

Tire Noise Optimization Problem: a Mixed Integer Linear Programming Approach

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Abstract

We present a Mixed Integer Linear Programming (MILP) approach in order to model the non-linear problem of minimizing the tire noise function. In a recent work, we proposed an exact solution for the Tire Noise Optimization Problem, dealing with an APproximation of the noise (TNOP-AP). Here we study the original non-linear problem modeling the EXact - or real - noise (TNOP-EX) and propose a new scheme to obtain a solution for the TNOP-EX. Relying on the solution for the TNOP-AP, we use a Branch&Cut framework and develop an exact algorithm to solve the TNOP-EX. We also take more industrial constraints into account. Finally, we compare our experimental results with those obtained by other methods.

Keywords: Mixed integer linear programming; branch-and-cut; tire shape optimization

1 Introduction

The reduction of tire noise in the vehicle interior has been a major field of research in the tire industry for many years. After the tire engineers have determined every aspect of a new tire prototype concerning the driving and safety characteristics, the last degree of freedom is the final design of the so-called pitch sequence. In [16], T. A. Williams provides a historical overview of the approaches used by different companies, based on some U.S. Patent documents, from 1934 to 1994. First studies show that the noise optimization does not give good results when the number of pitch types is small (a mono-space sequence is the noisiest) and the tire mold is difficult when this number is large. A realistic tire size is a sequence of around 60–70 pitches, hence a complete search is not feasible because of an exponential computational time. First methods concerned patented single pitch sequences and a randomization of the sequence design. In the 1990's, some heuristic optimization algorithms were used in [9], based on genetic algorithms. In [11], the road surface is also taken into account when modeling the noise of certain pitch patterns. In [14], the

authors explain the sound generation mechanisms of a tire as well as the mathematical noise model. Further works deal with application of intelligent heuristic optimization algorithms (such as genetic algorithms and algorithms based on neural networks) on problems with one pitch sequence, with multiple pitch sequences, and on tires with spikes (cf. [1], [2] and references within those articles). An approach based on artificial immune systems combined with genetic algorithms is presented in [4] and [13]. In [10], a pitch sequence with five pitch types and 50 pitches is optimized using a genetic algorithm. There, by contrast with other approaches, the noise of the tire pattern is assessed via an image based algorithm. During the optimization procedure, the only free parameter is the pitch sequence. The quality of each sequence is assessed via a spectral analysis of the candidate sequences. The optimization of the pattern of a single pitch via particle swarm (PSO) is studied in [5].

The tire noise is approximated by a mathematical function, based on Fourier coefficients computed on a space of functions that is not a vector space and hence cannot be linearized in a natural way. Nevertheless, a pitch sequence has to satisfy some constraints (pitch length ratios, number of pitches, etc.), and a Mixed Integer Linear Programming (MILP) approach can be considered once a convenient noise approximation is defined.

In a previous work [2], we proposed in a very first approach a linear approximation of the tire noise and a MILP solving the tire noise optimization problem: the noise value computed for a given sequence is only an approximation of the modeled real noise and the sequences we found are in general not the optimal ones. Nevertheless, the ratio between the computed value and the modeled real noise is bounded by $\sqrt{2}$.

In the present paper, we deepen our study and focus on the modeled real noise. The MILP defined in [2] only allows for an approximated solution. Here we complete the MILP with a Branch-and-Cut algorithm to obtain the optimal solution. Furthermore, we add some new constraints to the MILP such as incompatibility or symmetry breaking etc. We also present a graph model of the problem, in particular the tire optimization problem can be interpreted as a path problem. For an easier understanding, we clarify the notations of [2] and present the mathematical model in more detail. We conclude with some experimental results.

2 The Tire Noise Optimization Problem

2.1 Brief description of the problem

The surface of a tire consists of different tracks, each track built out of a sequence of pitches of different sizes (see Figure 1a). A pitch is considered as the juxtaposition of an elevated part and a low part, called groove. The tire noise is produced by the grooves when coming into contact with the road surface. All pitches have the same height h and the groove is a constant fractional part q of the pitch length (Figure 1b). Let $P := \{1, \dots, r\}$ be the finite set of pitch types with lengths $\bar{l}_1 < \dots < \bar{l}_r$. Then, a pitch sequence can be modeled by an integer vector on P . For instance, for $|P| = 3$, the pitch sequence of Figure 1c is modeled by $(3, 2, 3, 1, \dots)$.

In this paper, we only study single track tires, thus the tire profile can be considered as a step function. The modeled exact tire noise is then defined as a scalar multiple of the largest Fourier coefficient in absolute value of that function (see Section 2.2.1 for a formal definition). From now on, we mean by tire noise the modeled exact tire noise.

The Tire Noise Optimization Problem for the EXact noise (TNOP-EX) consists in finding a pitch sequence producing the lowest noise, respecting given parameters provided by engineers. For

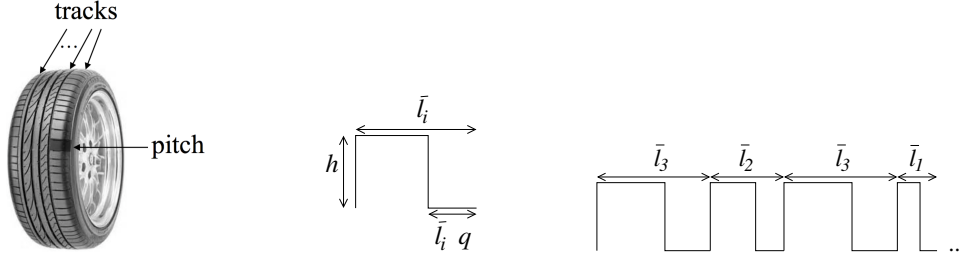


Figure 1: (a) Tracks and pitches, (b) a pitch of size \bar{l}_i and (c) a pitch sequence representing a tire track

the basic model, they are:

- the number of different pitch types
- the length ratios of the different pitch types
- the total number of pitches composing the tire.

Notice that the length of each type is not given as an absolute physical number in SI unit, but as a relative length ratio between the types. Note also that only the total number of pitches is fixed but not the number of pitches of each type, therefore the length of the pitch sequence, corresponding to the tire circumference, can vary. However, this circumference can always be normalized to some standard industrial tire size since the tire noise only depends on the relative pitch lengths.

To ensure stability in driving and wearing, we also consider other constraints such as:

- ctMinMaxOcc: the minimal and maximal number of occurrences of each pitch type in a pitch sequence is bounded
- ctIncompatibility: pitches of given types cannot be neighbors in a sequence (*e.g.* the shortest and the longest ones)
- ctMaxSeq: the length of a subsequence of pitches of the same type is bounded.

2.2 Definition of a discrete model

Here we describe a continuous mathematical model for the tire noise and its discretization. In particular, we define length units to transform the relative ratios into integral pitch length thus allowing for a MILP model. Notice that this MILP provides an approximated noise value. The Tire Noise Optimization Problem for this APproximated noise is denoted TNOP-AP.

2.2.1 The noise function

The model we study is based on the computation of the Fourier coefficients a_n, b_n, c_n of a given function depending on the profile of the tire. Recall that, if f is a T -periodic function of one real variable, then formally, for all $n \in \mathbb{N}$,

$$\begin{aligned}
a_n(f) &:= \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n}{T}x\right) dx \\
b_n(f) &:= \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T}x\right) dx \\
c_n(f) &:= \frac{1}{T} \int_0^T f(x) e^{\frac{-2i\pi n}{T}x} dx = \frac{1}{2}(a_n(f) - ib_n(f)).
\end{aligned}$$

Originally (see *e.g.* [3]), the real noise produced by a tire with one track is proportional to

$$2 \max_{n \geq 1} (|c_n(f)|),$$

where f is the profile function of the track.

Variable-rescaling does not change the value of the Fourier coefficients: if f is a T -periodic function, then for any real positive number δ , the function $f_\delta(x) := f(\delta x)$ is $\frac{T}{\delta}$ -periodic and, for every $n \in \mathbb{N}$, we have $c_n(f_\delta) = c_n(f)$ (in particular $a_n(f_\delta) = a_n(f)$ as well as $b_n(f_\delta) = b_n(f)$). This property has the following important consequence: instead of fixing the circumference T of the tire and let the pitch-lengths vary according to the number of pitches N present on the track, we may fix the pitch-lengths once for all and let the total circumference vary in terms of N ; the computed Fourier coefficients (and therefore the noise) will not change under that transformation.

Another important assumption deals with relative pitch length ratios that we assume to be rational, hence allowing the pitch lengths to be integers (see Section 2.2.3). When we denote a length by an integer, we implicitly assume it is expressed as an integer number of a fixed unit length that we do not need to make precise for the computations.

It is worth noticing that several models exist for the profile function. For instance, in [14], the profile function is described as a finite sum of Dirac functions (actually distributions) at positions $t_1 < \dots < t_N$, where N is the number of pitches. In that case the modulus of the Fourier coefficient c_n is given by

$$|c_n| = \frac{1}{T} \sqrt{\left(\sum_{i=1}^N \cos\left(\frac{2\pi n}{T}t_i\right)\right)^2 + \left(\sum_{i=1}^N \sin\left(\frac{2\pi n}{T}t_i\right)\right)^2}, n \in \{1, \dots, 200\}.$$

Remark that, in real life, a tire track can be considered as a sequence of $N = 60$ pitches at most. As explained in Section 4, the estimated noise can be computed before the $n = 3.5N$ th Fourier coefficient.

Trying to calculate the impact of exactly one pitch at a certain position t_m ($1 \leq m \leq N$) we can write $|c_n| = \frac{1}{T} \sqrt{C^2 + S^2}$, with

$$\begin{aligned}
C &= \sum_{i=1}^{m-1} \cos\left(\frac{2\pi n}{T}t_i\right) + \cos\left(\frac{2\pi n}{T}t_m\right) + \sum_{i=m+1}^N \cos\left(\frac{2\pi n}{T}t_i\right) \\
S &= \sum_{i=1}^{m-1} \sin\left(\frac{2\pi n}{T}t_i\right) + \sin\left(\frac{2\pi n}{T}t_m\right) + \sum_{i=m+1}^N \sin\left(\frac{2\pi n}{T}t_i\right),
\end{aligned}$$

where m corresponds to the position of the pitch and n to the measured frequency.

The specific harmonic cannot be calculated up to a position t_m because of the squares in the expression, which means we have to multiply *e.g.* the expression for a_n out, when we can compute $a_n^2/4$ as

$$\frac{1}{T^2} \left(\sum_{i=1}^{m-1} \cos\left(\frac{2\pi n}{T} t_i\right) + \cos\left(\frac{2\pi n}{T} t_m\right) + \sum_{i=m+1}^N \cos\left(\frac{2\pi n}{T} t_i\right) \right)^2$$

which makes everything dependent with everything (the part before position t_m has to be multiplied with the part after t_m , with contribution at t_m itself).

As mentioned in Section 1, several further models have been described in the literature. Here we are interested in the model that was proposed in [3] and [2]. The profile function f is provided by the profile of the tire, *i.e.* the step function given by the pitch sequence. In that case, the Fourier coefficients of a given track may be explicitly computed in terms of the pitch sequence of the track. From now on, we assume there are N pitches on the given track. We denote by $l(1), \dots, l(N)$ the lengths of the successive pitches of a given pitch sequence. Then, for any $n \geq 1$,

$$a_n(f) = \frac{h}{n\pi} \cdot \sum_{j=1}^N \sin(M_{j,n}) - \sin(N_{j,n}) \quad (1)$$

$$b_n(f) = -\frac{h}{n\pi} \cdot \sum_{j=1}^N \cos(M_{j,n}) - \cos(N_{j,n}), \quad (2)$$

where

$$M_{j,n} := \frac{2n\pi}{T} \cdot \left(\sum_{p=1}^{j-1} l(p) + (1-q)l(j) \right) \text{ and}$$

$$N_{j,n} := \frac{2n\pi}{T} \cdot \sum_{p=1}^{j-1} l(p) = M_{j,n} - (1-q)\frac{2n\pi}{T}l(j).$$

In particular,

$$c_n(f) = \frac{ih}{2n\pi} \cdot \sum_{j=1}^N e^{-iM_{j,n}} - e^{-iN_{j,n}}. \quad (3)$$

It is important to notice a few symmetry properties of the Fourier coefficients.

- The modulus of each Fourier coefficient c_n is invariant under translation, that is, applying a circular permutation to the pitches does not change the value of $|c_n|$. Note however that $|a_n|$ and $|b_n|$ change in general under such transformations. As a consequence, for the computation of the real noise involving $|c_n|$, we can assume that the first pitch in the sequence is of the first pitch type.
- Taking the mirror image of a given profile function keeps a_n unchanged while turning each b_n into $-b_n$, in particular it fixes $|a_n|$ and $|b_n|$ – and thus also $|c_n|$.

In order to handle the noise with Mixed Integer Linear Programming (MILP), the authors of [2] defined the approximated noise of a given one-tracked-tire to be proportional to

$$\max_{n \geq 1} (\max(|a_n(f)|, |b_n(f)|)),$$

where neither squares nor square-roots enter. Though *a priori* different, both formulae are roughly equivalent in the following sense: for any real numbers a, b ,

$$\max(|a|, |b|) \leq \sqrt{a^2 + b^2} \leq \sqrt{2} \cdot \max(|a|, |b|). \quad (4)$$

Therefore, the noises computed by both formulae cannot be too far from each other, the ratio between both lying in $[1, \sqrt{2}]$. Moreover, $\max(|a|, |b|) = \sqrt{a^2 + b^2}$ if and only if $a = 0$ or $b = 0$ and $\sqrt{a^2 + b^2} = \sqrt{2} \cdot \max(|a|, |b|)$ if and only if $|a| = |b|$.

2.2.2 Fourier precision

In theory, all Fourier coefficients must be computed in order to determine the noise produced by a one-tracked-tire. However, it is sufficient to compute the c_n coefficients only for n bounded in terms of N . A regular tire composed of N pitches has a peak at the frequency of N as well as at multiples of N , so called harmonics. Humans are able to hear only frequencies between 20 Hz and 20 kHz, so only the first harmonics of the tire signal are in the main human audible frequency range, and at most the second harmonics come close to the 20kHz frequency border. Furthermore, the amplitude of higher frequencies of a regular signal decreases quickly, so there is no need to compute more than 1.5 to 3.0 times N coefficients, around 200 ($= n$) in the case of a realistic tire pattern with $N = 60$ pitches.

Moreover, because of the very specific profile functions we deal with, it is an easy consequence of (3) that, for every $n \geq 1$,

$$|c_n(f)| \leq \frac{Nh}{n\pi},$$

where $h > 0$ is the pitch height. In particular, if N is fixed, then $|c_n|$ (as well as $|a_n|$ and $|b_n|$) is an $O(\frac{1}{n})$ and thus is small for large n .

2.2.3 Units and integer pitch lengths

Recall that the set of different pitch types $P = \{1, \dots, r\}$ is finite and $\bar{l}_1 < \dots < \bar{l}_r$ are the different pitch lengths. Recall that the ratio of any two pitch lengths is supposed to be rational and hence we can assume the existence of a largest length (called unit) dividing each pitch length into an integral number of units. Formally, let $\bar{l}_i = \frac{p_i}{q_i} \cdot \bar{l}_r$ with $p_i, q_i \in \mathbb{N}$ and $\text{gcd}(p_i, q_i) = 1$ for all $1 \leq i \leq r$ (note that $p_r = q_r = 1$). We define the unit length by $u := \frac{\bar{l}_r}{\text{lcm}(q_1, \dots, q_r)}$, where $\text{lcm}(q_1, \dots, q_r)$ is the least common multiple of the integers q_1, \dots, q_r . From now on, we study tires with reduced pitch lengths $l_1 < \dots < l_r$, where $l_i := \frac{\bar{l}_i}{u} \in \mathbb{N}$ for all $i \in \{1, \dots, r\}$. We call $l_{\min} := N \cdot l_1 \in \mathbb{N}$ and $l_{\max} := N \cdot l_r \in \mathbb{N}$ the minimal and maximal tire-length, respectively. From the industrial perspective, the sequence of pitch lengths is given as a sequence of ratios with respect to the smallest pitch length. Most of the results presented in Section 4 concern ratios $\{1.0, 1.25, 1.5\}$ reduced to lengths $\{4, 5, 6\}$ in the ILP (here $u = 0.25$). Notice also that we do not study tires of a fixed length T but tires with lengths varying from l_{\min} to l_{\max} , while N is fixed.

3 Optimization models

In the previous section, we described the tire noise optimization problem and the corresponding model for the approximated noise (TNOP-AP). In this section, we first highlight the link with scheduling, graph and mathematical programming. We then propose a linear optimization model for the TNOP-AP. In Section 3.3, we present a method to solve the Tire Noise Optimization Problem for the real noise, that we call TNOP-EX.

3.1 Graph presentation

In order to illustrate the model, we consider here tires of length $T = l_{\max}$. From Section 3.2 on, we let the tire length vary and solve the TNOP-AP for each tire size from l_{\min} to l_{\max} . Recall that a tire track is a sequence of consecutive pitches of different sizes assumed to be integers. Then, on each position, a pitch can or cannot start. A sequence of pitches can be seen as a scheduling problem on a single machine where the pitches are the jobs and the criteria to minimize depends on the beginning of each pitch. This problem also can be modeled as a graph problem. Let T denote the length of a tire expressed as an integral number of the above units and let $G = (V, A)$ be the oriented graph defined as follows:

- $V = \{v_p^i \mid \forall i \in \{1, \dots, T\}; \forall p \in \{1, \dots, r\}\} \cup \{s, t\}$ where i denotes a position on the tire and p denotes a pitch type,
- $A = A_s \cup A_{\text{int}} \cup A_t$, where

$$\begin{aligned} A_s &= \{(s, v_p^1) \mid p \in \{1, \dots, r\}\}, \\ A_{\text{int}} &= \{(v_p^i, v_{p'}^{i+\text{length}(p)}) \mid \forall i \in \{1, \dots, T\}; \forall p, p' \in \{1, \dots, r\}, \\ &\quad i + \text{length}(p) \leq T\}, \\ A_t &= \{(v_p^{T-\text{length}(p)}, t) \mid p \in \{1, \dots, r\}\}. \end{aligned}$$

An example for a tire of length 6 built of pitches of lengths 2, 3 and 4 is presented on Figure 2 (solid, dashed and dotted arrows, respectively). To each arc a starting at v_p^i in A and each $k \in \mathbb{N}$, we associate a weight $(w_a^k(\mathbf{a}), w_b^k(\mathbf{a}))$, corresponding to the a_k - and b_k -contributions to the noise produced by a pitch of type p starting at position i . Actually, $w_a^k(\mathbf{a}) = A_{k,0,i,p}$ and $w_b^k(\mathbf{a}) = B_{k,0,i,p}$, where $A_{k,0,i,p}$ (*resp.* $B_{k,0,i,p}$) denotes the contribution to the k^{th} Fourier coefficient a_k (*resp.* b_k) of the function having only one pitch of type p starting on the i^{th} unit (see Section 3.2 for the formal definitions).

Let \mathbf{P} be the set of all paths of N pitches starting at s and ending at t . Then, the TNOP-AP consists in finding a path $\mathbf{p} \in \mathbf{P}$ such that the maximum noise $\max_k \{|\sum_{\mathbf{a} \in \mathbf{p}} w_a^k(\mathbf{a})|, |\sum_{\mathbf{a} \in \mathbf{p}} w_b^k(\mathbf{a})|\}$ is minimal.

In the example of Figure 2, the paths of \mathbf{p} are highlighted by bold arrows. For

- $N = 3$: $\mathbf{P} = \{(s, v_1^1)(v_1^1, v_1^3)(v_1^3, v_1^5)(v_1^5, t)\}$
- $N = 2$: $\mathbf{P} = \{(s, v_1^1)(v_1^1, v_3^3)(v_3^3, t),$
 $(s, v_2^1)(v_2^1, v_2^4)(v_2^4, t),$
 $(s, v_3^1)(v_3^1, v_1^5)(v_1^5, t)\}.$

We propose in the next section a mixed integer linear program based on the path problem defined above.

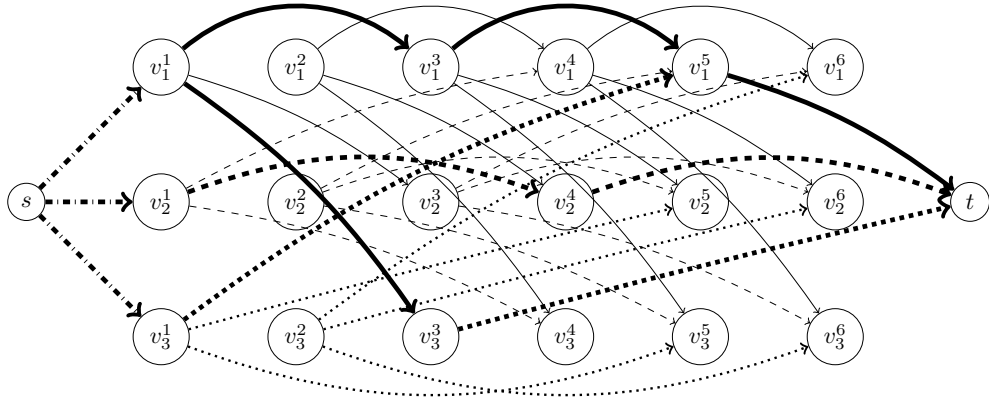


Figure 2: Graph model for a tire of length 6.

3.2 Mixed Integer Linear Program (MILP) solving TNOP-AP

We propose a MILP to solve the problem for a fixed tire length between l_{\min} and l_{\max} , considering hence some empty unit slots at the end of the sequence. Recall that the noise we compute differs from the real noise by a factor at most $\sqrt{2}$, as described in Section 2.2.1.

We use the following notations:

- N is the number of pitches,
- $P = \{1, 2, \dots, r\}$ is the set of pitch types and $l_1 < l_2 < \dots < l_r$ are the reduced pitch lengths, supposed to be integers,
- $L = \{0, \dots, l_{\max} - l_{\min}\}$ denotes the set of possible numbers of empty unit slots at the end, where l_{\max} and l_{\min} denote respectively the maximal and minimal tire lengths, and $|L|$ is then the number of possible tire lengths,
- $L_p^j = \{1, \dots, l_{\max} - l_p - j\}$ denotes the set of positions at which a pitch of type $p \in P$ can start on a sequence with j empty unit slots at the end,
- $T_j = l_{\max} - j$ denotes the length of a tire with j empty unit slots at the end,
- $K = \{1, \dots, 3N\}$ where $3N$ denotes the Fourier coefficient precision (see Section 2.2.2),
- $A_{k,j,i,p}$ denotes the contribution to the k^{th} Fourier coefficient a_k of the function having only one pitch of type p starting on the i^{th} unit and such that the j last units do not bear any pitch. Formally, they are computed as follows:

$$A_{0,j,i,p} = \frac{1}{T_j} \cdot h \cdot (1 - q)l_p = \frac{(1 - q) \cdot h \cdot l_p}{(l_{\max} - j)}$$

and for any $k \geq 1$,

$$A_{k,j,i,p} = \frac{h}{k\pi} \cdot \left(\sin \left(\frac{2k((i-1) + (1-q)l_p)\pi}{(l_{\max} - j)} \right) - \sin \left(\frac{2k(i-1)\pi}{(l_{\max} - j)} \right) \right).$$

These identities can be deduced from (1) described in Section 2.2.1.

- $B_{k,j,i,p}$ denotes the contribution to the Fourier coefficient b_k under the same conditions as for $A_{k,j,i,p}$, and they are computed as follows:

$$B_{k,j,i,p} = -\frac{h}{k\pi} \cdot \left(\cos \left(\frac{2k((i-1) + (1-q)l_p)\pi}{(l_{\max} - j)} \right) - \cos \left(\frac{2k(i-1)\pi}{(l_{\max} - j)} \right) \right).$$

These identities can be deduced from (2) described in Section 2.2.1.

- minOcc_p : the minimum number of pitches of type p in a sequence;
- maxOcc_p : the maximum number of pitches of type p in a sequence;
- max_p : the maximal length of a subsequence composed of pitches of a same type p .

Let us now introduce the binary variables

- $x_i^p \in \{0, 1\}$:

$$x_i^p = \begin{cases} 1 & \text{if a pitch of type } p \text{ starts at position } i \\ 0 & \text{otherwise} \end{cases}$$

for all $p \in P$ and $i \in L_p^0$

and the continuous variables

- $z^j \in \mathbb{R}_+$: the noise produced by a pitch sequence with j empty units at the end, for all $j \in L$,
- $za_k^j \in \mathbb{R}$: the value of the k -th Fourier coefficient a_k of a given pitch sequence with j empty units at the end, for all $j \in L$ and $k \in K$,
- $zb_k^j \in \mathbb{R}$: the value of the k -th Fourier coefficient b_k of a given pitch sequence with j empty units at the end, for all $j \in L$ and $k \in K$.

Let us fix the number of empty units $j \in L$ at the end of the sequence and hence consider tires of length T_j . The MILP below solves the TNOP-AP for this fixed tire length.

The objective function z^j represents actually the maximum of $|za_k^j|$ and $|zb_k^j|$ for all $k \in K$. Note that thanks to additivity of integration on intervals, the Fourier coefficients za_k^j and zb_k^j are the sums of the contribution of each pitch in the sequence.

Notice that, in order to solve the TNOP-AP, the MILP has to be solved for all $j \in L$.

Mixed Integer Linear Program (\mathcal{P}):

$$\min z^j$$

$$za_k^j \leq z^j \quad \forall k \in K, \quad (5)$$

$$-za_k^j \leq z^j \quad \forall k \in K, \quad (6)$$

$$zb_k^j \leq z^j \quad \forall k \in K, \quad (7)$$

$$-zb_k^j \leq z^j \quad \forall k \in K, \quad (8)$$

$$za_k^j = \sum_{p \in P} \sum_{i \in L_p^j} A_{k,j,i,p} x_i^p \quad \forall k \in K, \quad (9)$$

$$zb_k^j = \sum_{p \in P} \sum_{i \in L_p^j} B_{k,j,i,p} x_i^p \quad \forall k \in K, \quad (10)$$

$$\sum_{p \in P} x_i^p \leq 1 \quad \forall i \in \{1, \dots, T_j\}, \quad (11)$$

$$\sum_{p \in P} \sum_{i \in L_p^j} l_p x_i^p = T_j, \quad (12)$$

$$\sum_{\substack{p=1 \\ i-l_p > 0}}^r x_{i-l_p}^p = \sum_{p=1}^r x_i^p, \quad \forall i \in \{1, \dots, T_j\}, \quad (13)$$

$$\sum_{p \in P} \sum_{i \in L_p^j} x_i^p = N, \quad (14)$$

$$x_i^p \in \{0, 1\} \quad \forall p \in P, \forall i \in \{1, \dots, T_j - l_p + 1\}, \quad (15)$$

$$z^j \geq 0 \quad (16)$$

$$za_k^j, zb_k^j \in \mathbb{R}, \quad \forall k \in K \quad (17)$$

The objective function and inequalities (5)–(8) ensure that z^j is a maximum of the absolute value of za_k^j and zb_k^j for all $k \in K$. Equalities (9) and (10) compute the noise contribution of the sequence found by the MILP. The inequality (11) ensures that at most one pitch can start at each position. The equality (12) ensures that the non-empty part is completely filled with pitches, that is, we take into account all possible combinations having a fixed tire length. The equalities (13) prevent a pitch from starting inside another pitch, *i.e.* there is no overlapping. Remark that, this constraint corresponds to the existence of paths of length N in the graph presented in Section 3.1. The equality (14) ensures that exactly N pitches compose the tire.

The above (\mathcal{P}) is a model for the basic TNOP-AP problem. We can however easily add some more constraints such as those mentioned in Section 2.1:

$$\sum_{i \in L_p^j} x_i^p \geq \text{minOcc}_p \quad \forall p \in P, \quad (18)$$

$$\sum_{i \in L_p^j} x_i^p \leq \text{maxOcc}_p \quad \forall p \in P, \quad (19)$$

$$\sum_{k=0}^{\text{max}_p - 1} x_{i+kl_p}^p \leq \text{max}_p \quad \forall p \in P, \forall i \in \{1, \dots, T_j - l_p + 1\}, \quad (20)$$

$$x_i^1 + x_{i+l_1}^r \leq 1, \quad \forall i \in \{1, \dots, T_j - l_r - l_1 + 1\}, \quad (21)$$

$$x_i^r + x_{i+l_r}^1 \leq 1, \quad \forall i \in \{1, \dots, T_j - l_r - l_1 + 1\}, \quad (22)$$

where

- inequalities (18) and (19) correspond to the constraint ctMinMaxOcc,
- inequalities (20) express ctMaxSeq,
- inequalities (21) and (22) translate ctIncompatibility for the incompatibility of the smallest and the largest pitch types.

3.3 Exact algorithm to solve TNOP-EX

Recall that the MILP (\mathcal{P}) presented in the previous section allows to solve TNOP-AP. In this section, we propose an algorithm based on (\mathcal{P}) allowing to find the optimal solution of the TNOP-EX regarding the real noise (see Section 2.2.1).

From the MILP (\mathcal{P}) we use the Branch-and-Cut framework of ILP solvers to ensure that the solution is optimal for the TNOP-EX. Let z_{ub} be an upper bound for the approximated noise. We initialize z_{ub} by the heuristic algorithm described in [3]. During the execution of the Branch-and-Bound algorithm associated with the solving of (\mathcal{P}), for each integral feasible solution \tilde{x} found by the Branch-and-Bound, the real noise is computed and denoted by $\text{noise}_{c_n}(\tilde{x})$. If $\text{noise}_{c_n}(\tilde{x}) < z_{ub}$ then the solution found is better, so that we improve the upper bound by setting $z_{ub} = \text{noise}_{c_n}(\tilde{x})$. As \tilde{x} is momentarily the best solution found, we save it in x^* and we add the inequalities

$$z^j \leq z_{ub} \quad (23)$$

and

$$\sum_{p \in P} \sum_{i \in \{1, \dots, T_j\}} \tilde{x}_i^p x_i^p \leq N - 1 \quad (24)$$

to the current MILP. We then continue the Branch-and-Bound algorithm. The inequality (23) updates the upper bound of z^j . Adding the inequality (24) makes the solution \tilde{x} unfeasible for the MILP.

This procedure can be easily done using specific callback in the ILP solver (*e.g.* lazy constraint callback for CPLEX). The idea of inequalities (23) is similar to the no-good cut inequalities but we also cut valid solutions in order to converge to the best solution (see *e.g.* [7]).

The algorithm stops when no solution exists and the best solution is x^* with a noise $noise_{c_n}(x^*)$. As we consider the Branch-and-Bound on the MILP (\mathcal{P}) allowing to solve TNOP-AP, the linear relaxation still provides a valid lower bound, and thus all solutions need not be enumerated.

Recall that, as mentioned in Section 2.2.1, the noise is invariant under circular permutations and, as $\min Occ_p > 0$ for each $p \in P$, we can always assume that the first pitch is of type $p = 1$. Hence, we add to (\mathcal{P}) the equality

$$x_1^1 = 1 \tag{25}$$

as a constraint, breaking some of the symmetries and thus reducing the space of feasible solutions. Mind that this can only be done for the computation of the exact noise and not for the approximated noise.

To break the symmetry, we also propose the following method: Given the sequence \tilde{x} associated to constraint (24), we can add $N - 1$ constraints (24) associated to each sequence obtained from \tilde{x} by circular permutation. Note that further symmetry-defeating methods have been introduced in the literature (see *e.g.* [15], [8]) but those do not apply to our situation since our model does not allow for these particular symmetries.

4 Implementation and experimental results

In order to compare experimental results produced by the different methods, the tests have been carried out using the same parameters as for the case of Genetic Algorithms (GA), studied in [1]:

- There are three different pitch types of length ratios 1, 1.25 and 1.5, respectively.
- The height of a pitch is $h = 100$.
- The groove is $q = 0.1$.

Notice that these parameters correspond to realistic values, the same pitch types have been studied in [6], for the Goodyear patent described in [12].

To solve the TNOP-EX with GA [1], JAVA is used for the implementation. The main parameters of the GA are: a maximal population size of 1500 and a crossover probability of 0.3. The crossover is performed randomly. The mutation probability is 0.15 and the mutation is also random. Roulette and Ranking selection has been used, with a selection pressure of 0.4. For performance reasons, the coding of the genes has been realized as final Byte Array instead of a binary encoding. The value semantics of the genes save memory usage and enable the use of the efficient "==" operator. A chromosome then is composed of genes.

To solve the TNOP-AP with the MILP (\mathcal{P}) we used the CPLEX 12.7.1 solver. The iterative algorithm solving (\mathcal{P}) for each tire length T_j was implemented in JAVA. The exact algorithm to solve TNOP-EX presented in Section 3.3 was implemented in JAVA using CPLEX and the lazy constraint callback. The numerical tests have been carried out on an Intel Core i5 of 3.4 GHz in a Linux environment.

In Tables 1, 2 and 3, an instance is considered as a triple $(N, \min Occ_p, \max Occ_p)$. According to the tests made with heuristic methods, we do not take into account the constraints (20), (21) and (22). The computational times are given in *sec*, *min:sec* and *hrs:min:sec*.

Table 1 presents some experimental results for the heuristic method (GA) and the MILP with B&C, the latter computing the exact noise. We can remark that the sequences found coincide up

Table 1: Experimental results – exact solutions

instance	noise GA = noise MILP	pitch sequence GA	time GA	pitch sequence MILP with B&C	time MILP with B&C
(10,1,8)	9.019	1311323331	5	1323331131	7
(10,2,6)	9.247	1231123333	3	1233331231	6
(10,2,4)	9.268	2213111333	3	1333221311	3
(10,3,4)	9.368	1233321123	3	1231233321	2
(15,1,13)	7.027	311113311133312	6	111333123111133	25:27
(15,2,11)	7.236	112131223333111	5	131223333111112	27:30
(15,4,7)	7.261	111222123333111	5	122212333311111	13:13
(15,4,6)	7.439	332112331112213	4	112331112213332	10:25
(20,6,8)	6.444	11123332312321132112	26	12321132112111233323	153:53:39
(60,1,58)	4.131 (noise GA)	21231123311232313111 21112311231111131332 13323323333331212131	02:21:34	–	–
(60,10,40)	3.899 (noise GA)	12311331212133312113 21131112223333311111 23311231333113331121	03:35:87	–	–

to circular permutation and hence we can also confirm that the solutions found by the heuristic method are also exact. The computational time remains small for GA but increases in a significant way for the MILP with B&C. Indeed, for 10 pitches, the computation times are very close for GA and B&C. For 15 pitches the computation time for B&C ranges from 10 to 25 minutes which still remains acceptable, but from 20 pitches on, it increases considerably. Notice that the genetic algorithm cannot ensure to find the optimal solution, *e.g.* the noise found for the instance (60, 10, 40) is 3.899 and the one for the instance (60, 1, 58) is 4.132, whereas the noise of the latter should be smaller since its space of feasible solutions is larger than that of the former.

Table 2 shows the results found for the computation of the approximated noise, based on the same instances. The presented values are the following: *noise MILP* is the approximated noise, *real noise* is the exact noise of the sequence found by the MILP and the *optimal noise* is the noise of the optimal sequence, presented in Table 1. The gap is computed between the *real noise* and the *optimal noise*, with respect to the latter. Notice that, for each of the sequences found, the real noise is an upper bound for the optimal noise. We can also remark that, from 15 pitches on, the solution found is not optimal but the computational time is significantly shorter than that of the exact algorithm (MILP with B&C) and the gap might be considered as reasonable. Moreover, although the theoretical gap between the real and the optimal noises is bounded by 42%, we notice that for our instances the gap computed is much lower, ranging from 3 to 13.2%.

In Section 3.3, we proposed two possible improvements due to the invariance of the noise computing under symmetries. In Table 3, we compare the computation times when taking them into account: symmetry 1 corresponds to constraint (25) and symmetry 2 to the N constraints coming from circular permutations. We remark that none of the two symmetries allows to reduce the computation time.

Table 2: Experimental results – approximated solutions

instance	pitch sequence	time	noise MILP	real noise	optimal noise	gap (%)
(10,1,8)	1121133231	4	7.100	9.540	9.019	5.5
(10,2,6)	1121133231	3	7.100	9.540	9.247	3
(10,2,4)	1223331113	1	7.177	9.638	9.268	3.8
(10,3,4)	1332311212	1	7.397	9.921	9.367	5.9
(15,1,13)	223333121111131	1:12	5.613	7.852	7.027	10.5
(15,2,11)	13122333111112	1:11	5.613	7.852	7.236	7
(15,4,7)	123333131222111	52	5.949	8.362	7.260	13.2
(15,4,6)	123333131222111	37	5.949	8.362	7.439	11
(20,6,8)	22323333113221221111	24:13	5.094	6.931	6.444	7

Table 3: Computational times using symmetries

instance	no symmetry	symmetry 1	symmetry 2
(10,1,8)	7	8	10
(10,2,6)	6	7	8
(10,2,4)	3	4	5
(10,3,4)	2	2	3
(15,1,13)	25:27	4:38:48	52:12
(15,4,6)	10:25	1:52:05	20:29

5 Conclusion

We had started in [2] with a new approach using a MILP for the Tire Noise Optimization Problem. Indeed, the tire shape has to satisfy several constraints *a priori* allowing for linear programming. However, the objective function depends a very non-linear way on the data. In [2], we proposed an approximated tire noise model allowing for its linearisation. The approximated noise is guaranteed to differ from the optimal noise by a ratio of at most 42%. In the present paper, we take some more constraints into account and provide an exact method to minimize the noise thanks to a Branch&Cut algorithm, where we also consider possible symmetries in the pitch sequences. We have tested all proposed algorithms and compared their performances. The MILP approach always finds a solution in reasonable time, up to a ratio of 14%, which is much lower than the theoretical 42% one. The first results on Branch&Cut algorithm show that our algorithm can provide an exact solution for at most 20 pitches. As an extension, it would be interesting to enhance the optimization part to be able to compute the noise for realistic instances, *e.g.* by developing a dedicated algorithm or by adding valid inequalities to strengthen the linear relaxation. This opens a new research field on the tire noise problem.

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