{n-1}{2}\overline{\frac{\mathrm{grad}(|\psi|^2)}{|\psi|^2} \cdot \phi}) \\
&= |\psi|^2(-\overline{\frac{\mathrm{grad}(|\psi|)}{|\psi|^2} \cdot \psi} + \frac{1}{|\psi|}\overline{D\psi} - \frac{n-1}{2}\overline{\frac{\mathrm{grad}(|\psi|^2)}{|\psi|^2} \cdot \phi}) \\
&= |\psi|\overline{D\psi} - \frac{n}{2}\overline{\mathrm{grad}(|\psi|^2) \cdot \phi},
\end{aligned}$$

so that, for any $p \in Z_\psi$, $|\overline{D}\,\overline{\phi}|(x) \underset{x\to p}{\longrightarrow} 0$ and by (A.10) for $\overline{g}$ and $\overline{\phi}$ one obtains $S_{\overline{g}}(x) \underset{x\to p}{\longrightarrow} 0$ (both w.r.t. the topology given by $g$ on $M$). Applying again Proposition A.2.1.3.e), since $S_{\overline{g}}$ is constant it must vanish identically, hence $\overline{\phi}$ is parallel on $(\overline{M}^n, \overline{g})$ as soon as $Z_\psi \neq \varnothing$. □

Note that the equivalent statement in dimension $n = 2$ does not hold because of Proposition A.2.3.

## A.3 Classification results for manifolds with twistor-spinors

Corollary A.2.4 induces a dichotomy in the classification of $n(\geq 3)$-dimensional Riemannian spin manifolds $M$ carrying a non-zero twistor-spinor $\psi$: either $Z_\psi = \varnothing$ and then up to conformal change of metric $M$ belongs to the class of manifolds admitting Killing spinors (which is studied in greater detail in Section A.4), or $Z_\psi \neq \varnothing$. In the latter case and for closed $M$, using the solution to the Yamabe problem about the existence of a constant scalar curvature metric in a conformal class A. Lichnerowicz showed:

**Theorem A.3.1 (A. Lichnerowicz [183])** *Let $(M^n, g)$, $n \geq 2$, be a closed Riemannian spin manifold carrying a non-trivial twistor-spinor $\psi$ with non-empty zero-set $Z_\psi$. Then $|Z_\psi| = 1$ and $(M^n, g)$ is conformally equivalent to $(\mathbb{S}^n, \mathrm{can})$.*

A relatively simple proof of Theorem A.3.1 can be found in [176].

For general $M$ W. Kühnel and H.-B. Rademacher proved that the Ricci-flat metric $\frac{1}{|\psi|^4}g$ on $M^n \setminus Z_\psi$ is either flat or locally irreducible, more precisely:

**Theorem A.3.2 (W. Kühnel and H.-B. Rademacher [175])** *Let $(M^n, g)$ be a simply-connected Riemannian spin manifold carrying a non-trivial twistor-spinor $\psi$ with non-empty zero-set $Z_\psi$ and assume that the metric is not conformally flat. Then the following holds:*

1. *Every non-zero twistor-spinor on $(M^n, g)$ vanishes exactly at $Z_\psi$.*

2. *For $N := \dim(\mathrm{Ker}(P))$ and the reduced holonomy group $\overline{\mathrm{Hol}} := \mathrm{Hol}(\overline{M}^n, \overline{g})$ of the Ricci-flat metric $\overline{g} := \frac{1}{|\psi|^4}g$ on $\overline{M}^n := M^n \setminus Z_\psi$ one has one of the following:*

   a) $n = 2m \geq 4$, $\overline{\mathrm{Hol}} = \mathrm{SU}_m$ *and* $N = 2$.

   b) $n = 4m \geq 8$, $\overline{\mathrm{Hol}} = \mathrm{Sp}_m$ *and* $N = m + 1$.

   c) $n = 7$, $\overline{\mathrm{Hol}} = \mathrm{G}_2$ *and* $N = 1$.

   d) $n = 8$, $\overline{\mathrm{Hol}} = \mathrm{Spin}_7$ *and* $N = 1$.

Theorem A.3.2, a proof of which can be found in the beautiful paper [176], actually requires Mc.K. Wang's classification of manifolds with non-zero parallel spinors, see Theorem A.4.2 below. Besides, we mention that up to now no example with reduced holonomy of type b), c) or d) has been described (an example with $\overline{\mathrm{Hol}} = \mathrm{SU}_m$ is constructed in [174]).

## A.4 Classification results for manifolds with Killing spinors

We now come to the geometric properties specifically implied by the existence of a non-zero Killing spinor (Definition A.1.1.*ii*)).

**Proposition A.4.1** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional Riemannian spin manifold admitting a non-zero $\alpha$-Killing spinor $\psi$ for some $\alpha \in \mathbb{C}$.*

1. *The zero-set of $\psi$ is empty. If furthermore $\alpha$ is real then $|\psi|$ is constant on $M$.*

2. *The space of $\alpha$-Killing spinors on $(M^n, g)$ is at most $2^{[\frac{n}{2}]}$-dimensional.*

3. *The manifold $(M^n, g)$ is Einstein with scalar curvature $S = 4n(n-1)\alpha^2$. In particular $\alpha$ must be real or purely imaginary.*

*Proof*: By definition $\psi$ is an $\alpha$-Killing spinor if and only if it is a parallel section of $\Sigma M$ w.r.t. the covariant derivative $X \mapsto \nabla_X - \alpha X\cdot$; moreover that covariant derivative is metric as soon as $\alpha$ is real. This shows 1. and 2.
Assuming $n \geq 3$ it follows from (A.4) that, for every $X \in TM$,

$$\mathrm{Ric}(X) \cdot \psi = (2(n-2)\alpha^2 + \frac{S}{2(n-1)})X \cdot \psi$$

(remember that $\psi$ is a twistor-spinor satisfying $D\psi = -n\alpha\psi$). Since $\psi$ has no zero we obtain that $(M^n, g)$ is Einstein with scalar curvature $S = 4n(n-1)\alpha^2$. In dimension $n = 2$ the equation (A.7) for $\psi$ is of the form

$$R^{\nabla}_{X,Y}\psi = \alpha^2(X \cdot Y - Y \cdot X) \cdot \psi,$$

hence comparing with (1.8) we obtain $S = 8\alpha^2$, which concludes the proof of 3. □

In particular Myers' theorem implies that a complete Riemannian spin manifold without boundary and carrying a non-zero real Killing spinor must be compact.

Complete simply-connected Riemannian spin manifolds $(M^n, g)$ carrying a non-zero space of $\alpha$-Killing spinors have been classified by Mc.K. Wang [238] for $\alpha = 0$, C. Bär [40] for $\alpha \in \mathbb{R}^*$ and H. Baum [53] for $\alpha \in i\mathbb{R}^*$ respectively. First note that from Note A.1.2 one may assume, up to rescaling the metric, that $\alpha \in \{\pm\frac{1}{2}, 0, \pm\frac{i}{2}\}$. In the case $\alpha = 0$, a parallel spinor must be a fixed point of the action of the lift of the reduced holonomy group to $\mathrm{Spin}_n$. Excluding the trivial example $(M^n, g) = (\mathbb{R}^n, \mathrm{can})$, which has a maximal number of linearly independent parallel spinors, as well as local products (products of manifolds with parallel spinors carry themselves parallel spinors), the classification can be deduced from Berger-Simons' list of Riemannian holonomy groups.

**Theorem A.4.2 (McK. Wang[238])** *Let $(M^n, g)$ be an $(n \geq 2)$-dimensional simply-connected complete irreducible Riemannian spin manifold without boundary. Let $N$ denote the dimension of* $\mathrm{Ker}(\nabla)$*. Then the manifold $(M^n, g)$ carries a non-zero parallel spinor if and only if its reduced holonomy group* $\mathrm{Hol} := \mathrm{Hol}(M, g)$ *belongs to the following list:*

a) $\mathrm{Hol} = \mathrm{SU}_m$*, $n = 2m \geq 4$, and in that case $N = 2$.*

b) $\mathrm{Hol} = \mathrm{Sp}_m$*, $n = 4m \geq 8$, and in that case $N = m + 1$.*

c) $\mathrm{Hol} = \mathrm{G}_2$*, $n = 7$, and in that case $N = 1$.*

d) $\mathrm{Hol} = \mathrm{Spin}_7$*, $n = 8$, and in that case $N = 1$.*

There also exists a classification in the non-flat non-simply-connected case in terms of lifts the holonomy group to the spin group, see [209] where the proof of Theorem A.4.2 can also be found.

The classification when $\alpha = \pm\frac{1}{2}$ relies on Mc.K. Wang's one using the following clever remark due to C. Bär and based on a geometric construction by S. Gallot (see reference in [40]): a spinor field is a $\frac{1}{2}$-Killing spinor on the manifold $(M^n, g)$ if and only if the induced spinor field on its Riemannian cone $(M \times \mathbb{R}_+^*, t^2 g \oplus dt^2)$ is parallel. Hence C. Bär proved:

**Theorem A.4.3 (C. Bär [40])** *Let $(M^n, g)$ be an $(n \geq 2)$-dimensional simply-connected closed Riemannian spin manifold. Let $p$ (resp. $q$) denote the dimension of the space of $\frac{1}{2}$- (resp. $-\frac{1}{2}$-) Killing spinors on $(M^n, g)$. Then the manifold $(M^n, g)$ carries up to scaling a non-zero $\pm\frac{1}{2}$-Killing spinor if and only if it is either the round sphere* $(\mathbb{S}^n, \mathrm{can})$ *(in which case $(p, q) = (2^{[\frac{n}{2}]}, 2^{[\frac{n}{2}]})$) or one of the following:*

a) $(4m+1)$*-dimensional Einstein-Sasaki, $m \geq 1$, and in that case* $(p,q) = (1,1)$.

b) $(4m+3)$*-dimensional Einstein-Sasaki but not 3-Sasaki, $m \geq 2$, and in that case* $(p,q) = (0,2)$.

c) $(4m+3)$*-dimensional 3-Sasaki, $m \geq 2$, and in that case* $(p,q) = (0, m+2)$.

d) 6*-dimensional nearly Kähler non-Kähler, and in that case* $(p,q) = (1,1)$.

e) 7*-dimensional with a nice 3-form $\phi$ satisfying $\nabla\phi = *\phi$ but not Sasaki, and in that case* $(p,q) = (0,1)$.

f) 7*-dimensional Sasaki but not 3-Sasaki, and in that case* $(p,q) = (0,2)$.

g) 7*-dimensional 3-Sasaki, and in that case* $(p,q) = (0,3)$.

For the definitions of 3-Sasaki structures and nice forms as well as the proof of Theorem A.4.3 we refer to [40]. Parts of this classification had already been obtained in [92, 133, 93, 94, 95, 120, 90]. As an interesting fact, two higher eigenvalues of $(n = 4m+3)$-dimensional 3-Sasaki manifolds can be explicitly computed in terms of the scalar curvature: A. Moroianu showed [205] that on such manifolds both $-\sqrt{\frac{nS}{4(n-1)}} - 1$ and $\sqrt{\frac{nS}{4(n-1)}} + 2$ are Dirac eigenvalues with multiplicities at least $3m$ and $m$ respectively. The proof relies on a clever combination of the Killing vector fields provided by the 3-Sasaki structure and the Killing spinors.

In the last case ($\alpha = \pm\frac{i}{2}$) the classification turns out to rely on totally different arguments. Studying in detail the level sets of the length function of an imaginary Killing spinor H. Baum proved the following theorem, which relies on Theorem A.4.2 but where the assumption $\pi_1(M) = 1$ turns out not to be necessary.

**Theorem A.4.4 (H. Baum [53])** *Let $(M^n, g)$ be an $(n \geq 2)$-dimensional connected complete Riemannian spin manifold without boundary. Then $(M^n, g)$ admits a non-trivial $\alpha$-Killing spinor with $\alpha \in i\mathbb{R}^*$ if and only if it is isometric to a warped product of the form*

$$(N \times \mathbb{R}, e^{4i\alpha t} h \oplus dt^2),$$

*where $(N^{n-1}, h)$ is a complete connected Riemannian spin manifold carrying a non-zero parallel spinor.*

Of course the $n$-dimensional real hyperbolic space can be obtained as a warped product of this form (take $(N, h) = (\mathbb{R}^{n-1}, \mathrm{can})$); in the disk model, this corresponds to the foliation by horospheres tangential to a fixed point on the boundary at infinity.

It was noticed by O. Hijazi, S. Montiel and A. Roldán [144] that the geometric part of Theorem A.4.4 - i.e., that $(M^n, g)$ must be a pseudo-hyperbolic space - follows from a classical argument by Yoshihiro Tashiro (see reference in [144]), namely from the existence of a smooth non-zero real-valued function $f$ on $M$ such that

$$\mathrm{Hess}(f) - fg = 0.$$

Here, up to rescaling $g$, the function $f := |\psi|^2$, where $\psi$ is a non-zero $\alpha$-Killing spinor on $(M^n, g)$, satisfies that equation (use (A.9) when $n \geq 3$). Nevertheless this argument does not describe the correspondence between spinor fields on $M$ and those on the warped product, see [53] for a rigorous treatment of that point.

Theorem A.4.4 generalizes to the situation where the constant $\alpha$ is replaced by a smooth imaginary-valued function, in which case a similar statement on the structure of the underlying manifold holds.

**Theorem A.4.5 (H.-B. Rademacher [215])** *Let $(M^n, g)$ be an $n(\geq 2)$-dimensional connected complete Riemannian spin manifold without boundary. For a given non-zero $\alpha \in C^\infty(M, i\mathbb{R})$ assume the existence of a non-zero section $\psi$ of $\Sigma M$ satisfying*

$$\nabla_X \psi = \alpha X \cdot \psi$$

*for all $X \in TM$. Then $(M^n, g)$ is isometric either to the real hyperbolic space of constant sectional curvature $4\alpha^2$ (in particular $\alpha$ must be constant) or to a warped product of the form $(N \times \mathbb{R}, e^{4i\int_0^t \alpha(s)ds} h \oplus dt^2)$, where $(N^{n-1}, h)$ is a complete connected Riemannian spin manifold admitting a non-zero parallel spinor and $\alpha \in C^\infty(\mathbb{R}, i\mathbb{R})$.*

Conversely, for any given $\alpha \in C^\infty(\mathbb{R}, i\mathbb{R})$ and $(N^{n-1}, h)$ as above, the warped product $(N \times \mathbb{R}, e^{4i\int_0^t \alpha(s)ds} h \oplus dt^2)$ admits a non-zero section $\psi$ of $\Sigma M$ satisfying $\nabla_X \psi = \alpha X \cdot \psi$ for all $X \in TM$, where $\alpha$ is extended by a constant onto the $N$-factor. Interestingly enough, there exist compact quotients admitting such spinors for some necessarily non-constant $\alpha$'s, see [215, Thm. 1] and references therein. The proof of Theorem A.4.5 relies on the classification of complete Riemannian manifolds carrying a non-isometric conformal closed Killing field, see [215] for details.

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