Linear wave equations

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Introduction

This lecture deals with linear wave equations on Lorentzian manifolds. We first recall the physical origin of that equation which describes the propagation of a wave in space. Consider $\mathbb{R}^3$ with its canonical cartesian coordinates and let $u(t,x)$ denote the height of the wave at $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and at time $t \in \mathbb{R}$. Then u solves

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2},$$

i.e., $\Box u = 0$, where $\Box := \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the so-called d'Alembert operator on the 4-dimensional Minkowski space $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$.
What can be said on the solutions u to the wave equation, or equivalently on the kernel of $\Box$? The operator $\Box$ is obviously linear, so that its kernel is a vector space. The functions $(t,x) \mapsto \cos(nt)\cos(nx_1)$, for n running over $\mathbb{Z}$, all belong to $\mathrm{Ker}(\Box)$ so that it is infinite dimensional. However, if one prescribes the height and the speed of the wave at some fixed time then it is well-known (see also Sec. 4.3) that the corresponding solution must be unique.

Our aim here is to handle wave equations associated to some kind of generalized d'Alembert operators on any Lorentzian manifold. In particular we want to discuss the local and global existence of solutions as well as give a short motivation on how those provide the fundamental background for some quantization theory.

The first section makes the concept of generalized d'Alembert operator more precise and recalls the central role of fundamental solutions for differential operators. Fundamental solutions for the d'Alembert operator on the Minkowski space can be obtained from the so-called Riesz distributions: this is the object of Section 2. They are called advanced or retarded fundamental solutions according to their support being contained in the causal future or past of the origin. Turning to "curved" spacetimes, i.e., to Lorentzian manifolds, there does not exist any analog of Riesz distribution, at least globally. Nevertheless using normal coordinates it is always possible to transport Riesz distributions from the tangent space at a point to a neighbourhood of this point. The distributions obtained do not lead to local fundamental solutions for the classical d'Alembert operator in a straightforward manner (Sec. 3.1), however combining linearly an infinite number of them, solving the wave equation formally (Sec. 3.2) and correcting the formal series using a cut-off function one obtains a local fundamental solution up to an error term (Proposition 2 in Sec. 3.3). General methods of functional analysis then allow one to get rid of this error term and construct true local fundamental solutions for any generalized d'Alembert operator (Corollary 2). Those fundamental solutions are in some sense near to the formal series from which they are constructed (Corollary 3).

The global aspect of the theory is based on a completely different approach. First it would be illusory to construct global fundamental solutions on any spacetime, therefore we restrict the issue to globally hyperbolic spacetimes, which can be thought of as the analog of complete Riemannian manifolds in the Lorentzian setting. In that case global fundamental solutions for generalized d'Alembert operators are provided by the solutions to the so-called Cauchy problem associated to such operators, see Section 4. After discussing uniqueness of fundamental solutions (Sec. 4.1) we show how local and then global solutions to the Cauchy problem can be constructed (Sec. 4.2 and 4.3) and global fundamental solutions be deduced from them (Sec. 4.4). Here it should be pointed out that the local existence of fundamental solutions (Sec. 3.3) actually enters this global construction in a crucial way since it provides (local) solutions to the so-called inhomogeneous wave equation, see Proposition 6.
We end this survey by introducing the concept of (advanced or retarded) Green's operators associated to generalized d'Alembert operators and by showing their elementary properties, in particular their tight relationship with fundamental solutions (Sec. 5). Green's operators constitute the starting point for the so-called local approach to quantization, since their existence together with a few additional assumptions on a given spacetime directly provide a C^* -algebra in a functorial way, see K. Fredenhagen's lecture for more details.

This lecture is intended as an introduction to the subject for students from the first or second university level. Only the main results and some ideas are presented, nevertheless most proofs are left aside. We shall also exclusively deal with scalar operators, although all results of Sections 3 to 5 can be extended to generalized d'Alembert operators acting on sections of vector bundles. For a thorough and complete introduction to the topic as well as a list of references we refer to [1], on which this survey is widely based.

1 General setting

In this section we describe the general frame in which we want to work.

1.1 Generalized wave equations

In the following and unless explicitly mentioned (M^n, g) will denote an n-dimensional Lorentzian manifold and $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$.

Definition 1. *A* generalized d'Alembert operator *on M is a linear differential operator of second order P on M whose principal symbol is given by minus the metric.*

In the scalar setting, a generalized d'Alembert operator P is a linear differential operator of second order which can be written in local coordinates

$$P = -\sum_{i,j=0}^{n-1} g^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=0}^{n-1} a_j(x)\frac{\partial}{\partial x_j} + b_1(x)$$

where a_j and b_1 are smooth $\mathbb{K}$-valued functions of x and $(g^{ij})_{i,j} := (g_{kl})_{k,l}^{-1}$. In particular, if P is a generalized d'Alembert operator on M then so is its formal adjoint P^*.

Examples

1. The d'Alembert operator of (M^n, g) is defined on smooth functions by

$$\Box f := -\mathrm{tr}_g(\mathrm{Hess}(f)),$$

where $\mathrm{Hess}(f)(X,Y) := \langle \nabla_X \mathrm{grad} f, Y\rangle$ and tr_g is the trace w.r.t. the metric g. Here we denote as usual by ∇ the Levi-Civita covariant derivative on TM and by $\mathrm{grad} f$ the gradient vector field of the (real- or complex-valued) function f. In normal coordinates about $x \in M$

$$\Box f = -\mu_x^{-1}\sum_{j=0}^{n-1}\frac{\partial}{\partial x_j}(\mu_x(\mathrm{grad} f)_j),$$

with $\mu_x := |\det((g_{ij})_{i,j})|^{\frac{1}{2}}$, therefore the principal symbol of $\Box$ is given by minus the metric. For instance on the Minkowski space $M = \mathbb{R}^n$ one has $\mu_x = 1$, hence

$$\Box f = -\sum_{j=0}^{n-1}\frac{\partial}{\partial x_j}(\varepsilon_j \frac{\partial f}{\partial x_j}) = \frac{\partial^2 f}{\partial x_0^2} - \sum_{j=1}^{n-1}\frac{\partial^2 f}{\partial x_j^2},$$

where $\varepsilon_0 := -1$ and $\varepsilon_j := 1$ for every $j \geq 1$.

2. The general form of a generalized d'Alembert operator is actually given by $\Box + a + b$, where a and b are linear differential operators of first and zero order respectively (b is a smooth function on M). In particular the Klein-Gordon operator $\Box + m^2$, where $m > 0$ is a constant, is a generalized d'Alembert operator.

Other examples of generalized d'Alembert operators are given by the square of any generalized Dirac operator on a Clifford bundle (see [3] for definitions) such as the classical Dirac operator acting on spinors or the Euler operator acting on differential forms. For the sake of simplicity we do not deal with vector bundles, hence we restrict the whole discussion to scalar operators, i.e., to operators acting on scalar-valued functions. From now on any differential operator will be implicitly assumed to be scalar.

Definition 2. *Let P be a generalized d'Alembert operator on a Lorentzian manifold M. The* wave equation *associated to P is*

$$Pu = f$$

for a given $f \in C^\infty(M, \mathbb{K})$.

We want to prove existence and uniqueness results - locally as well as globally - for waves, i.e., for solutions $u \in C^\infty(M, \mathbb{K})$ to this generalized wave equation for given data f lying in a particular class of functions. In this context we recall the central role played by fundamental solutions.

1.2 Fundamental solutions

We first recall what we need about distributions.

Definition 3. *The space of $\mathbb{K}$-valued distributions on M is defined as*

$$\mathcal{D}'(M, \mathbb{K}) := \{T : \mathcal{D}(M, \mathbb{K}) \longrightarrow \mathbb{K} \text{ linear and continuous}\},$$

where $\mathcal{D}(M, \mathbb{K}) := \{\varphi \in C^\infty(M, \mathbb{K}), \operatorname{supp}(\varphi) \text{ compact}\}$ denotes the space of $\mathbb{K}$-valued test-functions on M.

For the definition of the topology of $\mathcal{D}(M, \mathbb{K})$ we refer to e.g. [1, Chap. 1]. We next describe how functions can be understood as distributions and how differential operators act on distributions:

- For any fixed $f \in C^\infty(M, \mathbb{K})$ the map $\varphi \mapsto \int_M f(x)\varphi(x)dx$, $\mathcal{D}(M, \mathbb{K}) \to \mathbb{K}$, defines a $\mathbb{K}$-valued distribution on M. Here and in the following we denote by dx the canonical measure associated to the metric g on M. We denote this distribution again by f, i.e., we identify $C^\infty(M, \mathbb{K})$ as a subspace of $\mathcal{D}'(M, \mathbb{K})$.

- Given $T \in \mathcal{D}'(M, \mathbb{K})$ and a linear differential operator P on M one can define
$$PT[\varphi] := T[P^*\varphi]$$
for any $\varphi \in \mathcal{D}(M, \mathbb{K})$, where P^* denotes the formal adjoint of P. It is an easy exercise using the definition of the topology of $\mathcal{D}(M, \mathbb{K})$ to show that $PT \in \mathcal{D}'(M, \mathbb{K})$.

Definition 4. *Let P be a generalized d'Alembert operator on M and $x \in M$. A* fundamental solution *for P at x on M is a distribution $F \in \mathcal{D}'(M, \mathbb{K})$ with*

$$PF = \delta_x,$$

where δ_x is the Dirac distribution in x (i.e., $\delta_x[\varphi] := \varphi(x)$ for all $\varphi \in \mathcal{D}(M, \mathbb{K})$).

What do fundamental solutions for P - which are distributions - have to do with solutions of the wave equation $Pu = f$ - which we wish to be smooth functions? The idea is that one can construct from the fundamental solutions for P solutions u to the wave equation $Pu = f$ for "any" given f. We state this in a bit more precise but purely formal manner, see e.g. Proposition 6 for a situation where the following computation can be carried out under some further assumptions.
Assume namely one had at each $x \in M$ a fundamental solution $F(x) \in \mathcal{D}'(M, \mathbb{K})$ for P at x on M and moreover that $F(x)$ depends continuously on x, meaning that $x \mapsto F(x)[\varphi]$ is a continuous function for all $\varphi \in \mathcal{D}(M, \mathbb{K})$. Fix $f \in C^\infty(M, \mathbb{K})$ and consider

$$u[\varphi] := \int_M f(x)F(x)[\varphi]dx$$

for all $\varphi \in \mathcal{D}(M, \mathbb{K})$. In other words, u is some kind of convolution product of f with F. Assume u to be a well-defined distribution, then for every $\varphi \in \mathcal{D}(M, \mathbb{K})$ one has

$$\begin{aligned} Pu[\varphi] &= u[P^*\varphi] \\ &= \int_M f(x)F(x)[P^*\varphi]dx \\ &= \int_M f(x)PF(x)[\varphi]dx \\ &= \int_M f(x)\varphi(x)dx \\ &= f[\varphi], \end{aligned}$$

that is, $Pu = f$ in the distributional sense. Thus every wave equation associated to P can be solved on M, at least in $\mathcal{D}'(M, \mathbb{K})$.

Therefore we momentarily forget about the wave equation itself and concentrate on the search for fundamental solutions. As we shall already see in the next section, if there exists one fundamental solution then there exist many of them in general, hence one has to fix an extra condition to single one particular fundamental solution out. The most natural condition here deals with the support of the fundamental solution (recall that the support of a distribution on a manifold M is the complementary subset of the largest open subset of M on which the distribution vanishes). Namely, assuming that M is a spacetime (connected timeoriented Lorentzian manifold), we look for fundamental solutions $F_+(x), F_-(x) \in \mathcal{D}'(M, \mathbb{K})$ for P at x on M such that

$$\mathrm{supp}(F_+(x)) \subset J_+^M(x) \qquad \text{resp.} \qquad \mathrm{supp}(F_-(x)) \subset J_-^M(x), \tag{1}$$

where $J_+^M(x)$ and $J_-^M(x)$ are the causal future and past of x in M respectively. Such an $F_+(x)$ (resp. $F_-(x)$) will be called *advanced* (resp. *retarded*) fundamental solution for P at x on M. In physics this condition has to do with the finiteness of the propagation speed of a wave.

Remark. The most naive condition would be to require the support to be compact. There exists unfortunately no fundamental solution with compact support in general, as the example of $P = \Box$ on M already shows. Indeed if such a distribution F existed it could be extended to a continuous linear form on $C^\infty(M, \mathbb{K})$ (see e.g. A. Strohmaier's lecture), hence any non-zero constant φ would satisfy

$$\varphi(x) = \Box F[\varphi] = F[\underbrace{\Box\varphi}_{0}] = 0,$$

which would be a contradiction. In particular there exists no fundamental solution for $\Box$ on compact Lorentzian manifolds.

2 Riesz distributions on the Minkowski space

In this section we describe the fundamental solutions for the d'Alembert operator at 0 on the Minkowski space $(\mathbb{R}^n, \langle\langle\cdot\,,\cdot\rangle\rangle_0)$ (recall that $\langle\langle x, y\rangle\rangle_0 := -x_0y_0 + \sum_{j=1}^{n-1} x_jy_j$ for all $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ in $\mathbb{R}^n$) for $n \geq 2$.

Definition 5. *For $\alpha \in \mathbb{C}$ with $\Re\mathrm{e}(\alpha) > n$ let $R_+(\alpha)$ and $R_-(\alpha)$ be the functions defined on $\mathbb{R}^n$ by*

$$R_\pm(\alpha)(x) := \begin{cases} C(\alpha, n)\gamma(x)^{\frac{\alpha-n}{2}} & \text{if } x \in J_\pm(0) \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma := -\langle\langle\cdot\,,\cdot\rangle\rangle_0$, $C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}$ and $z \mapsto (z-1)!$ is the Gamma-function.

Recall that the Gamma-function is defined by $\{z \in \mathbb{C},\, \Re e(z) > 0\} \to \mathbb{C}$, $z \mapsto \int_0^\infty t^{z-1}e^{-t}dt$. It is a holomorphic nowhere vanishing function on $\{\Re e(\alpha) > 0\}$ and satisfies

$$z! = z \cdot (z-1)! \tag{2}$$

for every $z \in \mathbb{C}$ with $\Re e(z) > 0$.

The function $R_\pm(\alpha)$ is well-defined because of $\gamma \geq 0$ on $J_\pm(0)$, it is continuous on $\mathbb{R}^n$ and C^k as soon as $\Re e(\alpha) > n + 2k$. For any fixed $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ is holomorphic on $\{\Re e(\alpha) > n\}$. Moreover $R_\pm(\alpha)$ satisfies the first important property:

Lemma 1. *For all $\alpha \in \mathbb{C}$ with $\Re e(\alpha) > n$ one has*

$$\Box R_\pm(\alpha + 2) = R_\pm(\alpha). \tag{3}$$

In particular the map $\alpha \mapsto R_\pm(\alpha)$, $\{\Re e(\alpha) > n\} \to \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$ can be holomorphically extended on $\mathbb{C}$ such that (3) *holds for every $\alpha \in \mathbb{C}$.*

Of course by holomorphic extension we mean that, for every fixed $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$, the function $\alpha \mapsto R_\pm(\alpha)[\varphi]$ can be holomorphic extended on $\mathbb{C}$.

Proof. The identity (3) follows from the two following ones:

- 1^{st} *identity*:

$$\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2) R_\pm(\alpha + 2). \tag{4}$$

  Proof of (4): Both l.h.s and r.h.s vanish outside $J_\pm(0)$ so that we just have to prove the identity on $J_\pm(0)$. By Definition 6 one has on $J_\pm(0)$:

$$\begin{aligned} \gamma \cdot R_\pm(\alpha) &= C(\alpha, n)\gamma^{\frac{\alpha+2-n}{2}} \\ &= \frac{C(\alpha, n)}{C(\alpha+2, n)} R_\pm(\alpha + 2), \end{aligned}$$

  with

$$\begin{aligned} \frac{C(\alpha, n)}{C(\alpha+2, n)} &= \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!} \cdot \frac{(\frac{\alpha}{2}-1+1)!(\frac{\alpha+2-n}{2})!}{2^{1-\alpha-2}\pi^{\frac{2-n}{2}}} \\ &\overset{(2)}{=} 4 \cdot \frac{\alpha}{2} \cdot \frac{\alpha+2-n}{2} \\ &= \alpha(\alpha - n + 2), \end{aligned} \tag{5}$$

  which achieves the proof of (4).

- 2^{nd} *identity*: for every $X \in \mathbb{R}^n$,

$$\partial_X \gamma \cdot R_\pm(\alpha) = 2\alpha\, \partial_X R_\pm(\alpha + 2). \tag{6}$$

  Proof of (6): From its definition, $R_\pm(\alpha + 2)$ is C^1 on $\mathbb{R}^n$. We show the identity on $I_\pm(0)$. On this domain and for every $X \in \mathbb{R}^n$,

$$\begin{aligned}\partial_X\gamma\cdot R_\pm(\alpha) &= C(\alpha,n)\gamma^{\frac{\alpha-n}{2}}\cdot\partial_X\gamma\\ &= \frac{2C(\alpha,n)}{\alpha+2-n}\partial_X(\gamma^{\frac{\alpha+2-n}{2}}),\end{aligned}$$

with $\frac{2C(\alpha,n)}{\alpha+2-n}\stackrel{(5)}{=}2\alpha C(\alpha+2,n)$, so that

$$\begin{aligned}\partial_X\gamma\cdot R_\pm(\alpha) &= 2\alpha C(\alpha+2,n)\partial_X(\gamma^{\frac{\alpha+2-n}{2}})\\ &= 2\alpha\,\partial_X R_\pm(\alpha+2)\end{aligned}$$

which is (6).

We now prove (3). Let first $\alpha\in\mathbb{C}$ with $\Re e(\alpha)>n+2$. The function $R_\pm(\alpha+2)$ is then C^2 on $\mathbb{R}^n$ and for every $X\in\mathbb{R}^n$,

$$\begin{aligned}\partial_X^2 R_\pm(\alpha+2) &\stackrel{(6)}{=} \frac{1}{2\alpha}\partial_X(\partial_X\gamma\cdot R_\pm(\alpha))\\ &= \frac{1}{2\alpha}\Big(\partial_X^2\gamma\cdot R_\pm(\alpha)+\partial_X\gamma\cdot\partial_X R_\pm(\alpha)\Big)\\ &\stackrel{(6)}{=} \frac{1}{2\alpha}\Big(\partial_X^2\gamma\cdot R_\pm(\alpha)+\frac{1}{2(\alpha-2)}(\partial_X\gamma)^2\cdot R_\pm(\alpha-2)\Big),\end{aligned}$$

with $(\partial_X\gamma)_x=-2\langle\langle x,X\rangle\rangle_0$ and $\partial_X^2\gamma=2\gamma(X)$ for any $x\in\mathbb{R}^n$. Hence, if $(e_j)_{0\le j\le n}$ denotes the canonical basis of $\mathbb{R}^n$,

$$\begin{aligned}\Box R_\pm(\alpha+2) &= -\sum_{j=0}^n\varepsilon_j\frac{\partial^2 R_\pm(\alpha+2)}{\partial x_j^2}\\ &= -\frac{1}{2\alpha}\sum_{j=0}^n\Big(2\varepsilon_j\gamma(e_j)R_\pm(\alpha)+\frac{2}{\alpha-2}\varepsilon_j\langle\langle\cdot\,,e_j\rangle\rangle_0^2 R_\pm(\alpha-2)\Big)\\ &= -\frac{1}{\alpha}\Big(-nR_\pm(\alpha)-\frac{1}{\alpha-2}\gamma\cdot R_\pm(\alpha-2)\Big)\\ &\stackrel{(4)}{=} \frac{1}{\alpha}\Big(nR_\pm(\alpha)+(\alpha-n)R_\pm(\alpha)\Big)\\ &= R_\pm(\alpha),\end{aligned}$$

where, as usual, $\varepsilon_0=\langle\langle e_0,e_0\rangle\rangle_0=-1$ and $\varepsilon_j=\langle\langle e_j,e_j\rangle\rangle_0=1$ for every $1\le j\le n-1$. This proves (3) for $\Re e(\alpha)>n+2$. It follows from the holomorphic dependence in α of both distributions $R_\pm(\alpha)$ and $\Box R_\pm(\alpha+2)$ that (3) must actually hold on the whole domain $\{\Re e(\alpha)>n\}$.

Equation (3) allows one to define inductively $R_\pm(\alpha)$ for every α with $\Re e(\alpha)>n-2k$ with $k\in\mathbb{N}$. Indeed one can define the distribution $R_\pm(\alpha):=\Box R_\pm(\alpha+2)$ for all $\alpha\in\mathbb{C}$ with $\Re e(\alpha)>n-2$. For $\Re e(\alpha)>n$ this is of course not a definition but simply coincides with (3). Fix now $\varphi\in\mathcal{D}(\mathbb{R}^n,\mathbb{C})$. Since $\alpha\mapsto R_\pm(\alpha)[\varphi]$ is holomorphic on $\{\Re e(\alpha)>n\}$ then so is $\alpha\mapsto\Box R_\pm(\alpha)[\varphi]$. Hence the extension of $\alpha\mapsto R_\pm(\alpha)[\varphi]$ onto $\{\Re e(\alpha)>n-2\}$ is again holomorphic and (3) is again trivially satisfied for those α. This shows the first step of the induction and achieves the proof of Lemma 1.

Definition 6. *The* advanced *(resp. the* retarded*)* Riesz distribution *to the parameter $\alpha \in \mathbb{C}$ is defined to be $R_+(\alpha)$ (resp. $R_-(\alpha)$).*

The second important properties for our purpose are the following:

Lemma 2. *The Riesz distributions satisfy:*

1. *For any $\alpha \in \mathbb{C}$ one has* $\mathrm{supp}(R_\pm(\alpha)) \subset J_\pm(0)$.
2. $R_\pm(0) = \delta_0$*, the Dirac distribution at the origin.*

The first assertion follows directly from the definition of the Riesz distributions and from $\Box$ being a differential operator. The second one requires a more technical and detailed study of the distribution $R_\pm(2)$, we refer to [1, Prop. 1.2.4] for a proof. Note also that, although $R_\pm(\alpha)$ is complex-valued on $\mathcal{D}(M,\mathbb{C})$, its restriction to $\mathcal{D}(M,\mathbb{R})$ for real α gives a real-valued distribution.

As a consequence of Lemmas 1 and 2 we obtain:

Corollary 1. *The Riesz distribution $R_\pm(2)$ satisfies*

$$\left| \begin{array}{ll} \Box R_\pm(2) & = \delta_0 \\ \\ \mathrm{supp}(R_\pm(2)) \subset J_\pm(0). & \end{array} \right.$$

In particular $R_+(2)$ (resp. $R_-(2)$) is an advanced (resp. retarded) fundamental solution for $\Box$ at 0 on $\mathbb{R}^n$.

Remark. The set of fundamental solutions for a generalized d'Alembert operator P at a point is an affine subspace of $\mathcal{D}'(M,\mathbb{K})$ with direction $\mathrm{Ker}(P)$. For $M = \mathbb{R}^n$, since $\mathrm{Ker}(\Box)$ contains all constant extensions of harmonic functions on the spacelike slice $\mathbb{R}^{n-1}$, the space of fundamental solutions for $\Box$ at 0 on $\mathbb{R}^n$ is at least 2-dimensional for $n = 2$ and is infinite dimensional for $n \geq 3$ (remember that holomorphic functions on $\mathbb{C}$ are harmonic). This shows evidence that there exist significantly more than one fundamental solution as soon as there is one. Actually even if one keeps the support conditions (1) there may exist more than one fundamental solution. Consider for instance $P := \Box$ on $M := \mathbb{R}\times]-1,1[\subset \mathbb{R}^2$ with the induced Lorentzian metric. The restriction F_+ of $R_+(2)$ on M is an advanced fundamental solution for $\Box$ at 0 on M. On the other hand the map $\varphi \mapsto R_+(2)[\varphi \circ ((0,3) + \mathrm{Id}]$, $\mathcal{D}(\mathbb{R}^2,\mathbb{C}) \to \mathbb{C}$ defines a distribution on $\mathbb{R}^2$ which is obviously an advanced fundamental solution for $\Box$ at $x = (0,3)$ on $\mathbb{R}^2$, thus its restriction G_+ onto M lies in $\mathrm{Ker}(\Box)$ with support contained in $J_+(3) \cap M$ which is again a subset of $J_+^M(0)$. Therefore $F_+ + G_+$ is another advanced fundamental solution for $\Box$ at 0 in M.
However we shall see in Corollary 4 that, on $\mathbb{R}^n$, there exists exactly one advanced and one retarded fundamental solution for $\Box$ at 0.

3 Local fundamental solutions

In this section we come back to the general setting and construct local fundamental solutions for any generalized d'Alembert operator on any Lorentzian manifold.

3.1 Attempt

We first examine the case where $P = \square$ on M. From the local point of view the most naive attempt to obtain fundamental solutions for $\square$ on M consists in pulling the Riesz distributions $R_\pm(2)$ back from the tangent space at a point onto a neighbourhood of that point:

Definition 7. *Let Ω be a geodesically starshaped neighbourhood of a point x in a Lorentzian manifold (M^n, g). Let $\exp_x : \exp_x^{-1}(\Omega) \to \Omega$ be the exponential map and $\mu_x : \Omega \to \mathbb{R}$, $\mu_x := |\det((g_{ij})_{i,j})|^{\frac{1}{2}}$. The Riesz distribution at x on Ω to the parameter $\alpha \in \mathbb{C}$ is defined by*

$$\begin{aligned} R_\pm^\Omega(\alpha, x) : \mathcal{D}(\Omega, \mathbb{C}) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto R_\pm(\alpha)[(\mu_x \varphi) \circ \exp_x], \end{aligned}$$

where $R_\pm(\alpha)$ denotes the Riesz distribution to the parameter α.

The factor μ_x enters the definition of $R_\pm^\Omega(\alpha, x)$ in order to take the difference between the volume form of M and that of T_xM (w.r.t. g_x) into account: indeed $\mathrm{dvol}_g = \mu_x (\exp_x^{-1})^* \mathrm{dvol}_{g_x}$ on Ω.

By definition $R_\pm^\Omega(\alpha, x)$ is a distribution on Ω. It can be relatively easily deduced from its definition and from Lemmas 1 and 2 that $R_\pm^\Omega(\alpha, x)$ satisfies the following [1, Prop. 1.4.2]:

Lemma 3. *Let Ω be a geodesically starshaped neighbourhood of a point x in a Lorentzian manifold (M^n, g). Then the Riesz distributions at x on Ω satisfy:*

1. $R_\pm^\Omega(0, x) = \delta_x$.
2. $\mathrm{supp}(R_\pm^\Omega(\alpha, x)) \subset J_\pm^\Omega(x)$.
3. $\square R_\pm^\Omega(\alpha + 2, x) = (\frac{\square \Gamma_x - 2n}{2\alpha} + 1) R_\pm^\Omega(\alpha, x)$ *for every $\alpha \neq 0$, where $\Gamma_x := \gamma \circ \exp_x^{-1}$ on Ω.*

Although the first two properties make $R_\pm^\Omega(2, x)$ a good candidate to become a fundamental solution for $\square$ at x, the third one sweeps this hope away since the term $\square \Gamma_x - 2n$ does not vanish in general. Therefore one has to look for another approach to find fundamental solutions for $\square$.

3.2 Formal ansatz

Considering again any generalized d'Alembert operator P we fix a point $x \in M$ and a geodesically starshaped neighbourhood of x in (M^n, g). We look for fundamental solutions for P of the form

$$T_\pm := \sum_{k=0}^{\infty} V_x^k \cdot R_\pm^\Omega(2k+2, x),$$

where, for each k, V_x^k is a smooth coefficient depending on x. Of course this series is *a priori* only formal. Nevertheless if one plugs it into the equation $PT_\pm = \delta_x$, differentiates it termwise, uses relations satisfied by the Riesz distributions such as (4), (6) or Lemma 3 and identifies the coefficients standing in front of the $R_\pm^\Omega(2k+2,x)$ then one obtains [1, Sec. 2.1]

$$\nabla_{\mathrm{grad}\Gamma_x} V_x^k - (\frac{1}{2}\Box\Gamma_x - n + 2k)V_x^k = 2kPV_x^{k-1} \tag{7}$$

for every $k \geq 1$ as well as $V_{x,x}^0 = 1$. This leads to the following

Definition 8. *Let $\Omega \subset M$ be convex. A sequence of* Hadamard coefficients *for P on Ω is a sequence $(V^k)_{k\geq 0}$ of $C^\infty(\Omega\times\Omega, \mathbb{C})$ which fulfills* (7) *and $V_{x,x}^0 = 1$, for all $x \in \Omega$ and $k \geq 1$, where we denote by $V_x^k := V_{x,\cdot}^k \in C^\infty(\Omega, \mathbb{C})$.*

The equation (7) that Hadamard coefficients must satisfy turns out to be a singular differential equation and can be solved without any further assumption [1, Sec. 2.2 & 2.3]. For the sake of simplicity we give a formula for the Hadamard coefficients only in the case where the operator P has no term of first order (the general formula involves the parallel transport of the connection which is canonically associated to P, see [1, Lemmas 1.5.5 & 2.2.2]).

Proposition 1. *Let $\Omega \subset M$ be a convex open subset in a Lorentzian manifold (M^n, g) and P be a generalized d'Alembert operator on M of the form $P = \Box + b$, where $b \in C^\infty(M, \mathbb{K})$. Then there exists a unique sequence of Hadamard coefficients for P on Ω. It is given for all $x, y \in \Omega$ by*

$$V_{x,y}^0 = \mu_x^{-\frac{1}{2}}(y)$$

and, for all $k \geq 1$,

$$V_{x,y}^k = -k\mu_x^{-\frac{1}{2}}(y)\int_0^1 \mu_x^{\frac{1}{2}}(\Phi(y,s))s^{k-1} \cdot (P_{(2)}V_x^{k-1}(\Phi(y,s))ds,$$

where $\Phi(y,s) := \exp_x(s\exp_x^{-1}(y))$, $\Phi : \Omega\times[0,1] \to \Omega$.

The index "(2)" in $P_{(2)}V_x^{k-1}$ stands for P acting on $z \mapsto V^{k-1}(x,z)$. The existence of Hadamard coefficients leads to the following definition:

Definition 9. *Let $\Omega \subset M$ be a convex open subset in a Lorentzian manifold (M^n, g) and P be a generalized d'Alembert operator on M. Let $(V^k)_{k\geq 0}$ be the sequence of Hadamard coefficients for P on Ω. The (advanced or retarded)* formal fundamental solution *for P at $x \in \Omega$ is the formal series*

$$R_\pm^\Omega(x) := \sum_{k=0}^{\infty} V_x^k \cdot R_\pm^\Omega(2k+2, x).$$

3.3 Exact local fundamental solutions

The existence of Hadamard coefficients still does not provide any (local) fundamental solution, since the series defining $R_\pm^\Omega(x)$ may diverge. The idea presented here for the construction of local fundamental solutions (which is that of [1]) consists in keeping the first terms of the formal fundamental solutions unchanged while multiplying the higher ones by a cut-off function.
More precisely, consider again a convex open subset Ω' in M. Let $\sigma : \mathbb{R} \to [0,1]$ be a smooth function with $\mathrm{supp}(\sigma) \subset [-1,1]$ and $\sigma_{|_{[-\frac{1}{2},\frac{1}{2}]}} = 1$. Fix an integer $N \geq \frac{n}{2}$ (this is just to ensure $R_\pm^{\Omega'}(2k+2, x)$ be a continuous function for any $k \geq N$) and a sequence $(\varepsilon_j)_{j\geq N}$ of positive real numbers. Set

$$\widetilde{R}_\pm(x) := \sum_{j=0}^{N-1} V_x^j \cdot R_\pm^{\Omega'}(2j+2, x) + \sum_{j=N}^{\infty} \sigma(\frac{\Gamma_x}{\varepsilon_j}) \cdot V_x^j \cdot R_\pm^{\Omega'}(2j+2, x) \qquad (8)$$

for every $x \in \Omega$ (recall that $\Gamma_x := \gamma \circ \exp_x^{-1}$ with $\gamma := -\langle\langle \cdot\,, \cdot \rangle\rangle_0$). The identity (8) does not *a priori* define a fundamental solution since it does not even define a distribution. However, for ε_j small enough both conditions are almost fulfilled (see [1, Lemmas 2.4.2-2.4.4]):

Proposition 2. *Let $\Omega' \subset M$ be convex and $\Omega \subset\subset \Omega'$ be relatively compact. Fix an integer $N \geq \frac{n}{2}$. Then there exists a sequence $(\varepsilon_j)_{j\geq N}$ of positive real numbers such that, for all $x \in \overline{\Omega}$, $\widetilde{R}_\pm(x)$ defines a distribution on Ω satisfying:*

a) $P_{(2)}\widetilde{R}_\pm(x) - \delta_x = K_\pm(x, \cdot)$, *where* $K_\pm \in C^\infty(\overline{\Omega} \times \overline{\Omega}, \mathbb{C})$,
b) $\mathrm{supp}(\widetilde{R}_\pm(x)) \subset J_\pm^{\Omega'}(x)$,
c) $y \mapsto \widetilde{R}_\pm(y)[\varphi]$ *is smooth on* Ω *for all* $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$.

In other words, choosing suitably the ε_j leads to a distribution depending smoothly on the base point and which is near to a fundamental solution in the sense that the difference $P_{(2)}\widetilde{R}_\pm(x) - \delta_x$ is a smooth function. How to obtain now a "true" fundamental solution? The main idea is to get rid of the error term using methods of functional analysis. Namely setting, for all $u \in C^0(\overline{\Omega}, \mathbb{C})$,

$$\mathcal{K}_\pm u := \int_{\overline{\Omega}} K_\pm(\cdot, y) u(y) dy,$$

the identity a) of Proposition 2 can be rewritten in the form

$$P_{(2)}\widetilde{R}_{\pm}(\cdot)[\varphi] = (\mathrm{Id} + \mathcal{K}_{\pm})\varphi$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$. One can therefore look for an *inverse* to the operator $\mathrm{Id} + \mathcal{K}_{\pm}$. It is well-known that, given a bounded endomorphism A of a Banach space, the operator $\mathrm{Id} + A$ is invertible as soon as $\|A\| < 1$. This is the main idea underlying the following proposition (see [1, Lemma 2.4.8]):

Proposition 3. *Let $\Omega \subset\subset \Omega'$ be a relatively compact causal domain in Ω' and assume that $\mathrm{Vol}(\overline{\Omega}) \cdot \|K_{\pm}\|_{C^0(\overline{\Omega}\times\overline{\Omega})} < 1$. Then $\mathrm{Id} + \mathcal{K}_{\pm}$ is an isomorphism $C^k(\overline{\Omega}, \mathbb{C}) \to C^k(\overline{\Omega}, \mathbb{C})$ for all $k \in \mathbb{N}$.*

Setting

$$F^{\Omega}_{\pm}(\cdot)[\varphi] := (\mathrm{Id} + \mathcal{K}_{\pm})^{-1}(y \mapsto \widetilde{R}_{\pm}(y)[\varphi]).$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$, we really obtain what we wanted: for any $x \in \Omega$ and $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$,

$$\begin{aligned}
(PF^{\Omega}_{\pm}(x))[\varphi] &= F^{\Omega}_{\pm}(x)[P^*\varphi] \\
&= \{(\mathrm{Id} + \mathcal{K}_{\pm})^{-1}(y \mapsto \widetilde{R}_{\pm}(y)[P^*\varphi])\}(x) \\
&= \{(\mathrm{Id} + \mathcal{K}_{\pm})^{-1}\underbrace{(y \mapsto P_{(2)}\widetilde{R}_{\pm}(y)[\varphi])}_{(\mathrm{Id}+\mathcal{K}_{\pm})\varphi}\}(x) \\
&= \varphi(x),
\end{aligned}$$

that is, $PF^{\Omega}_{\pm}(x) = \delta_x$. The other properties $F^{\Omega}_{\pm}(x)$ should satisfy can be relatively easily checked, hence we can state the following

Proposition 4. *Under the assumptions of* Proposition 3 *the map $\varphi \mapsto F^{\Omega}_{+}(x)[\varphi]$ is an advanced fundamental solution on Ω for P at $x \in \Omega$ and the map $\varphi \mapsto F^{\Omega}_{-}(x)[\varphi]$ is a retarded one.*

To sum up:

Corollary 2. *Let P be a generalized d'Alembert operator on a Lorentzian manifold (M^n, g). Then every point of M possesses a relatively compact causal neighbourhood Ω such that, for every $x \in \Omega$, there exist fundamental solutions $F^{\Omega}_{\pm}(x)$ on Ω for P at x satisfying*

a) $\mathrm{supp}(F^{\Omega}_{\pm}(x)) \subset J^{\Omega}_{\pm}(x)$ *and*
b) $x \mapsto F^{\Omega}_{\pm}(x)[\varphi]$ *is smooth, for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$.*

3.4 Comparison of formal and exact local fundamental solutions

In this section we show that the formal fundamental solutions obtained in Section 3.2 is asymptotic to the true one along the light cone. More precisely [1, Prop. 2.5.1]:

Proposition 5. *Let* $\Omega \subset\subset \Omega'$ *be as in* Proposition 3. *Fix* $N \geq \frac{n}{2}$ *and for* $k \in \mathbb{N}$ *set*

$$\mathcal{R}_{\pm}^{N+k}(x) := \sum_{j=0}^{N+k-1} V_x^j \cdot R_{\pm}^{\Omega'}(2j+2, x),$$

where $(V^j)_{j\geq 0}$ *is the sequence of Hadamard coefficients for* P *on* Ω'. *Then for every* $k \in \mathbb{N}$ *the map*

$$(x, y) \mapsto (F_{\pm}^{\Omega}(x) - \mathcal{R}_{\pm}^{N+k}(x))(y)$$

is a C^k*-function on* $\Omega \times \Omega$, *where* $F_{\pm}^{\Omega}$ *is given by* Proposition 4.

This a strong statement, since both $\mathcal{R}_{\pm}^{N+k}(x)$ and $F_{\pm}^{\Omega}(x)$ are singular along the light cone $\{y \in \Omega \,|\, \Gamma_x(y) = 0\}$ based at x (see [1, Prop. 1.4.2]). Using an elementary argument of differential geometry [1, Lemma 2.5.4] one can deduce the following

Corollary 3. *Under the assumptions of* Proposition 5, *there exists for every* $k \in \mathbb{N}$ *a constant* C_k *such that*

$$\|(F_{\pm}^{\Omega}(x) - \mathcal{R}_{\pm}^{N+k}(x))(y)\| \leq C_k \cdot |\Gamma_x(y)|^k$$

for all $(x, y) \in \Omega \times \Omega$.

4 The Cauchy problem and global fundamental solutions

In this section we want to construct global fundamental solutions. A naive idea would consist in taking the local fundamental solutions constructed above and gluing them together using a partition of unity. A quick reflection convinces one of the difficulties which then may occur. Namely it is already not clear which equation should be solved in each coordinate patch not containing the point at which the fundamental solutions are sought after. Studying this question in more detail one immediatly observes that the global topology and geometry of the manifold could set up serious problems. As we have already seen at the end of Section 1.2 there cannot exist any fundamental solution for $\Box$ on compact spacetimes. Even if the manifold is not compact the possible existence of closed or almost-closed causal curves can make the very definition of fundamental solutions ill-posed; indeed it could theoretically happen that a wave overlaps itself after finite time.
Since we want to avoid this kind of situation we have to first properly choose our Lorentzian manifolds. There exists a “good” class of Lorentzian manifolds in this respect, which are called *globally hyperbolic* (see [1, Def. 1.3.8]). We restrict the discussion of the issue to the globally hyperbolic setting, although uniqueness as well as existence results may each be extended to broader classes of spacetimes, see [1, Sec. 3.1 & 3.5].

In this case we make what at first seems to be a detour: we solve the so-called *Cauchy problem*, which in analogy with ordinary differential equations consists in solving a wave equation fixing initial conditions on a subset of the manifold. Since generalized d'Alembert operators are differential operators of second order, two conditions have to be fixed:

Definition 10. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime (M^n, g) and $S \subset M$ be a (smooth) spacelike hypersurface with unit normal vector field ν. Let $f \in C^\infty(M, \mathbb{K})$ and $u_0, u_1 \in C^\infty(S, \mathbb{K})$. The* Cauchy problem *for P with Cauchy data (f, u_0, u_1) is the system of equations*

$$\left| \begin{array}{ll} Pu &= f \quad \text{on } M \\ u_{|S} &= u_0 \\ \partial_\nu u &= u_1 \quad \text{on } S. \end{array} \right.$$

We shall be interested in solving the Cauchy problem in $C^\infty(M, \mathbb{K})$ and with compactly supported data (see [2] for less regular solutions). The link with fundamental solutions will be explained in Section 4.4.

4.1 Uniqueness of fundamental solutions

We first show the uniqueness of advanced and retarded fundamental solutions at a point on globally hyperbolic spacetimes. One of the main ingredients involved is the local solvability of the following inhomogeneous wave equation.

Proposition 6. *Under the assumptions of* Proposition 3*, there exists for every $v \in \mathcal{D}(\Omega, \mathbb{C})$ a function $u_\pm \in C^\infty(\Omega, \mathbb{C})$ such that*

$$\left| \begin{array}{ll} Pu_\pm &= v \\ \mathrm{supp}(u_\pm) &\subset J_\pm^\Omega(\mathrm{supp}(v)). \end{array} \right.$$

Sketch of proof. It follows from Proposition 4 that, for every $x \in \Omega$, there exist fundamental solutions $F_\pm^\Omega(x)$ for P at x on Ω. As in Section 1.2 we set

$$u_\pm[\varphi] := \int_\Omega v(x) F_\pm^\Omega(x)[\varphi] dx$$

for every $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$. There are three assertions to be shown.

- *The map $u_\pm : \mathcal{D}(\Omega, \mathbb{C}) \to \mathbb{C}$ is a solution of $Pu_\pm = v$ in $\mathcal{D}'(\Omega, \mathbb{C})$*: that $u_\pm$ defines a distribution follows from $F_\pm^\Omega(x)$ being one, from $x \mapsto F_\pm^\Omega(x)[\varphi]$ being smooth for every $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$ and from an uniform estimate of the order of the distributions $F_\pm^\Omega(x)$ for x running in Ω, see [1, Lemma 2.4.4]. As for $Pu_\pm = v$ in the distributional sense, this is exactly the computation carried out in Section 1.2 and which is justified.

- *The support condition*: Let $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$ be such that $u_\pm[\varphi] \neq 0$, then there exists an $x \in \Omega$ such that $F^\Omega_\pm(x)[\varphi]v(x) \neq 0$, which implies $\mathrm{supp}(\varphi) \cap \mathrm{supp}(F^\Omega_\pm(x)) \neq \varnothing$ and $x \in \mathrm{supp}(v)$. Hence $\mathrm{supp}(\varphi) \cap J^\Omega_\pm(x) \neq \varnothing$, i.e., $x \in J^\Omega_\mp(\mathrm{supp}(\varphi))$, so that $J^\Omega_\mp(\mathrm{supp}(\varphi)) \cap \mathrm{supp}(v) \neq \varnothing$, or equivalently $\mathrm{supp}(\varphi) \cap J^\Omega_\pm(\mathrm{supp}(v)) \neq \varnothing$, which was to be proved.
- *The distribution $u_\pm$ is in fact a smooth section*: this is the technical part of the proof, which actually relies not only on Corollary 2 but also on the explicit form of the local fundamental solutions on Ω, see [1, Sec. 2.6] for details.

We state the main result of this section:

Theorem 1. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime (M^n, g). Then every solution $u \in \mathcal{D}'(M, \mathbb{C})$ with past- or future-compact support of the equation $Pu = 0$ vanishes.*

Sketch of proof. Assume u has past-compact support (the other case is completely analogous). One has to show that $u[\varphi] = 0$ for every $\varphi \in \mathcal{D}(M, \mathbb{C})$. The idea is to apply Proposition 6 and solve the inhomogeneous wave equation to the operator P^* (which is of generalized d'Alembert type, see Sec. 1.1)

$$\left| \begin{array}{ll} P^*\psi & = \varphi \\ \mathrm{supp}(\psi) & \subset J^\Omega_-(\mathrm{supp}(\varphi)) \end{array} \right.$$

for any fixed φ with "small" support (small enough in order to Proposition 6 be applied). Since u has past-compact support one can hope for the intersection $\mathrm{supp}(u) \cap J^\Omega_-(\mathrm{supp}(\varphi))$ be compact and hence for

$$\begin{aligned} u[\varphi] &= u[P^*\psi] \\ &= \underbrace{Pu}_{0}[\psi] \\ &= 0, \end{aligned}$$

which would be the result. The whole work is to justify this computation as well as the assumptions on $\mathrm{supp}(\varphi)$, which requires global properties of the causality relation, see [1, Thm. 3.1.1]. This completes the sketch of proof of Theorem 1.

Since $J^M_+(K)$ (resp. $J^M_-(K)$) is past- (resp. future-) compact for any compact subset K of a globally hyperbolic spacetime M, we obtain:

Corollary 4. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime (M^n, g) and $x \in M$. Then there exists at most one advanced (resp. retarded) fundamental solution for P at x.*

4.2 Cauchy problem: local solvability

The existence of solutions to the Cauchy problem is a local-to-global construction. In this section we deal with the local aspect, which of course does not require global hyperbolicity of the manifold.

Theorem 2. *Let (M^n, g) be a spacetime and S be a smooth spacelike hypersurface with (timelike) unit normal vector field ν. Then for each open subset Ω of M satisfying the hypotheses of* Proposition 3 *and such that $S \cap \Omega$ is a Cauchy hypersurface of Ω the following holds: for all $u_0, u_1 \in \mathcal{D}(S \cap \Omega, \mathbb{C})$ and each $f \in \mathcal{D}(\Omega, \mathbb{C})$, there exists a unique $u \in C^\infty(\Omega, \mathbb{C})$ with*

$$\left| \begin{aligned} Pu &= f \\ u_{|S} &= u_0 \\ \partial_\nu u &= u_1. \end{aligned} \right.$$

Furthermore $\mathrm{supp}(u) \subset J_+^\Omega(K) \cup J_-^\Omega(K)$*, where*

$$K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f).$$

Sketch of proof. Although Proposition 6 naturally enters the proof, it does not straightforward imply the result. One has to formulate a separate ansatz. Namely writing Ω as the product $\mathbb{R} \times (S \cap \Omega)$ (which is possible since $S \cap \Omega$ is assumed to be a Cauchy hypersurface in Ω and because of [1, Thm. 1.3.10]), one looks for a solution of the form

$$\sum_{j=0}^{\infty} t^j u_j(x),$$

where $(t, x) \in \mathbb{R} \times (S \cap \Omega)$ and $u_j \in C^\infty(S \cap \Omega, \mathbb{C})$. As for the Hadamard coefficients one obtains inductive relations of the form $u_j = \mathcal{F}(u_0, u_1, \ldots, u_{j-1})$ for every $j \geq 2$, which then stand as definition for the u_j in terms of u_0 and u_1 on the whole of $S \cap \Omega$. Coming back to the inhomogeneous equation, one introduces a cut-off function σ as in the construction of local fundamental solutions (see Sec. 3.3) and sets

$$\hat{u} := \sum_{j=0}^{\infty} \sigma(\frac{t}{\varepsilon_j}) t^j u_j$$

with the $u_j \in C^\infty(S \cap \Omega, \mathbb{C})$ found above and a sequence $(\varepsilon_j)_{j \geq 0}$ of positive real numbers. Choosing the ε_j suitably small one obtains a smooth function $\hat{u}$ on Ω such that $P\hat{u} - f$ vanishes not on all Ω but at least on $S \cap \Omega$ at infinite order. Proposition 6 then provides smooth solutions $\tilde{u}_+$ and $\tilde{u}_-$ of the respective inhomogeneous problems on Ω:

$$\left| \begin{array}{ll} P\tilde{u}_\pm & = w_\pm \\ \mathrm{supp}(\tilde{u}_\pm) \subset J_\pm^\Omega(\mathrm{supp}(w_\pm)), \end{array} \right.$$

where $w_\pm|_{J_\pm^\Omega(S\cap\Omega)} := P\hat{u} - f$ and vanishes on the rest of Ω. The last step consists in showing that $u_\pm := \hat{u} - \tilde{u}_\pm$ solves the wave equation $Pu_\pm = f$ on $J_\pm^\Omega(S\cap\Omega)$ and vanishes on $J_\mp^\Omega(S\cap\Omega)$. The function u defined by

$$u := \left| \begin{array}{ll} u_+ & \text{on } J_+^\Omega(S\cap\Omega) \\ u_- & \text{on } J_-^\Omega(S\cap\Omega) \end{array} \right.$$

is then a smooth function on Ω solving the requested Cauchy problem.
The uniqueness - which is actually needed for the last step of the existence just above - follows from an independent argument, which is an integral formula for solutions of the Cauchy problem on Ω: namely, if $u \in C^\infty(\Omega, \mathbb{C})$ solves $Pu = 0$, then

$$\int_\Omega u(x)\varphi(x)dx = \int_{S\cap\Omega} (\partial_\nu(F^\Omega[\varphi])u_0 - F^\Omega[\varphi]u_1)ds$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$, where $F^\Omega[\varphi] : \mathcal{D}(\Omega, \mathbb{C}) \to \mathbb{C}$, $\psi \mapsto \int_\Omega \varphi(x)(F_+^\Omega(x)[\psi] - F_-^\Omega(x)[\psi])dx$ and ds is the induced measure on $S\cap\Omega$, see [1, Lemma 3.2.2]. Therefore if u_0 and u_1 vanish on $S\cap\Omega$ then u vanishes - as distribution and hence as function - on Ω.
The control of the support of the solution follows from the corresponding one for the inhomogeneous problem (Proposition 6) and from the integral formula just above. This completes the sketch of proof of Theorem 2.

Note that, since every point on a spacelike hypersurface S in M admits a basis of neighbourhoods Ω_j in M such that $S\cap\Omega_j$ is a Cauchy hypersurface of Ω_j (roughly speaking one just has to consider the Cauchy development of $S\cap U$ in an open subset U meeting S, see e.g. [1, Lemma A.5.6]), Theorem 2 actually proves that the Cauchy problem with compactly supported data is always locally solvable.

4.3 Cauchy problem: global solvability

We come to the central result of this survey:

Theorem 3. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime M and $S \subset M$ be a spacelike Cauchy hypersurface in M with (timelike) unit normal ν.*

i) For all $(f, u_0, u_1) \in \mathcal{D}(M, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C})$, there exists a unique $u \in C^\infty(M, \mathbb{C})$ such that

$$\left| \begin{aligned} Pu &= f \\ u_{|S} &= u_0 \\ \partial_\nu u &= u_1. \end{aligned} \right. \tag{9}$$

Moreover $\mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$ *with*

$$K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f).$$

ii) The map

$$\begin{aligned} \mathcal{D}(M, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C}) \oplus \mathcal{D}(S, \mathbb{C}) &\to C^\infty(M, \mathbb{C}) \\ (f, u_0, u_1) &\mapsto u, \end{aligned}$$

where $u \in C^\infty(M, \mathbb{C})$ *is the solution of* (9)*, is linear continuous.*

Sketch of proof. The existence of u in $i)$, which is rather technical, is proved in two main steps. First one constructs a solution u in a strip $]-\varepsilon, \varepsilon[\times S$ (where M is identified with $\mathbb{R} \times S$) for some $\varepsilon > 0$: this is the easier step, since it roughly means gluing together local solutions obtained by Theorem 2 along the hypersurface $S \simeq \{0\} \times S$. There is only a finite number of them to be taken into account since $(J_+^M(K) \cup J_-^M(K)) \cap S$ is compact (outside this intersection u should vanish along S). In the second step one shows that u can be extended in the whole future and past of the strip. The core of the global theory lies here. Namely using the local theory and the global hyperbolicity of M it can be shown that u can be continued in the future or past "independently" of the behaviour of the already existing u; in other words, no explosion can occur. We refer to [1, Thm. 3.2.11] for a clean and thorough argumentation.
The uniqueness of u follows from another technical argument based on the local integral formula described in the proof of Theorem 2, see [1, Cor. 3.2.4]. This shows $i)$.
Statement $ii)$, which should be interpreted as a stability result for waves (solutions of the Cauchy problem depend continuously on the data), is a not-so-direct application of the open mapping theorem using the continuity of linear differential operators w.r.t. the topology of $C^\infty(M, \mathbb{K})$ or of $\mathcal{D}(M, \mathbb{K})$ (beware that the latter is not Fréchet). This completes the sketch of proof of Theorem 3.

Corollary 5. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime M and $S \subset M$ be a spacelike Cauchy hypersurface in M with (timelike) unit normal ν. Then for all $(f, u_0, u_1) \in C^\infty(M, \mathbb{C}) \oplus C^\infty(S, \mathbb{C}) \oplus C^\infty(S, \mathbb{C})$, there exists a unique $u \in C^\infty(M, \mathbb{C})$ solving* (9)*. Moreover* $\mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$ *with* $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

Proof. Uniqueness already follows from Theorem 3. Let $(K_n)_n$ be a sequence of compact subsets of S with $K_n \subset \mathring{K}_{n+1}$ and $\cup_n K_n = S$. Identify M with $\mathbb{R} \times$

S and set $\widetilde{K}_n := D(\overset{\circ}{K}_n) \cap (]-n,n[\times S)$, where $D(\overset{\circ}{K}_n)$ is the so-called Cauchy-development of $\overset{\circ}{K}_n$ in M, see [1, Def. 1.3.5]. Then $(\widetilde{K}_n)_n$ is an increasing sequence of relatively compact and globally hyperbolic open subsets of M with $\cup_n \widetilde{K}_n = M$. Furthermore $\overset{\circ}{K}_n \subset S \simeq \{0\} \times S$ is a Cauchy hypersurface of $\widetilde{K}_n$ for every n. Let χ_n be a smooth function with compact support on M such that $\chi_{n|_{\widetilde{K}_n}} = 1$. From Theorem 3 there exists a unique solution $v_n \in C^\infty(M,\mathbb{C})$ to the Cauchy problem

$$\left| \begin{aligned} P v_n &= \chi_n f \\ v_{n|_S} &= \chi_n u_0 \\ \partial_\nu v_n &= \chi_n u_1. \end{aligned} \right.$$

If $m > n$ then $v := v_m - v_n$ solves $Pv = 0$ on the globally hyperbolic manifold $\widetilde{K}_n$ with $v = \partial_\nu v = 0$ on the Cauchy hypersurface $\overset{\circ}{K}_n$ of $\widetilde{K}_n$, therefore Theorem 3 implies $v = 0$ on $\widetilde{K}_n$. Hence $u(x) := v_n(x)$ for $x \in \widetilde{K}_n$ defines a smooth function u on M solving (9). The statement on the support is also a straightforward consequence of Theorem 3. This shows Corollary 5.

4.4 Global existence of fundamental solutions on globally hyperbolic spacetimes

We come back to the issue of finding global fundamental solutions on globally hyperbolic spacetimes. Although they seem to be far away, Theorem 3 makes fundamental solutions easily accessible:

Theorem 4. *Let P be a generalized d'Alembert operator on a globally hyperbolic spacetime M. Then there exists for each $x \in M$ a unique fundamental solution $F_+(x)$ with past-compact support for P at x and a unique one $F_-(x)$ with future-compact support.*
They satisfy:

- $\mathrm{supp}(F_\pm(x)) \subset J_\pm^M(x)$ *and*
- *for every $\varphi \in \mathcal{D}(M,\mathbb{C})$ the map $M \to \mathbb{C}$, $x \mapsto F_\pm(x)[\varphi]$ is a smooth function with*
$$P^*(x \mapsto F_\pm(x)[\varphi]) = \varphi.$$

Proof. Uniqueness has already been obtained in Corollary 4, so that we just have to prove existence. Identify M with $\mathbb{R} \times S$, where $\frac{\partial}{\partial t}$ is future-directed and each $\{s\} \times S$ is a smooth spacelike Cauchy hypersurface (this is always possible on globally hyperbolic spacetimes, see [1, Thm. 1.3.10]). Fix a smooth unit vector field ν normal to all $\{s\} \times S$. Set, for $x \in M$ and $\varphi \in \mathcal{D}(M,\mathbb{C})$,

$$F_+(x)[\varphi] := (\chi_\varphi)(x),$$

where χ_φ is the solution of the Cauchy problem

$$\left| \begin{array}{ll} P^*\chi_\varphi & = \varphi \\ \chi_{\varphi|_{\{t\}\times S}} & = 0 \\ \partial_\nu \chi_{\varphi|_{\{t\}\times S}} & = 0 \end{array} \right. \tag{10}$$

and t is chosen such that $\mathrm{supp}(\varphi) \subset I_-^M(\{t\} \times S)$ (such a t can be found because of the compactness of $\mathrm{supp}(\varphi)$). If t is fixed then the existence of a solution of (10) is guaranteed by Theorem 3. However one has to show that χ_φ is well-defined, i.e., does not depend on t. Let $t' \in \mathbb{R}$ be such that $\mathrm{supp}(\varphi) \subset I_-^M(\{t'\} \times S)$ and with, say, $t < t'$. Let $(\chi_\varphi)'$ be the solution of (10) with t' instead of t. We show that $(\chi_\varphi)'$ vanishes as well as its normal derivative on $\{t\} \times S$.

Since $\mathrm{supp}(\varphi)$ is compact there exists a $t_- < t$ such that $\mathrm{supp}(\varphi) \subset I_-^M(\{t_-\}\times S)$. Consider

$$M_{t_-} := \cup_{\tau > t_-}\{\tau\} \times S,$$

which is a globally hyperbolic spacetime in its own right and in which $\{t'\}\times S$ sits again as a Cauchy hypersurface. By assumption $\mathrm{supp}(\varphi)$ is contained in the complement of M_{t_-} in M, so that the restriction of $(\chi_\varphi)'$ onto M_{t_-} solves the Cauchy problem $Pu = 0$, $u_{|_{\{t'\}\times S}} = 0$ and $\partial_\nu u_{|_{\{t'\}\times S}} = 0$. The uniqueness of solutions (Theorem 3) implies that $(\chi_\varphi)'_{|_{M_{t_-}}} = 0$, in particular $(\chi_\varphi)'$ vanishes in a neighbourhood of $\{t\} \times S$.

Now the smooth function $\chi := \chi_\varphi - (\chi_\varphi)'$ satisfies $P^*\chi = 0$ on M and $\chi_{|_{\{t\}\times S}} = \partial_\nu \chi_{|_{\{t\}\times S}} = 0$, hence by Theorem 3 again one concludes that $\chi = 0$ on M. Therefore χ_φ (and thus $F_+(\cdot)[\varphi]$) is well-defined.

We next show that, for a fixed $x \in M$, the map $\varphi \mapsto F_+(x)[\varphi]$ is an advanced fundamental solution for P at x on M. The linearity as well as the continuity of $F_+(x)$ both directly follow from Theorem 3. On the other hand, given $\varphi \in \mathcal{D}(M, \mathbb{C})$, the function φ itself provides an obvious solution to $P^*u = P^*\varphi$ with $u_{|_{\{t\}\times S}} = \partial_\nu u_{|_{\{t\}\times S}} = 0$; since $\mathrm{supp}(P^*\varphi) \subset \mathrm{supp}(\varphi)$ is compact, Theorem 3 may be applied and we deduce that $\chi_{P^*\varphi} = \varphi$, which in turn implies from the definition of $F_+(x)$ that

$$\begin{aligned} PF_+(x)[\varphi] &= F_+(x)[P^*\varphi] \\ &= (\chi_{P^*\varphi})(x) \\ &= \varphi(x). \end{aligned}$$

This holds for all $\varphi \in \mathcal{D}(M, \mathbb{C})$, that is, $PF_+(x) = \delta_x$.

The support condition is equivalent to $\mathrm{supp}(\chi_\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ for every $\varphi \in \mathcal{D}(M, \mathbb{C})$. But for any such φ the open subset $M' := M \setminus J_-^M(\mathrm{supp}(\varphi))$ of M is again a globally hyperbolic manifold containing $\{t\} \times S$ as Cauchy hypersurface, where t is chosen as above (this follows from e.g. [1, Lemma A.5.8] and a short reflection). The function $u := \chi_{\varphi|_{M'}}$ satisfies $P^*u = 0$ with

$u_{|\{t\}\times S} = \partial_\nu u_{|\{t\}\times S} = 0$, hence Theorem 3 again implies that $\chi_{\varphi|_{M'}} = 0$, which was to be proved.
Thus $F_+(x)$ is an advanced fundamental solution for P at x on M. That $x \mapsto F_+(x)[\varphi]$ is smooth with $P^*(x \mapsto F_+(x)[\varphi]) = \varphi$ for any $\varphi \in \mathcal{D}(M,\mathbb{C})$ is trivially seen from the definition of $F_+(\cdot)$. The construction of F_- is completely analogous, replacing all "+" by "−" and vice-versa. This achieves the proof of Theorem 4.

In particular the wave equation $Pu = f$ with $f \in \mathcal{D}(M,\mathbb{C})$ possesses a unique solution $u_\pm \in C^\infty(M,\mathbb{C})$ with $\mathrm{supp}(u_\pm) \subset J_\pm^M(\mathrm{supp}(f))$, or equivalently with $\mathrm{supp}(u_+)$ (resp. $\mathrm{supp}(u_-)$) being past- (resp. future-) compact on a globally hyperbolic spacetime M.

Remark. Because of the definition of Riesz distributions we have only proved the existence of solutions to wave equations as well as fundamental solutions for $\mathbb{K} = \mathbb{C}$. In fact, all existence and uniqueness results from Corollary 1 to Theorem 4 still hold replacing $\mathbb{C}$ by $\mathbb{K} = \mathbb{R}$ for real-valued generalized d'Alembert operators P, where "real-valued" means that $Pu \in C^\infty(M,\mathbb{R})$ whenever $u \in C^\infty(M,\mathbb{R})$ (or, equivalently, that the coefficients a_j and b_1 in local coordinates are real-valued functions, see Section 1.1). Indeed, we have already noticed in Section 2 that Riesz distributions for real parameters are real-valued; moreover, if P is real-valued then all objects involved in the local construction are real-valued (e.g. the Riesz distributions $R_\pm^\Omega(2k+2,x)$ or the Hadamard coefficients, see Proposition 1) hence provide real-valued fundamental or classical solutions. In case uniqueness is available (such as in Theorems 3 and 4) the existence of real-valued solutions for such a P and for real-valued data straightforward follows from the corresponding result in the complex case, since the complex conjugate $\overline{u}$ of the distribution u then solves the same equation as u, hence $\overline{u} = u$.

5 Green's operators

We now briefly sketch how solutions of wave equations can be encoded into a pair of operators, which furthermore offer an entrance door to (local) quantum field theory for generalized d'Alembert operators.

Definition 11. *Let P be a generalized d'Alembert operator on a spacetime M. A linear map*

$$G_+ : \mathcal{D}(M,\mathbb{K}) \longrightarrow C^\infty(M,\mathbb{K}),$$

satisfying:

i) $P \circ G_+ = \mathrm{Id}_{\mathcal{D}(M,\mathbb{K})}$,
ii) $G_+ \circ P|_{\mathcal{D}(M,\mathbb{K})} = \mathrm{Id}_{\mathcal{D}(M,\mathbb{K})}$,
iii) $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$ *for all* $\varphi \in \mathcal{D}(M,\mathbb{K})$

is called advanced Green's operator *for P on M.*

A retarded Green's operator G_- *for P on M is a linear map* $\mathcal{D}(M,\mathbb{K}) \longrightarrow C^\infty(M,\mathbb{K})$ *satisfying i), ii) and* $\mathrm{supp}(G_-\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ *for all* $\varphi \in \mathcal{D}(M,\mathbb{K})$*.*

For P a (retarded or advanced) Green's operator is almost an inverse: it is a right inverse to P, however a left inverse to $P_{|_{\mathcal{D}(M,\mathbb{K})}}$ and not to P itself. This consideration reminds us of Section 1.2, where we have seen how fundamental solutions generally provide solutions to the corresponding wave equation for "every" right member. One could therefore expect a direct relationship between fundamental solutions and Green's operators. In fact Green's operators and fundamental solutions are two different versions of mainly the same concept:

Proposition 7. *Let P be a generalized d'Alembert operator on a spacetime M. Then advanced (resp. retarded) Green's operators for P stand in one-to-one correspondence with retarded (resp. advanced) fundamental solutions for P^*. More precisely, if there exists for every $x \in M$ a fundamental solution $F_\pm(x)$ for P^* at x on M with*

a) $\mathrm{supp}(F_\pm(x)) \subset J_\pm^M(x)$,
b) $x \mapsto F_\pm(x)[\varphi]$ *is smooth and*
c) $P(x \mapsto F_\pm(x)[\varphi]) = \varphi$

for each $\varphi \in \mathcal{D}(M,\mathbb{K})$, then the formula

$$(G_\mp\varphi)(x) = F_\pm(x)[\varphi] \qquad \forall\, x \in M,\ \forall\, \varphi \in \mathcal{D}(M,\mathbb{K}) \tag{11}$$

defines a linear map $G_\mp : \mathcal{D}(M,\mathbb{K}) \to C^\infty(M,\mathbb{K})$ satisfying i), ii) in Definition 11 *as well as* $\mathrm{supp}(G_\mp\varphi) \subset J_\mp^M(\mathrm{supp}(\varphi))$*.*
Conversely, every linear map $G_\mp : \mathcal{D}(M,\mathbb{K}) \to C^\infty(M,\mathbb{K})$ having those properties defines at each point $x \in M$ through (11) *a fundamental solution $F_\pm(x)$ for P^* satisfying* a), b) *and* c)*.*

The proof of Proposition 7 is an easy exercise left to the reader. Combining it with Theorem 4, we obtain the

Corollary 6. *Every generalized d'Alembert operator on a globally hyperbolic spacetime admits a unique advanced and a unique retarded Green's operator.*

We next list the properties which are needed for quantum field theory. On a given spacetime M we introduce the space

$$C^\infty_{\mathrm{sc}}(M,\mathbb{K}) := \{u \in C^\infty(M,\mathbb{K}) \,|\, \exists\, K \subset M \text{ compact s.t. } \mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)\}.$$

The "sc" stands for "spacelike compact", since in case M is globally hyperbolic the intersection of $J_+^M(K) \cup J_-^M(K)$ with any Cauchy hypersurface is compact.

It can be proved that $C^\infty_{\mathrm{sc}}(M,\mathbb{K})$ is a Fréchet vector space w.r.t. the topology for which a sequence $(u_j)_j$ converges towards 0 if and only if there exists a compact $K\subset M$ with $\mathrm{supp}(u_j)\subset J_+^M(K)\cup J_-^M(K)$ for all j and such that $(u_j)_j$ converges to 0 in every C^k-norm on any compact subset of M.

Proposition 8. *Let P be a generalized d'Alembert operator on a spacetime M. Let G_+, G_- be advanced and retarded Green's operators for P on M. Set $G:=G_+-G_-$. Then the following holds:*

i) The sequence

$$0\longrightarrow\mathcal{D}(M,\mathbb{K})\stackrel{P}{\longrightarrow}\mathcal{D}(M,\mathbb{K})\stackrel{G}{\longrightarrow}C^\infty_{\mathrm{sc}}(M,\mathbb{K})\stackrel{P}{\longrightarrow}C^\infty_{\mathrm{sc}}(M,\mathbb{K})\tag{12}$$

is a complex (i.e., the composition of any two successive maps is zero) which is exact at the first $\mathcal{D}(M,\mathbb{K})$.

ii) If M is globally hyperbolic, then the formal adjoint of $G_\pm$ coincides with $G^*_\mp$, where G^*_+, G^*_- are the Green's operators for P^* on M.

iii) If M is globally hyperbolic, then the complex (12) *is exact everywhere and all maps are sequentially continuous.*

Proof. By definition of Green's operators, (12) is obviously a complex. Furthermore, if $\varphi\in\mathcal{D}(M,\mathbb{K})$ solves $P\varphi=0$, then applying e.g. G_+ one has $G_+(P\varphi)=\varphi=0$, which shows exactness at the first $\mathcal{D}(M,\mathbb{K})$ and *i*).

Assume now that M is globally hyperbolic. Let $\varphi,\psi\in\mathcal{D}(M,\mathbb{K})$, then $\mathrm{supp}(G_\pm\varphi)\cap\mathrm{supp}(G^*_\mp\psi)$ is compact, so that the following computation is justified:

$$\begin{aligned}\int_M\langle G_\pm\varphi,\psi\rangle dx&=\int_M\langle G_\pm\varphi,P^*G^*_\mp(\psi)\rangle dx\\&=\int_M\langle PG_\pm\varphi,G^*_\mp\psi\rangle dx\\&=\int_M\langle\varphi,G^*_\mp\psi\rangle dx,\end{aligned}$$

where $\langle\cdot\,,\cdot\rangle$ denotes the natural Euclidean or Hermitian inner product on $\mathbb{K}$. This proves *ii*).

Still assuming M to be globally hyperbolic, let $\varphi\in\mathcal{D}(M,\mathbb{K})$ be such that $G\varphi=0$. Then the function $\psi:=G_+\varphi=G_-\varphi$ is smooth with $\mathrm{supp}(\psi)\subset J_+^M(\mathrm{supp}(\varphi))\cap J_-^M(\mathrm{supp}(\varphi))$, which is compact. Moreover, $P\psi=PG_+\varphi=\varphi$. This shows exactness at the second $\mathcal{D}(M,\mathbb{K})$. Let now $u\in C^\infty_{\mathrm{sc}}(M,\mathbb{K})$ solve $Pu=0$. After possibly enlarging K we may assume that a compact subset K of M exists such that $\mathrm{supp}(u)\subset I_+^M(K)\cup I_-^M(K)$. Let $\{\chi_+,\chi_-\}$ be a partition of unity subordinated to the open covering $\{I_+^M(K),I_-^M(K)\}$ of $I_+^M(K)\cup I_-^M(K)$. Setting $u_\pm:=\chi_\pm u$ we obtain $u=u_++u_-$, where $u_\pm$ is smooth with $\mathrm{supp}(u_\pm)\subset I_\pm^M(K)$. Set now $\varphi:=Pu_+=-Pu_-$. It is a smooth function with $\mathrm{supp}(\varphi)\subset J_+^M(K)\cap J_-^M(K)$, which is compact, hence $\varphi \in \mathcal{D}(M, \mathbb{K})$. We check that $G\varphi = u$. Although $u_\pm$ does not have compact support, we may integrate $G_\pm\varphi$ against any $\psi \in \mathcal{D}(M, \mathbb{K})$; using the compacity of $\mathrm{supp}(u_\pm) \cap J^M_\mp(\mathrm{supp}(\psi))$ and *ii*) we obtain

$$\begin{aligned}\int_M \langle G_\pm\varphi, \psi\rangle dx &= \int_M \langle \varphi, (G_\pm)^*\psi\rangle dx \\ &= \int_M \langle \varphi, G^*_\pm\psi\rangle dx \\ &= \pm \int_M \langle Pu_\pm, G^*_\mp\psi\rangle dx \\ &= \pm \int_M \langle u_\pm, P^* G^*_\mp\psi\rangle dx \\ &= \pm \int_M \langle u_\pm, \psi\rangle dx,\end{aligned}$$

that is, $G_\pm\varphi = \pm u_\pm$, so that $G\varphi = u_+ + u_- = u$. This shows exactness at the first $C^\infty_{\mathrm{sc}}(M, \mathbb{K})$. The sequential continuity of all maps of (12) follows from P being a differential operator and from Theorem 3. This shows *iii*) and completes the proof of Proposition 8.

One of the reasons why Green's operators are so important for quantum field theory is the following: given a formally self-adjoint generalized d'Alembert operator P on a globally hyperbolic spacetime M, one can form a symplectic vector space in a canonical way. Namely, set

$$V := \mathcal{D}(M, \mathbb{K})/\mathrm{Ker}(G),$$

where $G := G_+ - G_-$ as above and G_+, G_- are the Green's operators for P. From Proposition 8.ii) the map $(\varphi, \psi) \mapsto \int_M \langle G\varphi, \psi\rangle dx$ defines a skew-symmetric bilinear form on V, which is by definition non-degenerate and hence a symplectic form on V. Now independently of this there also exists a canonical way to produce a C^*-algebra out of a symplectic vector space, which consists in defining its so-called CCR-representation, where CCR stands for "canonical commutation relations". Composing both one obtains a kind of map - actually a functor - associating to each pair (M, P) a C^*-algebra. Of course this construction is made so as to translate into algebraic properties the analytical ones of the operator and the geometric ones of the underlying manifold; for example, an inclusion of manifolds corresponds to an inclusion of algebras (this has to do with functoriality) and if two globally hyperbolic open subsets of M are causally independent (i.e., if there is no causal curve from the closure of one to the closure of the other one) then the corresponding algebras commute. Since quantization is not the topic of this lecture, we stop here and refer the reader to other introductory lectures such as K. Fredenhagen's one and [1, Chap. 4].

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