ON THE CAUCHY PROBLEM FOR FRIEDRICHS SYSTEMS ON GLOBALLY HYPERBOLIC MANIFOLDS WITH TIMELIKE BOUNDARY

by

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Abstract

In this paper, the Cauchy problem for a Friedrichs system on a globally hyperbolic manifold with a timelike boundary is investigated. By imposing admissible boundary conditions, the existence and the uniqueness of strong solutions are shown. Furthermore, if the Friedrichs system is hyperbolic, the Cauchy problem is proved to be well-posed in the sense of Hadamard. Finally, examples of Friedrichs systems with admissible boundary conditions are provided.

Keywords: symmetric hyperbolic systems, symmetric positive systems, admissible boundary conditions, Dirac operator, normally hyperbolic operator, Klein-Gordon operator, heat operator, reaction-diffusion operator, globally hyperbolic manifolds with timelike boundary.

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Contents

1 Introduction

The Cauchy problem for hyperbolic partial differential equations on curved spacetimes has been and continues to be at the forefront of scientific research. While for a generic spacetime the well-posedness of the Cauchy problem cannot be expected, in the class of *globally hyperbolic manifolds* (with empty boundary) it has been proved that any hyperbolic PDE admits a unique smooth solution which depends continuously on the smooth Cauchy data, see the founding article [61] as well as [5, 8]. Even though globally hyperbolic spacetimes have plenty of applications to physics, there exist also important and interesting situations which require the spacetimes to have a non-trivial boundary. For example, experimental setups for studying the Casimir effect confine quantum field theories between several metal plates, which may be modeled theoretically by introducing timelike boundaries to the system. From a PDE viewpoint, this suggests that the Cauchy problem could be well-posed once suitable boundary conditions are introduced. In the last two decades, the well-posedness of the mixed initial-boundary problem for hyperbolic operators has been investigated in different geometric settings: see e.g. [16,65] for general surveys, [3, 26, 29, 41, 49, 53, 68, 69] for asymptotically anti-de Sitter spacetimes, [27, 28, 47] for the Klein-Gordon, the Maxwell and the Dirac operator on stationary spacetimes and [39, 57] for the case of Dirac operator on globally hyperbolic spacetimes.

The aim of this paper is to prove well-posedness of the Cauchy problem, not only for hyperbolic PDE on globally hyperbolic manifolds with timelike boundary (*cf.* Definition 2.1), but for a larger class, known as *Friedrichs systems* (*cf.* Definition 2.5). Friedrichs systems were developed by K.O. Friedrichs in [50, 52] and include a large variety of PDE. The classical Dirac operator is an example and many second-order PDEs (like wave equations and the heat equation) can be reduced to a Friedrichs system. Our first main result is the existence of strong solutions for the forward Cauchy problem for Friedrichs system coupled with future admissible boundary conditions (*cf.* Definition 2.13).

Theorem 1.1 (Strong solutions)**.** *Let* M *be a globally hyperbolic manifold with timelike boundary and let* $t\colon \mathsf{M} \to \mathbb{R}$ *be a Cauchy temporal function. For any* $0 < T \in \mathbb{R}$ *denote with* $\mathsf{M}_\mathsf{T} := t^{-1}((0,T))$ *a time strip. Let* Σ_0 *be any smooth spacelike Cauchy hypersurface of* M*. Let* S *be a Friedrichs system with constant characteristic and denote with* G_B *a future admissible boundary condition. Then, there exists a unique strong solution of the Cauchy problem*

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \in \Gamma_c(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) \\ \Psi_{|\Sigma_0} = \mathfrak{h} \in \Gamma_c(\mathsf{E}_{|\Sigma_0}) \\ \Psi_{|\partial\mathsf{M}} \in \mathsf{B} := \ker(G_\mathsf{B})\,. \end{cases} \tag{1.1}$$

While full regularity of the strong solution cannot be expected for a generic Friedrichs system even for smooth Cauchy data (see Section 4.3 for more details), our second main result shows that the backward and the forward Cauchy problem for symmetric hyperbolic systems coupled with admissible boundary condition is well-posed.

Theorem 1.2 (Smooth solutions)**.** *Let* M *be a globally hyperbolic manifold with timelike boundary and let* S *be a symmetric hyperbolic system of constant characteristic. Assume* $\mathsf{B} = (\mathsf{B}_+, \mathsf{B}_-)$ *to* *be an admissible boundary condition for* S. *Let* Σ_0 *be any smooth spacelike Cauchy hypersurface of* M. *Then, for all* $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ *and* $\mathfrak{h} \in \Gamma_c(\mathsf{E}_{|\Sigma_0})$ *satisfying the compatibility conditions* (4.3) *and* (4.4) *up to any order, there exists a unique* $\Psi \in \Gamma(\mathsf{E})$ *satisfying the Cauchy problem*

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi_{|\Sigma_{t_0}} = \mathfrak{h} \\ \Psi_{|\partial\mathsf{M}\cap J^+(\Sigma_0)} \in \mathsf{B}_+ \\ \Psi_{|\partial\mathsf{M}\cap J^-(\Sigma_0)} \in \mathsf{B}_- \end{cases} \tag{1.2}$$

and the map $(\mathfrak{f}, \mathfrak{h}) \mapsto \Psi$ *sending a pair* $(\mathfrak{f}, \mathfrak{h}) \in \Gamma_c(\mathsf{E}) \times \Gamma_c(\mathsf{E}_{|\partial\mathsf{M}})$ *to the solution* $\Psi \in \Gamma(\mathsf{E})$ *of* (1.2), *is continuous.*

Roughly speaking, condition (4.3) up to some finite order k ensures that, when the support of initial data meets the boundary of Σ_0, the solution of the Cauchy problem is C^k.

Showing the well-posedness of the Cauchy problem is not the end of the story: Indeed an explicit construction of the evolution operator (as in [20–24] for the case of empty boundary) and a propagation of the singularity theorem should to be investigated. Clearly, the well-posedness of the Cauchy problem will guarantee the existence of Green operators (*cf.* Proposition 5.1) which play a pivotal role in the algebraic approach to linear quantum field theory, see e.g. [17, 54] for textbooks, [6, 7, 10, 48, 58] for recent reviews, [12–15] for homotopical approaches and [18–20, 30–35, 40, 45, 46] for some applications. Indeed, they fully characterize the space of solutions of a symmetric hyperbolic system [4, 31], they implement the canonical commutation/anticommutation relations typical of any linear quantum field theory [6, 10], and their difference, dubbed the casual propagator or Pauli-Jordan commutator, can be used to construct quantum states, see e.g. [11, 30, 40, 45, 46, 48].

Our strategy to prove the well-posedness of the Cauchy problem is as follows: first we derive a suitable energy inequality for a Friedrichs system in Section 3 which will be employed in Section 4.1 to show existence and uniqueness of weak solutions. In Section 4.2 we shall prove that any weak solution is actually a strong solution. This will be achieved by localizing the problem and then using the theory of mollifiers, see e.g. [63]. In Section 4.3, we discuss the regularity of solutions of symmetric hyperbolic systems and in Section 5 we prove the well-posedness of the Cauchy problem. Finally in Section 6 and Section 7 we provide some examples of Friedrichs systems with admissible boundary conditions.

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Notation and convention

- The symbol $\mathbb{K}$ denotes one of the elements of the set $\{\mathbb{R}, \mathbb{C}\}$.
- $\mathsf{M} := (\mathsf{M}, g)$ is a globally hyperbolic $n+1$-dimensional manifold with timelike boundary $\partial\mathsf{M}$ and we adopt the convention that g has the signature $(-, +, \dots, +)$.
- $t : \mathsf{M} \to \mathbb{R}$ is a Cauchy temporal function and $\mathsf{M}_\mathsf{T} := t^{-1}(0, T)$ is a time strip.
- $\mathtt{n}$ is the outward unit normal vector to $\partial\mathsf{M}$.
- $\flat : \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$ and $\sharp : \mathsf{T}^*\mathsf{M} \to \mathsf{TM}$ are the musical isomorphisms.

- E is a $\mathbb{K}$-vector bundle over M with N-dimensional fibers, denoted by E_p for $p \in \mathsf{M}$, and endowed with a Hermitian fiber metric $\prec \cdot | \cdot \succ_p$.
- $\Gamma_c(\mathsf{E}), \Gamma_{sc}(\mathsf{E})$ resp. $\Gamma(\mathsf{E})$ denote the spaces of compactly supported, spacelike compactly supported resp. smooth sections of E.
- S is a symmetric system of constant characteristic and $\mathsf{S}^\dagger$ denotes the formal adjoint operator with respect to the fiber metric $\prec \, | \, \succ_p$.
- G_B and $G_{\mathsf{B}^\dagger}$ are (future) admissible boundary conditions for S and $\mathsf{S}^\dagger$ respectively and $\mathsf{B} := \ker \mathsf{G}_\mathsf{B}$ and $\mathsf{B}^\dagger := \ker \mathsf{G}_{\mathsf{B}^\dagger}$.

2 Geometric preliminaries

Let M be a connected, oriented, time-oriented smooth manifold with boundary. We assume M to be endowed with a smooth Lorentzian metric g. Here and in the following we shall assume that the boundary is timelike, *i.e.* the pullback of g with respect to the natural inclusion $\iota : \partial\mathsf{M} \to \mathsf{M}$ defines a Lorentzian metric $\iota^* g$ on the boundary. In the class of Lorentzian manifolds with timelike boundary, those called globally hyperbolic provide a suitable background where to analyze the Cauchy problem for hyperbolic operators.

Definition 2.1. [2, Definition 2.14] A *globally hyperbolic manifold with timelike boundary* is a $(n+1)$-dimensional, oriented, time-oriented, smooth Lorentzian manifold M with timelike boundary $\partial\mathsf{M}$ such that

(i) M is causal, *i.e.* there are no closed causal curves;

(ii) for every point $p, q \in \mathsf{M}$, $J^+(p) \cap J^-(q)$ is compact, where $J^+(p)$ (*resp.* $J^-(p)$) denotes the causal future (*resp.* past) of p (*resp.* q).

Remark 2.2. In case of an empty boundary, this definition agrees with the standard one, see e.g. [9, Section 3.2] or [8, Section 1.3].

Recently, Aké, Flores and Sánchez gave a characterization of globally hyperbolic manifolds with timelike boundary:

Theorem 2.3 ([2], Theorem 1.1)**.** *Any globally hyperbolic manifold with timelike boundary admits a Cauchy temporal function* $t\colon \mathsf{M} \to \mathbb{R}$ *with gradient tangent to* $\partial\mathsf{M}$*. This implies that* M *splits into* $\mathbb{R} \times \Sigma$ *with metric*

$$g = -\beta^2 dt^2 + h_t$$

where $\beta : \mathbb{R} \times \Sigma \to \mathbb{R}$ *is a smooth positive function,* h_t *is a Riemannian metric on each slice* $\Sigma_t := \{t\} \times \Sigma$ *varying smoothly with* t*, and these slices are spacelike Cauchy hypersurfaces with boundary* $\partial\Sigma_t := \{t\} \times \partial\Sigma$*, namely achronal sets intersected exactly once by every inextensible timelike curve.*

2.1 Friedrichs systems of constant characteristic

Let $\mathsf{E} \to \mathsf{M}$ be a Hermitian vector bundle over a globally hyperbolic manifold with timelike boundary M, namely a $\mathbb{K}$-vector bundle with finite rank N endowed with a positive definite Hermitian fiber metric $\prec \cdot | \cdot \succ_p\colon \mathsf{E}_p \times \mathsf{E}_p \to \mathbb{K}$.

Definition 2.4. A linear differential operator $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ of first order is called a *symmetric system* over M if

(S) The principal symbol $\sigma_\mathsf{S}(\xi)\colon \mathsf{E}_p \to \mathsf{E}_p$ is hermitian with respect to $\prec \cdot | \cdot \succ_p$ for every $\xi \in \mathsf{T}_p^*\mathsf{M}$ and for every $p \in \mathsf{M}$.

Additionally, we say that S is *hyperbolic* respectively *positive* if it holds:

(H) For every future-directed timelike covector $\tau \in \mathsf{T}^*_p\mathsf{M}$, the bilinear form $\prec \sigma_{\mathsf{S}}(\tau) \cdot \,|\, \cdot \succ_p$ is positive definite on E_p for every $p \in \mathsf{M}$;

(P) For any Cauchy hypersurface $\Sigma_t \subset \mathsf{M}$, the quadratic form $\phi \mapsto \prec \Re e(\mathsf{S}^\dagger + \mathsf{S})\phi \,|\, \phi \succ$ on Σ_t is uniformely bounded from below by a positive scalar multiple c_t of the quadratic form $\phi \mapsto \prec \phi \,|\, \phi \succ$ which depends continuously on t;

Definition 2.5. We call *Friedrichs system*, any symmetric system S which is hyperbolic or positive. Furthermore, we say that S is *of constant characteristic* if $\dim\ker\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ is constant. In particular, if $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ has maximal rank we say that S is *nowhere characteristic.*

Remark 2.6. Notice that Definition 2.4 depends on the fiber metric $\prec \cdot \,|\, \cdot \succ_p$.

Example 2.7. Consider the $n+1$-dimensional Minkowski spacetime

$$\mathsf{M} = \mathbb{R} \times \mathbb{R}^n \qquad \eta = -dt^2 + \sum_{j=1}^n dx_j^2$$

and let $\mathsf{E} := \mathsf{M} \times \mathbb{C}^N$ be a trivial vector bundle with the canonical fiber metric $\prec \cdot \,|\, \cdot \succ_{\mathbb{C}^N}$. Any linear differential operator $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ of first order reads in a point $p \in \mathsf{M}$ as

$$\mathsf{S} := A_0(p)\partial_t + \sum_{j=1}^n A_j(p)\partial_{x_j} + C(p)$$

where the coefficients A_0, A_j, C are $N \times N$ matrices, with N being the rank of E, depending smoothly on $p \in \mathsf{M}$. In these coordinates, Condition (S) in Definition 2.4 reduces to

$$A_0 = A_0^\dagger \qquad \text{and} \qquad A_j = A_j^\dagger$$

for $j = 1, \ldots, n$, where $\dagger$ is the complex conjugate of the transposed matrix. Condition (H) and (P) can be stated respectively as follows:

$$(A_0 + \sum_{j=1}^n \alpha_j A_j) > 0 \quad \text{is positive definite for} \sum_{j=1}^n \alpha_j^2 < 1 \;,$$

$$\Re e(C + C^\dagger - \frac{\partial_t(\sqrt{g}A_0)}{\sqrt{g}} - \sum_{j=1}^n \frac{\partial_{x_j}(\sqrt{g}A_j)}{\sqrt{g}}) \quad \text{is positive definite,}$$

where g is the absolute value of the determinant of the Lorentzian metric.

As we shall see in Section 6.2, a prototype example of a first order system is the so-called classical Dirac operator. In this setting, the naturally defined fiber metric on the spinor bundle is indefinite rather than Hermitian. Therefore, in order to include this important example in our framework it would be desirable to require the fiber metric simply to be sesquilinear and nondegenerate. It turns out that assuming the fiber metric to be positive-definite is not a loss of generality for a *symmetric hyperbolic system.*

Lemma 2.8. *Let E be a $\mathbb{K}$-vector bundle endowed with an indefinite nondegenerate sesquilinear fiber metric $\prec \cdot \,|\, \cdot \succ_p$ and let S be a symmetric hyperbolic system with respect to $\prec \cdot \,|\, \cdot \succ_p$. The operator $\mathfrak{S}_\beta := \sigma_{\mathsf{S}}(dt)^{-1}\mathsf{S}$ is a symmetric hyperbolic system with respect to the positive-definite Hermitian fiber metric*

$$\langle \cdot \,|\, \cdot \rangle_\beta := \beta \prec \sigma_{\mathsf{S}}(dt) \cdot \,|\, \cdot \succ_p \,, \tag{2.1}$$

where $\beta : \mathsf{M} \to \mathbb{R}^+$ is chosen on account of Theorem 2.3. *Moreover, for any boundary space B, the Cauchy problem for the operator $\mathfrak{S}_\beta$ is equivalent to the Cauchy problem for S.*

Proof. On account of Properties (S),the fiber metric (2.1) is a Hermitian fiber metric. In particular, for any $\xi \in \mathsf{T}_p^*\mathsf{M}$ it holds

$$\begin{aligned} \langle \sigma_{\mathfrak{S}}(\xi) \cdot \,|\, \cdot \rangle_\beta &= \langle \sigma_{\mathsf{S}}(dt)^{-1}\sigma_{\mathsf{S}}(\xi) \cdot \,|\, \cdot \rangle_\beta \\ &= \beta \prec \sigma_{\mathsf{S}}(\xi) \cdot \,|\, \cdot \succ_p \\ &= \beta \prec \cdot \,|\, \sigma_{\mathsf{S}}(\xi) \cdot \succ_p \\ &= \beta \prec \cdot \,|\, \sigma_{\mathsf{S}}(dt)\sigma_{\mathsf{S}}(dt)^{-1}\sigma_{\mathsf{S}}(\xi) \cdot \succ_p \\ &= \langle \cdot \,|\, \sigma_{\mathsf{S}}(dt)^{-1}\sigma_{\mathsf{S}}(\xi) \cdot \rangle_\beta \\ &= \langle \cdot \,|\, \sigma_{\mathfrak{S}}(\xi) \cdot \rangle_\beta \,, \end{aligned}$$

where we used Property (S) in the second and fourth equalities. Moreover, any solution of the Cauchy problem for S is a solution of the Cauchy problem for $\mathfrak{S}_\beta$ where the right-hand side is given by $(\frac{1}{\beta}\sigma_{\mathsf{S}}(dt)^{-1}\mathfrak{f}, \mathfrak{h})$. ☐

The reader may wonder whether a symmetric system can be assumed to enjoy property (P). With the next lemma, we shall see that, at least on relatively compact subdomains, any symmetric hyperbolic system can be transformed into a symmetric positive system such that the corresponding Cauchy problems remain equivalent.

Lemma 2.9. *Let* M *be a globally hyperbolic manifold with timelike boundary. Let* t *be a Cauchy temporal function and denote with* M_T *a time strip, i.e.* $\mathsf{M}_\mathsf{T} := t^{-1}(t_0, t_1)$*. Finally let* S *be a symmetric hyperbolic system. Then, for all* $t_0, t_1 \in \mathbb{R}$ *and for any* $\lambda \in \mathbb{R}$*, the Cauchy problem for the symmetric system* $\mathsf{K}_\lambda \colon \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) \to \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ *defined by*

$$\mathsf{K}_\lambda := \mathsf{S} + \lambda \sigma_{\mathsf{S}}(dt)$$

is equivalent to the Cauchy problem for S*, namely*

$$\begin{cases} \mathsf{K}_\lambda \widetilde{\Psi} = \widetilde{\mathfrak{f}} \\ \widetilde{\Psi}|_{\Sigma_0} = \widetilde{\mathfrak{h}} \\ \widetilde{\Psi} \in \mathsf{B} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h} \\ \Psi \in \mathsf{B}, \end{cases}$$

where $\widetilde{\mathfrak{f}} = e^{-\lambda t}\mathfrak{f}$*,* $\widetilde{\mathfrak{h}} = \mathfrak{h}$ *and* $\widetilde{\Psi} = e^{-\lambda t}\Psi$*. Moreover, for any relative compact set* $U \subset \mathsf{M}$*, there exists a constant* $\lambda \equiv \lambda(U)$ *such that* K_λ *is a positive symmetric system.*

Proof. For every $\Psi \in \Gamma(\mathsf{E})$ and for every $t \in \mathbb{R}$, we have

$$\begin{aligned} \mathsf{K}_\lambda(e^{-\lambda t}\Psi) &= (\mathsf{S} + \lambda\sigma_{\mathsf{S}}(dt))\,(e^{-\lambda t}\Psi) \\ &= \sigma_{\mathsf{S}}(de^{-\lambda t})\Psi + e^{-\lambda t}\,(\mathsf{S} + \lambda\sigma_{\mathsf{S}}(dt))\,\Psi \\ &= -\lambda e^{-\lambda t}\sigma_{\mathsf{S}}(dt)\Psi + e^{-\lambda t}\,(\mathsf{S} + \lambda\sigma_{\mathsf{S}}(dt))\,\Psi \\ &= e^{-\lambda t}\mathsf{S}\Psi, \end{aligned}$$

which shows the correspondence between the Cauchy problems for S and K_λ. By assumption, $\mathsf{S} + \mathsf{S}^\dagger$ is a zero-order operator and $\Psi \mapsto \prec \sigma_{\mathsf{S}}(dt)\Psi \,|\, \Psi \succ_p$ is positive definite on E, therefore on every compact subset of M there exists a sufficiently large real λ (depending on the compact set) such that the operator $\mathsf{K}_\lambda + \mathsf{K}_\lambda{}^\dagger = \mathsf{S} + \mathsf{S}^\dagger + 2\lambda\sigma_S(dt)$ is positive definite. ☐

Remark 2.10. Actually the assumptions of Lemma 2.9 may be weakeaned as follows: it is namely sufficient to assume S to be symmetric, i.e. with $\sigma_{\mathsf{S}}(\xi)^* = \sigma_{\mathsf{S}}(\xi)$ for all $\xi \in \mathsf{T}^*\mathsf{M}$, and the family of pointwise quadratic forms $\Psi \mapsto \prec \sigma_{\mathsf{S}}(dt)\Psi \,|\, \Psi \succ$ to be uniformely bounded from below by a positive constant to get the result. However those assumptions are equivalent to S being symmetric hyperbolic for a perturbed Lorentzian metric $\bar{g}$ on M. For by continuity the quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(dt)\Psi \,|\, \Psi \succ$ remains positive definite for all ξ in an open neighborhood of dt in $\mathsf{T}^*\mathsf{M}$. We may assume without loss of generality that neighborhood to be an open cone in $\mathsf{T}^*\mathsf{M}$ which is contained in the set of future timelike covectors for g and which depends smoothly on the base-point. Now modifying the original metric g only in ∂_t-direction, it is possible to obtain a new Lorentzian metric $\bar{g}$ such that its set of future timelike covectors coincides with – or at least is in contained in – the above cone neighborhood. We may choose the same time orientation for that new metric $\bar{g}$. Since its future cone is contained in the one of g, the new Lorentzian metric is globally hyperbolic (any timelike curve for $\bar{g}$ is a timelike curve for g) and S becomes a symmetric hyperbolic operator on $(\mathsf{M}, \bar{g})$ by definition. For the general discussion of perturbations of globally hyperbolic metrics, we refer to e.g. [2, Sec. 4.2].

We conclude this section, by deriving the Green identity for any first-order linear differential operator. To this end, consider the scalar product defined by

$$(\Phi \,|\, \Psi)_{\mathsf{M}} := \int_{\mathsf{M}} \prec \Phi \,|\, \Psi \succ \; \mathrm{vol}_{\mathsf{M}} \,, \tag{2.2}$$

for all $\Psi, \Phi \in \Gamma(\mathsf{E})$ such that $\operatorname{supp}\Psi \cap \operatorname{supp}\Phi$ is compact, where $\mathrm{vol}_{\mathsf{M}}$ is the metric-induced volume element.

Lemma 2.11. *Let M be a manifold with Lipschitz boundary $\partial\mathsf{M}$ and S be any first-order linear differential operator acting on sections of some Hermitian vector bundle E over M. Denote by $\mathsf{S}^\dagger$ the formal adjoint of S. Then for every $\Phi \in \Gamma_c(\mathsf{E}_{|\mathsf{M}})$,*

$$\Re e\Big((\mathsf{S}\Phi \,|\, \Phi)_{\mathsf{M}} - (\Phi \,|\, \mathsf{S}^\dagger \Phi)_{\mathsf{M}} \Big) = \Re e (\Phi \,|\, \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi)_{\partial\mathsf{M}} \,, \tag{2.3}$$

where $\mathtt{n}$ is the outward unit normal vector to $\partial\mathsf{M}$ and $^\flat \colon \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$ denotes the musical isomorphism. If furthermore S is symmetric i.e., its principal symbol is Hermitian, then (2.3) *holds without taking the real parts on both sides.*

Proof. Let ∇ be any metric covariant derivative on E. Let $b_0, \ldots, b_n$ be a local tangent frame which is synchronous at the point under consideration, i.e. $\nabla b_j = 0$, and denote with $b_0^*, \ldots, b_n^*$ the dual basis. In such basis, the operator S and its formal adjoint $\mathsf{S}^\dagger$ read as

$$\mathsf{S} = \sum_{j=0}^{n} \sigma_{\mathsf{S}}(b_j^*)\nabla_{b_j} + C \,, \qquad \mathsf{S}^\dagger = \sum_{j=0}^{n} -\sigma_{\mathsf{S}}(b_j^*)^\dagger \nabla_{b_j} - \nabla_{b_j}\Big(\sigma_{\mathsf{S}}(b_j^*)^\dagger\Big) + C^\dagger \,,$$

where C is some zero-order operator. Consider now the real n-form on M given by

$$\omega := \sum_{j=0}^{n} \Re e \prec \sigma_{\mathsf{S}}(b_j^*)\Phi \,|\, \Phi \succ_p b_j \lrcorner \mathrm{vol}_{\mathsf{M}} \tag{2.4}$$

where $\lrcorner$ denotes denotes the insertion of a tangent vector into the first slot of a form. By straightforward computation we get

$$\begin{aligned} d\omega \;&=\; \Re e \sum_{j=0}^{n} \Big(\prec \nabla_{b_j}(\sigma_{\mathsf{S}}(b_j^*))\Phi \,|\, \Phi \succ_p + \prec \sigma_{\mathsf{S}}(b_j^*)\nabla_{b_j}\Phi \,|\, \Phi \succ_p \\ &\qquad - \prec \Phi \,|\, -\sigma_{\mathsf{S}}(b_j^*)^\dagger \nabla_{b_j}\Phi \succ_p \Big)\mathrm{vol}_{\mathsf{M}} \\ &=\; \Re e \Big(\prec \mathsf{S}\Phi \,|\, \Phi \succ_p - \prec \Phi \,|\, \mathsf{S}^\dagger\Phi \succ_p \Big)\, \mathrm{vol}_{\mathsf{M}}. \end{aligned}$$

Using Stokes' theorem for manifolds with Lipschitz boundary we obtain (2.3). □

Remark 2.12. In case S is symmetric, the differential form ω defined above is real, therefore we obtain (2.3) without the real parts on both sides.

2.2 Admissible boundary conditions

In this paper we are interested in sections subject to certain linear homogeneous boundary conditions, depending of course if we want to solve the forward or the backward Cauchy problem. We begin by fixing a Cauchy surface $\Sigma_0 := t^{-1}(\{0\})$ where we shall assign the initial data. To define these boundary conditions we associate with each boundary point $q \in \partial\mathsf{M}$ a pair of linear subspaces $(\mathsf{B}_\pm)_q \subset \mathsf{E}_q$ whose dimensions are the same at all points of $\partial\mathsf{M}$ and which vary smoothly with q. In particular, we shall focus on a class introduced by Friedrichs and Lax-Phillips in [52, 60], dubbed admissible boundary conditions.

Definition 2.13. A smooth linear bundle map $\mathsf{G}_{\mathsf{B}_+} : \mathsf{E}_{|\partial\mathsf{M}} \to \mathsf{E}_{|\partial\mathsf{M}}$ is said to be a *future admissible boundary condition* for a first-order Friedrichs system S if

- (i-f) the pointwise kernel B_+ of $\mathsf{G}_{\mathsf{B}_+}$ is a smooth subbundle of $\mathsf{E}_{|\partial\mathsf{M}}$;
- (ii-f) the quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ_p$ is positive semi-definite on B_+ ;
- (iii-f) the rank of B_+ is equal to the number of pointwise non-negative eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counting multiplicity.

Similarly we say that $\mathsf{G}_{\mathsf{B}_-} : \mathsf{E}_{|\partial\mathsf{M}} \to \mathsf{E}_{|\partial\mathsf{M}}$ is *past admissible* if

- (i-p) the pointwise kernel B_- of $\mathsf{G}_{\mathsf{B}_-}$ is a smooth subbundle of $\mathsf{E}_{|\partial\mathsf{M}}$;
- (ii-p) the quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ_p$ is negative semi-definite on B_-;
- (iii-p) the rank of B_- is equal to the number of pointwise non-positive eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counting multiplicity.

The pair $\mathsf{B} = (\mathsf{B}_+, \mathsf{B}_-)$ is called the *admissible boundary space* or *admissible boundary condition* for S.

Remark 2.14. The role of B_+ and B_- will become apparent when looking for energy estimates for symmetric hyperbolic S, see Theorem 3.2. It turns out that B_+ (resp. B_-) is only needed in the future (resp. past) of the chosen Cauchy hypersurface Σ_0.

Notice that if $\prec \cdot \,|\, \cdot \succ$ is not positive definite, by Lemma 2.8 the new symmetric hyperbolic system $\mathfrak{S}_\beta$ together with the Hermitian positive-definite fiber metric $\langle \cdot \,|\, \cdot \rangle_\beta$ can be defined such that the Cauchy problems for both $\mathfrak{S}_\beta$ and S become equivalent. In particular, being an admissible boundary condition for $\mathfrak{S}_\beta$ is equivalent to be admissible for S. Indeed it holds

$$\langle \sigma_{\mathfrak{S}_\beta}(\mathtt{n}^\flat)\Psi \,|\, \Psi \rangle_\beta = \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \,|\, \Psi \succ \ .$$

Conditions (ii-f) and (ii-p) are equivalent to require that the boundary conditions are *maximal* with respect to properties (iii-f) and (iii-p) respectively, see [59, Theorem D.1], namely no smooth vector subbundles $(\mathsf{B}')_\pm$ of E exist that properly contains $\mathsf{B}_\pm$ and such that for all $\Phi' \in (\mathsf{B}')_+$ and $\Phi'' \in (\mathsf{B}')_-$

$$\prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi' \,|\, \Phi' \succ \geq 0 \qquad \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi'' \,|\, \Phi'' \succ \leq 0$$

holds. The fact that we do not assume $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ to be invertible (which is the case in [59, App. D]) does not play any role. As a consequence, note that $\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)) \subset \mathsf{B}_+ \cap \mathsf{B}_-$.

Definition 2.15. Let G_B be a future or past admissible boundary condition for a given first-order Friedrichs system S on E. Assume S to be of constant characteristic along $\partial\mathsf{M}$. The *adjoint boundary condition* $\mathsf{G}^\dagger_\mathsf{B}$ is defined as the pointwise orthogonal projection onto $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)(\mathsf{B})$. In particular,

$$\mathsf{B}^\dagger := \ker(\mathsf{G}^\dagger_\mathsf{B}) = \Big(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)(\mathsf{B})\Big)^\perp \ .$$

If $\mathsf{B}^\dagger = \mathsf{B}$ then we say that B is a *self-adjoint future/past admissible boundary space.*

Similarly to [59, Theorem D.2], it can be shown that, if B is a future admissible boundary condition for instance, then the quadratic form $\Phi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \mid \Phi \succ_p$ is negative semi-definite on $\mathsf{B}^\dagger$, whose rank coincides with the number of nonpositive eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counted with multiplicities. Namely $\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat))$ must be contained in B by its maximality property, so that, pointwise,

$$\begin{aligned}\dim(\mathsf{B}^\dagger) &= \dim(\mathsf{E}_{|\partial\mathsf{M}}) - \dim(\mathsf{B}) + \dim(\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)) \cap \mathsf{B}) \\ &= \dim(\mathsf{E}_{|\partial\mathsf{M}}) - \dim(\mathsf{B}) + \dim(\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat))),\end{aligned}$$

which is precisely the number of nonpositive eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ counted with multiplicities. This in turn implies that $\mathsf{B}^\dagger$ is maximal such that $\Phi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Phi \mid \Phi \succ_p$ is negative semi-definite on $\mathsf{B}^\dagger$, because pointwise any subbundle containing $\mathsf{B}^\dagger$ and enjoying the same property cannot intersect the subbundle spanned by the eigenvectors associated to the positive eigenvalues of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ in a nontrivial way. Therefore it must have the same dimension as – and therefore coincide with – $\mathsf{B}^\dagger$. As a consequence, if B is future admissible for a given Friedrichs systems S, then $\mathsf{B}^\dagger$ is future admissible for $\mathsf{S}^\dagger$. Moreover, by construction of $\mathsf{B}^\dagger$, for all $(\Psi, \Phi) \in \mathsf{B} \times_{\partial\mathsf{M}} \mathsf{B}^\dagger$ it holds

$$\prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \mid \Phi \succ_p = 0\,.$$

2.3 The forward and the backward Cauchy problem

The backward Cauchy problem for a symmetric hyperbolic system S is equivalent to the forward Cauchy problem for $-\mathsf{S}$ on the time-reversed underground spacetime:

Lemma 2.16. *Let $t\colon \mathsf{M} \to \mathbb{R}$ be a Cauchy temporal function with gradient tangent to the boundary on a given globally hyperbolic spacetime with timelike boundary M. Let S be any symmetric hyperbolic system of constant characteristic along $\partial\mathsf{M}$ and with admissible boundary space $(\mathsf{B}_+, \mathsf{B}_-)$ along $\partial\mathsf{M}$.*
Then $\bar{\mathsf{S}} := -\mathsf{S}$ is symmetric hyperbolic on $\bar{\mathsf{M}}$, which is M with the same metric but with reversed time orientation. Moreover, $(\mathsf{B}_-, \mathsf{B}_+)$ is an admissible boundary space for $\bar{\mathsf{S}}$ along $\partial\mathsf{M}$.

Proof. A 1-form $\xi \in \mathsf{T}^*\bar{\mathsf{M}} = \mathsf{T}^*\mathsf{M}$ is future-oriented causal on $\bar{\mathsf{M}}$ if and only if it is past-oriented causal on M. Therefore, if ξ is future-oriented causal on $\bar{\mathsf{M}}$, then $\sigma_{\bar{\mathsf{S}}}(\xi) = \sigma_{\mathsf{S}}(-\xi)$ is a pointwise positive definite endormorphism of E because $-\xi$ is future-oriented on M. This shows the first statement.
The second statement follows from the fact that the outward unit normal $\mathtt{n}$ to $\partial\mathsf{M}$ does not change when the time orientation is reversed, so that, if $\psi \mapsto \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\psi \mid \psi \succ$ is positive (resp. negative) semi-definite on some subbundle of $\mathsf{E}_{|\partial\mathsf{M}}$, then $\psi \mapsto \prec \sigma_{\bar{\mathsf{S}}}(\mathtt{n}^\flat)\psi \mid \psi \succ = - \prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\psi \mid \psi \succ$ is negative (resp. positive) semi-definite on that same subbundle. □

As a consequence of Lemma 2.16, given any symmetric hyperbolic system S on M with admissible boundary space $(\mathsf{B}_+, \mathsf{B}_-)$ and given any $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ as well as $\mathfrak{h} \in \Gamma_c(\mathsf{E}_{|\Sigma_0})$, solving

$$\begin{cases} \mathsf{S}u & = \mathfrak{f} \quad \text{on } \mathsf{M} \\ u_{|\Sigma_0} & = \mathfrak{h} \quad \text{on } \Sigma_0 \\ u_{|J^+_{\mathsf{M}}(\Sigma_0)\cap\partial\mathsf{M}} & \in \mathsf{B}_+ \quad \text{along } J^+_{\mathsf{M}}(\Sigma_0) \cap \partial\mathsf{M} \\ u_{|J^-_{\mathsf{M}}(\Sigma_0)\cap\partial\mathsf{M}} & \in \mathsf{B}_- \quad \text{along } J^-_{\mathsf{M}}(\Sigma_0) \cap \partial\mathsf{M} \end{cases} \tag{2.5}$$

on M is equivalent to solving

$$\begin{cases} -\mathsf{S}u & = -\mathfrak{f} \quad \text{on } \bar{\mathsf{M}} \\ u_{|\Sigma_0} & = \mathfrak{h} \quad \text{on } \Sigma_0 \\ u_{|J^+_{\bar{\mathsf{M}}}(\Sigma_0)\cap\partial\mathsf{M}} & \in \mathsf{B}_- \quad \text{along } J^+_{\bar{\mathsf{M}}}(\Sigma_0) \cap \partial\bar{\mathsf{M}} \\ u_{|J^-(\Sigma_0)\cap\partial\mathsf{M}} & \in \mathsf{B}_+ \quad \text{along } J^-_{\bar{\mathsf{M}}}(\Sigma_0) \cap \partial\bar{\mathsf{M}} \end{cases} \tag{2.6}$$

on $\bar{\mathsf{M}}$: a section u of E solves (2.5) on M if and only if u solves (2.6) on $\bar{\mathsf{M}}$.

3 Energy Inequality

In this section we derive a suitable energy inequality for Friedrichs systems in any time strip $\mathsf{M}_\mathsf{T} := t^{-1}((0,T))$. By denoting with $\|\cdot\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})}$ the norm corresponding to the scalar product $(\cdot\,|\,\cdot)_{\mathsf{M}_\mathsf{T}}$ defined by Equation (2.2), the main result of this section is the following.

Theorem 3.1 (Energy Inequality)**.** *Let M be a globally hyperbolic manifold with timelike boundary, let $t\colon \mathsf{M}\to\mathbb{R}$ be a Cauchy temporal function and denote with M_T the time strip $\mathsf{M}_\mathsf{T} := t^{-1}((0,T))$. Let S be a Friedrichs system and denote by $\mathsf{S}^\dagger$ the formal adjoint operator. Assume M to be Cauchy-compact when S is symmetric hyperbolic. Finally denote by G_B a future admissible boundary condition and by $\mathsf{G}_\mathsf{B}^\dagger$ the adjoint boundary condition. Then there exists a constant $\widetilde{C} = \widetilde{C}(\mathsf{M}_\mathsf{T}) > 0$ such that, for all $\Phi \in \Gamma_c(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ satisfying $\Phi|_{\Sigma_0} = 0$, $\Phi|_{\Sigma_T} = 0$ and $\Phi \in \mathsf{B}^\dagger|_{\mathsf{M}_\mathsf{T}}$,*

$$\|\Phi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \leq \widetilde{C}\|\mathsf{S}^\dagger\Phi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})}\,. \tag{3.1}$$

Before proving our claim, we need some preliminary results on symmetric hyperbolic systems. Let $t\colon \mathsf{M}\to\mathbb{R}$ be a Cauchy temporal function and set $\Sigma_t^p := J^-(p)\cap\Sigma_t$ for $p\in\mathsf{M}$. Denote with $\langle\,|\,\rangle_\beta$ the normalized Hermitian scalar product (2.1) and let $|\cdot|_\beta$ be the corresponding norm. Finally let $d\mu_t$ be the volume density of Σ_t.

Theorem 3.2 (Energy estimates for symmetric hyperbolic systems)**.** *Let M be a globally hyperbolic manifold with timelike boundary and let S be a symmetric hyperbolic system of constant characteristic. Then, for each $p\in\mathsf{M}$ and all $t_0,t_1\in\mathbb{R}$ with $t_0\leq t_1$ there exists a constant $C = C(p,t_0,t_1) > 0$ such that*

$$\int_{\Sigma_{t_1}^p} |\Psi|_\beta^2 d\mu_{t_1} \leq Ce^{C(t_1-t_0)}\int_{t_0}^{t_1}\int_{\Sigma_s^p}|\mathsf{S}\Psi|_\beta^2 d\mu_s ds + e^{C(t_1-t_0)}\int_{\Sigma_{t_0}^p}|\Psi|_\beta^2 d\mu_{t_0} \tag{3.2}$$

holds for each $\Psi\in\Gamma(\mathsf{E})$ satisfying $\Psi_{|\partial\mathsf{M}}\in\ker\mathsf{G}_\mathsf{B}$, where G_B is a future admissible boundary condition for S. In particular, $C = C(t_0,t_1)$ if M is Cauchy-compact.

Proof. We shall reduce to the proof of [4, Theorem 5.3]. To this end, let us define the subset $K := J^-(p)\cap t^{-1}([t_0,t_1])\subset\mathsf{M}$ and consider the n-differential form defined by Equation (2.4). Stokes' theorem for manifold with Lipschitz boundary yields

$$\int_K d\omega = \int_{\partial K}\omega = \int_{\Sigma_{t_1}^p}\omega - \int_{\Sigma_{t_0}^p}\omega + \int_{K\cap\partial\mathsf{M}}\omega + \int_Y\omega$$

where $Y := \partial J^-(p)\cap t^{-1}([t_0,t_1])$. In order to reduce our proof to the one of [4, Theorem 5.3] we only need to show that

$$\int_{K\cap\partial\mathsf{M}}\omega \geq 0\,.$$

We choose a positively oriented orthonormal tangent basis $b_0,b_1,\dots,b_n$ of $\mathsf{T}_q\mathsf{M}$ in such a way that $b_0 = -\frac{1}{\beta}\partial_t$ and $b_1 = \mathtt{n}$, so that the restriction of ω to $\partial\mathsf{M}$ is given by

$$\iota^*\omega = \prec\sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi\,|\,\Psi\succ_p \mathtt{n}\lrcorner\mathrm{vol}_{K\cap\mathsf{M}} = \prec\sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi\,|\,\Psi\succ \mathrm{vol}_{K\cap\partial\mathsf{M}}\,.$$

Therefore

$$\int_{K\cap\partial\mathsf{M}}\omega = \int_{K\cap\partial\mathsf{M}}\prec\sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi\,|\,\Psi\succ_p \mathrm{vol}_{K\cap\partial\mathsf{M}}\,.$$

Since $\Psi|_{\partial\mathsf{M}}\in\mathsf{B}|_{\mathsf{M}_\mathsf{T}}$, property (i-f) of Definition 2.13 implies that the r.h.s. of the last identity is nonnegative, which concludes the proof. □

Combining Theorem 3.2 with Lemma 2.16, we immediately obtain that if there exists a solution to the Cauchy problem (1.1) it must be unique and it propagates with at most the speed of light. We recall this results for the sake of completeness.

Proposition 3.3 (Uniqueness and finite speed of propagation for symmetric hyperbolic system)**.** *Let* M *be a globally hyperbolic manifold with timelike boundary and let* S *be a symmetric hyperbolic system of constant characteristic coupled with admissible boundary conditions. If there exists* $\Psi \in \Gamma(\mathsf{E}|_{\mathsf{M}_\mathsf{T}})$ *satisfying the Cauchy problem* (1.1) *then it is unique and it propagates with at most the speed of light, i.e. its support is contained inside the region*

$$\mathcal{V} := \Big(J\big(\mathrm{supp}\ \mathfrak{f}\big) \cup J\big(\mathrm{supp}\, \mathfrak{h}\big)\Big)\,,$$

where $J(\cdot)$ *denote the causal future of a set.*

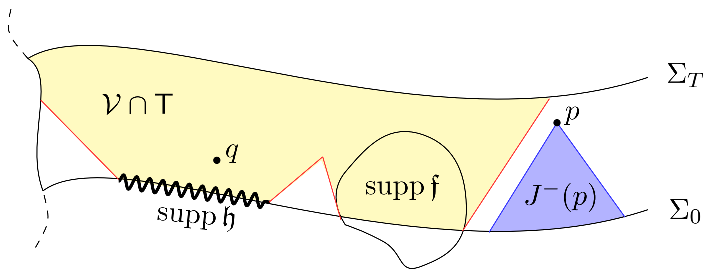

Figure 1: Finite propagation of speed – $\mathcal{V} \cap \mathsf{T}$.

Proof. Assume $q \in J^+(\Sigma_0)$ and consider any point p outside the region $\mathcal{V} \cap \mathsf{M}_\mathsf{T}$, with $\mathsf{M}_\mathsf{T} := t^{-1}(0,T)$ – *cf.* Figure 1. This means that there is no future-directed causal curve starting in supp $\mathfrak{f} \cup$ supp $\mathfrak{h}$, entirely contained in $\mathcal{V} \cap \mathsf{M}_\mathsf{T}$, which terminates at p. As a consequence, $\mathfrak{f}|_{J^-(p)} \equiv 0$ and $\mathfrak{h}|_{J^-(p)\cap\Sigma_0} \equiv 0$. Therefore, by Theorem 3.2, it follows that Ψ vanishes in $J^-(p)$.
The case $q \in J^-(\Sigma_0)$ is obtained with the time reversal (see also Lemma 2.16). Hence, Ψ vanishes outside $\mathcal{V}$.

Assume that there exist Ψ and Φ satisfying the same Cauchy problem (1.1). Then $\Psi - \Phi \in \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ is a solution of (1.1) with $\mathfrak{f} = 0$ and $\mathfrak{h} = 0$. As we have already shown, the supports of Ψ and Φ are contained in $\mathcal{V} \cap \mathsf{M}_\mathsf{T}$. Therefore, we can use Theorem 3.2 to conclude that $\Psi - \Phi$ vanishes identically. □

We notice that, as in the boundaryless case, solving the Cauchy problem associated to a symmetric hyperbolic system for Cauchy-compact or arbitrary globally hyperbolic manifolds with timelike boundary are equivalent.

Proposition 3.4. *Let* M *be a Cauchy-noncompact globally hyperbolic manifold with timelike boundary and let* $(\mathsf{M}, g) = (\mathbb{R} \times \Sigma_0, -\beta^2 dt^2 \oplus h_t)$ *be a splitting as in* Theorem 2.3*. If, for a given symmetric hyperbolic system* S *on* M*, the Cauchy problem* (1.1) *can be solved on* $(\mathbb{R} \times U, -\beta^2 dt^2 \oplus h_t)$ *for any relatively compact domain with smooth boundary* $U \subset \Sigma_0$ *and any admissible boundary condition along* ∂U*, then so can it on* M *itself.*

Proof. The proof virtually coincides with that of [5, Theorem 3.7.7]. Let $\mathfrak{f}, \mathfrak{h}$ be the Cauchy data in (1.1) and set $K :=$ supp $(\mathfrak{f}) \cup$ supp $(\mathfrak{h}) \subset \mathsf{M}$. Then K is a compact subset of M and therefore is included in some $\mathsf{M}_\mathsf{T} = (0,T) \times \Sigma_0$ for some real $0 < T$. Let $\hat{\Sigma}$ be the projection onto Σ_0 of the compact subset $J^\mathsf{M}(K) \cap ([0,T] \times \Sigma_0)$ w.r.t. the splitting $\mathsf{M} = \mathbb{R} \times \Sigma_0$. Then there exists a relatively compact open neighborhood U of $\hat{\Sigma}$ in Σ_0 with smooth boundary ∂U. Note that part of the boundary of U is contained in $\partial\mathsf{M}$ and it is only on that part that the support of the solution to (1.1) may meet ∂U. Consider now $\mathsf{M}' := \mathbb{R} \times \overline{U}$ with metric $g' := -\beta^2 dt^2 \oplus h_t$, where $\overline{U}$ is the closure of U in Σ_0. Then M' is a new Lorentzian manifold and is globally hyperbolic because of $\overline{U}$ being compact: it can be directly shown that every inextendible timelike curve in M' meets $\overline{U}$ exactly once; the main point is that, on $\overline{U}$, all metrics h_t for t in a compact interval are uniformly equivalent to some fixed metric. We refer to the proof of [8, Lemma A.5.14] that can be adapted to our situation. Now M' is a Cauchy-compact globally hyperbolic manifold with timelike boundary, therefore there exists a unique solution ψ to the Cauchy problem (1.1) on $\mathsf{M}'_\mathsf{T} := (0,T) \times \overline{U}$. Since by finite propagation speed the support of Ψ is contained in $J(K) \cap \mathsf{M}'_\mathsf{T}$, it meets $\partial \mathsf{M}'$ only along $\partial \mathsf{M}$ and, since by construction Ψ must vanish along the rest of ∂U, the section Ψ can be considered as a section of E on M_T. Therefore there exists a solution Ψ to (1.1) on M_T. Uniqueness holds on M_T as well since it holds on M'_T. Since this holds for any $0 < T \in \mathbb{R}$, we obtain global existence and uniqueness of solutions to (1.1) on M. □

Remark 3.5. Assume S to be a symmetric hyperbolic system on a globally hyperbolic manifold with noncompact Cauchy hypersurfaces. On account of Proposition 3.4, to solve the Cauchy problem for S in a time strip M_T it is enough solving it on $[0,T] \times \overline{U}$, where $\overline{U}$ is compact. Since $[0,T] \times \overline{U}$ is compact, then Lemma 2.9 guarantees the existence of a suitable λ such that the operator K_λ is a symmetric positive system and it has an equivalent Cauchy problem. Summing up, the Cauchy problem for a symmetric hyperbolic system S on a globally hyperbolic manifold M with non-compact Cauchy surfaces can be solved if the Cauchy problem for K_λ can be solved on $[0,T] \times \overline{U}$. Therefore, we may prove existence and uniqueness for the Cauchy problem for S via the auxiliary operator K_λ for sufficiently large λ. Note that this works only if M is Cauchy-compact.

We now have all the ingredients to prove Theorem 3.1.

Proof of Theorem 3.1. First, the proof of Theorem 3.1 for a symmetric hyperbolic system on a Cauchy-compact globally hyperbolic manifold is an immediate consequence of Theorem 3.2 since it suffices to integrate (3.2) on $[0,T]$ after choosing $-\mathsf{S}^\dagger$ – which is again symmetric hyperbolic – instead of S. Let S be now a symmetric positive system of constant characteristic.

By Lemma 2.11, Green identity reads

$$(\Phi \,|\, \mathsf{S}\Phi)_{\mathsf{M}_\mathsf{T}} - (\mathsf{S}^\dagger \Phi \,|\, \Phi)_{\mathsf{M}_\mathsf{T}} = (\Phi \,|\, \sigma_\mathsf{S}(\mathtt{n}^\flat)\Phi)_{\partial \mathsf{M}_\mathsf{T}}$$

where $(\cdot \,|\, \cdot)_{\partial \mathsf{M}_\mathsf{T}}$ is the induced L^2-product on $\partial \mathsf{M}_\mathsf{T}$. By adding $2(\mathsf{S}^\dagger \Phi \,|\, \Phi)_{\mathsf{M}_\mathsf{T}}$ and using that S is a symmetric positive system we thus obtain

$$(\Phi \,|\, \sigma_\mathsf{S}(\mathtt{n})\Phi)_{\partial \mathsf{M}_\mathsf{T}} + 2(\mathsf{S}^\dagger \Phi \,|\, \Phi)_{\mathsf{M}_\mathsf{T}} = (\Phi \,|\, (\mathsf{S} + \mathsf{S}^\dagger)\Phi)_{\mathsf{M}_\mathsf{T}} \geq c(\Phi \,|\, \Phi)_{\mathsf{M}_\mathsf{T}} \,, \tag{3.3}$$

for some $c > 0$, where we used condition (P) in Definition 2.4. Since $\Phi \in \mathsf{B}^\dagger|_{\mathsf{M}_\mathsf{T}}$, by definition of adjoint boundary space we have

$$(\Phi \,|\, \sigma_\mathsf{S}(\mathtt{n})\Phi)_{\partial \mathsf{M}_\mathsf{T}} \leq 0$$

Therefore, (3.3) reduces to

$$c(\Phi \,|\, \Phi)_{\mathsf{M}_\mathsf{T}} \leq 2(\Phi \,|\, \mathsf{S}^\dagger \Phi)_{\mathsf{M}_\mathsf{T}}$$

and by the Cauchy-Schwarz inequality we obtain the desired inequality. □

4 $\mathbf{L^2}$-well-posedness in a time strip

The aim of this section is to prove the L^2-well-posedness of the Cauchy problem for a Friedrichs system of constant characteristic in a time strip $\mathsf{M}_\mathsf{T} := t^{-1}((0,T))$ for $0 < T \in \mathbb{R}$. We shall achieve our goal in three steps: first, we shall prove the existence and uniqueness of weak solutions. Second, we shall prove that any weak solution can be approximated by a sequence of smooth sections by means of a localization procedure. Finally, we shall discuss the regularity of strong solutions.

To this end, let $\|\cdot\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})}$ be the norm corresponding to the scalar product (2.2) and denote L^2-completion of $\Gamma_c(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ by

$$L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) := \overline{\left(\Gamma_c(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}), (.\,|\,.)_{\mathsf{M}_\mathsf{T}}\right)}^{(.\,|\,.)_{\mathsf{M}_\mathsf{T}}}.$$

4.1 Weak solutions

Definition 4.1. We call $\Psi \in L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ a *weak solution* to the Cauchy problem (1.1) if the relation

$$(\Phi \,|\, \mathfrak{f})_{\mathsf{M}_\mathsf{T}} = (\mathsf{S}^\dagger \Phi \,|\, \Psi)_{\mathsf{M}_\mathsf{T}}$$

holds for all $\Phi \in \Gamma_c(\mathsf{E}|_{\mathsf{M}_\mathsf{T}})$ satisfying $\Phi|_{\Sigma_0} = 0$, $\Phi|_{\Sigma_T} = 0$ and $\Phi|_{\partial\mathsf{M}} \in \mathsf{B}^\dagger|_{\mathsf{M}_\mathsf{T}}$.

Theorem 4.2 (Weak existence)**.** *Let* M *be a globally hyperbolic manifold with timelike boundary and let* $t\colon \mathsf{M} \to \mathbb{R}$ *be a Cauchy temporal function. For any* $0 < T \in \mathbb{R}$ *denote with* $\mathsf{M}_\mathsf{T} := t^{-1}(0, T)$ *a time strip. Let finally* S *be a Friedrichs system and denote with* G_B *a future admissible boundary condition. Assume* M *to be Cauchy-compact when* S *is symmetric hyperbolic. Then there exists a unique weak solution* $\Psi \in L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ *to the Cauchy problem* (1.1) *restricted to* M_T*.*

Proof. By Theorem 3.1, we get that for every $\Phi \in \Gamma_c(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ satisfying $\Phi|_{\Sigma_0} = 0$, $\Phi|_{\Sigma_T} = 0$ and $\Phi|_{\partial\mathsf{M}} \in \mathsf{B}^\dagger|_{\mathsf{M}_\mathsf{T}}$ it holds

$$\|\Phi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \le \widetilde{C} \|\mathsf{S}^\dagger \Phi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})}\,. \tag{4.1}$$

The latter inequality implies that the kernel of the operator $\mathsf{S}^\dagger$ acting on

$$\operatorname{dom} \mathsf{S}^\dagger := \{\Phi \in \Gamma_c(\mathsf{E}|_{\mathsf{M}_\mathsf{T}}) \mid \Phi|_{\Sigma_0} = 0,\, \Phi|_{\Sigma_T} = 0,\, \Phi|_{\partial\mathsf{M}} \in \mathsf{B}^\dagger|_{\mathsf{M}_\mathsf{T}}\}$$

is trivial. Let now $\ell\colon \mathsf{S}^\dagger(\operatorname{dom} \mathsf{S}^\dagger) \to \mathbb{C}$ be the linear functional defined by

$$\ell(\Theta) = (\Phi \,|\, \mathfrak{f})_{\mathsf{M}_\mathsf{T}}$$

where Φ satisfies $\mathsf{S}^\dagger\Phi = \Theta$. By the energy inequality (4.1), ℓ is bounded:

$$\begin{aligned}\ell(\Theta) =& (\Phi \,|\, \mathfrak{f})_{\mathsf{M}_\mathsf{T}} \le \|\mathfrak{f}\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \, \|\Phi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \\ \le& \widetilde{C} \|\mathfrak{f}\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \|\mathsf{S}^\dagger\Phi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} = \widetilde{C}\|\mathfrak{f}\|_\mathsf{T} \|\Theta\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})},\end{aligned}$$

where in the first inequality we used the Cauchy-Schwarz inequality. Then ℓ can be extended to a continuous functional defined on the L^2-completion of $\mathsf{S}^\dagger(\operatorname{dom} \mathsf{S}^\dagger)$ denoted by $\mathcal{H} \subset L^2(\mathsf{E}|_{\mathsf{M}_\mathsf{T}})$. Finally, by Riesz's representation theorem, there exists a unique element $\Psi \in L^2(\mathsf{E}|_{\mathsf{M}_\mathsf{T}})$ such that

$$\ell(\Theta) = (\Theta \,|\, \Psi)_{\mathsf{M}_\mathsf{T}}\,.$$

for all $\Theta \in \mathsf{S}^\dagger(\mathrm{dom}\mathsf{S}^\dagger)$. Thus, we obtain

$$(\Phi \,|\, \mathfrak{f})_{\mathsf{M}_\mathsf{T}} = \ell(\Theta) = (\Theta \,|\, \Psi)_{\mathsf{M}_\mathsf{T}} = (\mathsf{S}^\dagger\Phi \,|\, \Psi)_{\mathsf{M}_\mathsf{T}}$$

for all $\Phi \in \mathrm{dom}\mathsf{S}^\dagger$, which proves the existence of a unique weak solution Ψ. □

4.2 Strong solutions

Definition 4.3. We call $\Psi \in L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ a *strong solution* of the initial-boundary value problem (1.1) if there exists a sequence of sections $\Psi_k \in \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) \cap L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ such that $\Psi_{k|\partial\mathsf{M}} \in \mathsf{B}|_{\mathsf{M}_\mathsf{T}}$ on $\partial\mathsf{M}_\mathsf{T}$ on Σ_0 and

$$\|\Psi_k - \Psi\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathsf{S}\Psi_k - \mathfrak{f}\|_{L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})} \xrightarrow{k\to\infty} 0.$$

In order to show that any weak solution is a strong solution, we begin by localizing the problem. Hence, consider an open covering $\{\mathsf{U}_j\}_j$ of M_T and let φ_j be a smooth partition of unity subordinated to U_j.

Lemma 4.4. *A section $\Psi \in L^2(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ is a weak solution of the Cauchy problem* (1.1) *if and only if for any j, $\Psi_j := \varphi_j\Psi$ is a weak solution of*

$$\begin{cases} \mathsf{S}\Psi_j = \mathfrak{f}_j := \varphi_j\mathfrak{f} + \sigma_\mathsf{S}(d\varphi_j)\Psi \\ \Psi_j|_{\Sigma_0} = \mathfrak{h}_j := \varphi_j\mathfrak{h} \\ \Psi_j|_{\partial\mathsf{M}} \in \mathsf{B}|_{\mathsf{M}_\mathsf{T}}. \end{cases} \tag{4.2}$$

Proof. To verify our claim, suppose Ψ satisfies $\mathsf{S}\Psi = \mathfrak{f}$ in a weak sense, *i.e.* for any $\Phi \in \Gamma_c(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ satisfying $\Phi_{|\partial\mathsf{M}} \in \mathsf{B}^\dagger|_{\mathsf{M}_\mathsf{T}}$, and $\Phi_{|\Sigma_0} = 0$, it holds $(\mathsf{S}^\dagger\Phi \,|\, \Psi)_\mathsf{T} = (\Phi \,|\, \mathfrak{f})_\mathsf{T}$. Using $(\Phi \,|\, \varphi_j\Psi)_{\mathsf{M}_\mathsf{T}} = (\varphi_j\Phi \,|\, \Psi)_{\mathsf{M}_\mathsf{T}}$ and then Leibniz rule, it follows that

$$\begin{aligned}(\mathsf{S}^\dagger\Phi \,|\, \Psi_j)_{\mathsf{M}_\mathsf{T}} &= (\varphi_j\mathsf{S}^\dagger\Phi \,|\, \Psi)_{\mathsf{M}_\mathsf{T}} = (\mathsf{S}^\dagger(\varphi_j\Phi) \,|\, \Psi)_{\mathsf{M}_\mathsf{T}} - ((\mathsf{S}^\dagger\varphi_j)\Phi \,|\, \Psi)_{\mathsf{M}_\mathsf{T}} = \\ &= (\varphi_j\Phi \,|\, \mathfrak{f})_{\mathsf{M}_\mathsf{T}} + (\sigma_\mathsf{S}(d\varphi_j)\Phi \,|\, \Psi)_{\mathsf{M}_\mathsf{T}} = (\Phi \,|\, \varphi_j\mathfrak{f} + \sigma_\mathsf{S}(d\varphi_j)\Psi)_{\mathsf{M}_\mathsf{T}}\end{aligned}$$

This shows that Ψ_j is a weak solution of the Cauchy problem 4.2. Conversely, suppose that Ψ_j is a weak solution of the Cauchy problem (4.2). Then by summing over j and using $\sum_j d\varphi_j = 0$, we find that a weak solution $\Psi = \sum_j \Psi_j$ is a weak solution of $\mathsf{S}\Psi = \mathfrak{f}$. □

Definition 4.5. Let $\mathsf{U} \subset \mathsf{M}_\mathsf{T}$ be a compact subset in M_T. We say that $\Psi \in L^2(\mathsf{E}_{|\mathsf{U}})$ is a *locally strong solution* of the Cauchy problem (1.1) if there exists a sequence of sections $\Psi_k \in \Gamma(\mathsf{E}_{|\mathsf{U}})$ such that $\Psi_{k|\partial\mathsf{M}} \in \mathsf{B}|_{\mathsf{M}_\mathsf{T}}$ on $\partial\mathsf{M}_\mathsf{T} \cap \mathsf{U}$ and

$$\|\Psi_k - \Psi\|_{L^2(\mathsf{E}_{|\mathsf{U}})} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathsf{S}\Psi_k - \mathfrak{f}\|_{L^2(\mathsf{E}_{|\mathsf{U}})} \xrightarrow{k\to\infty} 0.$$

We concentrate on points in the boundary $p \in \partial\mathsf{M} \cap \mathsf{M}_\mathsf{T}$ (the other points will even be easier to handle since we do not have to care about boundaries) and firstly define a convenient chart as follows, compare also Figure 2: Let Σ_p be the Cauchy surface of M_T to which p belongs to. For

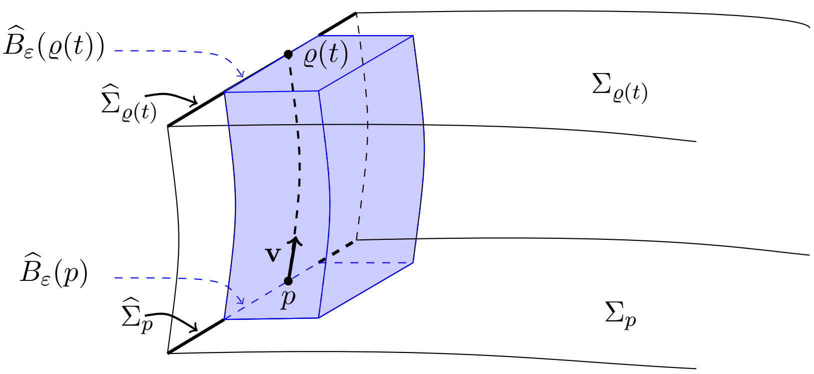

Figure 2: Fermi coordinates on each Cauchy surface.

$q \in \partial\mathsf{M} \cap \mathsf{M}_\mathsf{T}$ let $\widehat{\Sigma}_q := \Sigma_q \cap \partial\mathsf{M} \cap \mathsf{M}_\mathsf{T}$ be the corresponding Cauchy surface in the boundary. Let

$\varrho\colon [0,T] \to \partial\mathsf{M} \cap \mathsf{M}_\mathsf{T}$ be the timelike geodesic in $\partial\mathsf{M} \cap \mathsf{M}_\mathsf{T}$ starting at p with velocity $\mathbf{v} \in \mathsf{T}_p\partial\mathsf{M}$ where $\mathbf{v}$ is a normalized, future-directed, timelike vector perpendicular to $\widehat{\Sigma}_p$ in $\partial\mathsf{M} \cap \mathsf{M}_\mathsf{T}$. Let $\widehat{B}_\varepsilon(\varrho(t))$ be the ε-ball in $\partial\hat{\Sigma}_{\varrho(t)}$ around $\varrho(t)$. On these balls we choose geodesic normal coordinates $\widehat{\kappa}_t\colon B^{n-1}_\varepsilon(0) \subset \mathbb{R}^{n-1} \to \widehat{B}_\varepsilon(\varrho(t))$. Moreover, inside each $\Sigma_{\varrho(t)}$ we choose Fermi coordinates with base $\widehat{B}_\varepsilon(\varrho(t))$. Thus, we obtain a chart in $\Sigma_{\varrho(t)}$ around $\varrho(t)$ as

$$\begin{aligned} \widetilde{\kappa}_t\colon B^{n-1}_\varepsilon(0) \times [0,\varepsilon] \subset \mathbb{R}^n &\to U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) \\ (y,z) &\mapsto \exp^{\perp,\Sigma_{\varrho(t)}}_{\widehat{\kappa}_t(y)}(z) \end{aligned}$$

where $U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) := \{q \in \Sigma_{\varrho(t)} \mid \mathrm{dist}_{\Sigma_{\varrho(t)}}(q, \widehat{B}_\varepsilon(\varrho(t)) \leq \varepsilon\}$, $\exp^{\perp,\Sigma_{\varrho(t)}}_{\widehat{\kappa}_t(y)}(z)$ is the normal exponential map in $\Sigma_{\varrho(t)}$ starting at $\widehat{\kappa}_t(y)$ with velocity perpendicular to $\widehat{\Sigma}_{\varrho(t)} = \partial\Sigma_{\varrho(t)}$ pointing in the interior and with magnitude z. Putting all this together we obtain a chart

$$\begin{aligned} \kappa_p\colon [0,T] \times B^{n-1}_\varepsilon(0) \times [0,\varepsilon] \subset \mathbb{R}^{n+1} &\to \mathsf{U}_p := \bigcup_{t\in[0,\varepsilon]} U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) \subset \mathsf{M}_\mathsf{T} \\ (t,y,\overline{z}) &\mapsto \widetilde{\kappa}_t(y,\overline{z}). \end{aligned}$$

For us here, the only purpose of those charts is to specify coordinates such that near the point p the Cauchy problem is close enough to the Minkowski standard form.

We are finally in the position to prove our first main result.

Proof of Theorem 1.1. To prove our claim it remains to show that any weak solution Ψ of the Cauchy problem (1.1) is a strong solution. Let $\{\mathsf{U}_j\}_j$ be an open covering of M_T and let φ_j be a smooth partition of unity subordinated to $\{\mathsf{U}_j\}_j$. Set $\Psi_j := \varphi_j\Psi$. On account of Lemma 4.4, it is enough to check that for any j, Ψ_j is a locally strong solution. For every $p \in \mathsf{U}_j$, there exists a sufficiently small $\varepsilon > 0$ such that in the coordinates κ_p from above a symmetric hyperbolic systems S has the form

$$\mathsf{S} = \sigma_\mathsf{S}(dt)\partial_t + \sum_{j=1}^{n-1} \sigma_\mathsf{S}(dx^j)\partial_{x^j} + \sigma_\mathsf{S}(d\overline{z})\partial_{\overline{z}} + C(t,y,\overline{z})$$

for some zero-order operator C. For every $\mathsf{U}_j \cap \partial\mathsf{M} \neq \emptyset$ then [63, Theorem 4] ensures that any weak solution is a strong solution. If $\mathsf{U}_j \cap \partial\mathsf{M} = \emptyset$, then using the classical results of Friedrichs [51] we can conclude. Recall that, by Proposition 3.4, if S is a symmetric hyperbolic system, then M may be assumed to be Cauchy-compact. This ends the proof of Theorem 1.1. □

4.3 Differentiability of the solutions for symmetric hyperbolic systems

It is well-known that the Cauchy problem for the backward heat equation is not well-posed. This is because an initial data for the backward heat equation is a final condition for the forward heat equation. The latter equation has a smoothening effect on the initial data, *i.e.* the solution is smooth even if the initial data is only continuous. It is easy to understand that, there exists a class of smooth initial data for the backward heat equation generating non-smooth solutions. Since the heat equation can be reduced to a symmetric positive system (*cf.* Section 7.2), we cannot expect the existence of smooth solutions for a generic symmetric positive system. Hence, in this section we shall only focus only on the subclass of symmetric hyperbolic systems. In particular, we shall see that, if the Cauchy data $(\mathfrak{f}, \mathfrak{h})$ are smooth and a compatibility condition is imposed, then the strong solution is actually smooth. To this end, let $t\colon \mathsf{M} \to \mathbb{R}$ be a Cauchy temporal function with gradient tangent to the boundary, as in Theorem 2.3, and write the symmetric hyperbolic system S as

$$\mathsf{S} = \sigma_\mathsf{S}(dt)\nabla_t - \mathsf{H}$$

where H is a first-order linear differential operator which differentiates only in the directions that are tangent to Σ and where ∇ is any fixed metric connection for $\prec \cdot | \cdot \succ$. Finally let $G_{\mathsf{B}_+}, G_{\mathsf{B}_-} : \mathsf{E}_{|\partial\mathsf{M}} \longrightarrow \mathsf{E}_{|\partial\mathsf{M}}$ be future and past admissible boundary conditions for S, in particular $\mathsf{B}_\pm := \ker(G_{\mathsf{B}_\pm})$ defines the future resp. past admissible boundary space for S along $\partial\mathsf{M}$. The compatibility conditions of order $k \geq 0$ for $\mathfrak{h} \in \Gamma(\mathsf{E}_{|\Sigma_0})$ and $\mathfrak{f} \in \Gamma(\mathsf{E})$ read

$$\sum_{j=0}^{k} \frac{(k)!}{j!(k-j)!} \Big(\nabla_t^j G_{\mathsf{B}_+}\Big)\Big|_{\partial\Sigma_0} \mathfrak{h}_{k-j} = 0, \tag{4.3}$$

and

$$\sum_{j=0}^{k} \frac{(k)!}{j!(k-j)!} \Big(\nabla_t^j G_{\mathsf{B}_-}\Big)\Big|_{\partial\Sigma_0} \mathfrak{h}_{k-j} = 0, \tag{4.4}$$

where the sequence $(\mathfrak{h}_k)_k$ of sections of $E_{|\partial\Sigma_0}$ is defined inductively by $\mathfrak{h}_0 := \mathfrak{h}$ and

$$\mathfrak{h}_k := \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!} \mathsf{H}_{j\,|\partial\Sigma_0}\, \mathfrak{h}_{k-1-j} + \nabla_t^{k-1}\big(\sigma_{\mathsf{S}}^{-1}(dt)\mathfrak{f}\big)_{|\partial\Sigma_0} \qquad \text{for all } k \geq 1,$$

where $\mathsf{H}_j := [\nabla_t, \mathsf{H}_{j-1}]$ and $\mathsf{H}_0 := \sigma_{\mathsf{S}}(dt)^{-1}\mathsf{H}$.

Notation 4.6. We denote the space of data which satisfy the compatibility conditions as

$$\overline{\Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) \times \Gamma(\mathsf{E}_{|\Sigma_0})} := \{(\mathfrak{f}, \mathfrak{h}) \in \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) \times \Gamma(\mathsf{E}_{|\Sigma_0}) \mid \text{ (4.3) and (4.4) hold } \}.$$

The compatibility conditions (4.3) and (4.4) up to order k must be fulfilled in order for the solution of the Cauchy-problem, if it exists, to be C^k. Those conditions are sufficient for nowhere characteristic symmetric hyperbolic systems [64]. However, if the symmetric hyperbolic system is of nonvanishing constant characteristic, full regularity of the solution cannot be expected in general, see e.g. [67]. In that case, as shown in [63], there exists a good notion of tangential regularity. Given the Cauchy hypersurface $\Sigma_0 \subset \mathsf{M}$ where the initial condition is fixed, we say that a vector field $X \in \Gamma(\mathsf{T}\Sigma_0)$ is tangential to $\partial\Sigma_0$ if and only if $X_{|\partial\Sigma_0} \in \Gamma(\mathsf{T}\partial\Sigma_0)$, i.e. $g(X, \mathtt{n})_q = 0$ for every $q \in \partial\Sigma_0$. We denote the space of tangential vector fields as

$$\mathfrak{X}_{\tan}(\Sigma_0) := \{X \in \Gamma(\mathsf{T}\Sigma_0) \,|\, g(X, \mathtt{n})_q = 0 \text{ for all } q \in \partial\Sigma_0\}.$$

As in [66], we define – at least in the case where M is Cauchy-compact, otherwise a metric along Σ_0 has to be fixed – for any $m \geq 0$ the anisotropic Sobolev space $H^m_*(\mathsf{E}_{|\Sigma_0})$ as

$$\begin{aligned} H^m_*(\mathsf{E}_{|\Sigma_0}) \;\; := \;\; & \Big\{\phi \in \mathsf{L}^2(\mathsf{E}_{|\Sigma_0}),\, \nabla_{X_1} \cdots \nabla_{X_h} \nabla_{X_1'} \cdots \nabla_{X_k'} \phi \in \mathsf{L}^2(\mathsf{E}_{|\Sigma_0}) \\ & \forall\, X_1, \dots, X_h, X_1', \dots, X_k' \Big\}, \end{aligned}$$

where $X_1, \dots, X_h, X_1', \dots, X_k'$ are smooth tangent vector fields on Σ_0 with $X_1, \dots, X_h \in \mathfrak{X}_{\tan}(\Sigma_0)$, $X_1', \dots, X_k' \notin \mathfrak{X}_{\tan}(\Sigma_0)$ as well as $h + 2k \leq m$. The space $H^m_*(\mathsf{E}_{|\Sigma_0})$ can be endowed with a Hilbert-space structure, see [66, p. 673]. It is easy to see that $H^m_*(\mathsf{E}_{|\Sigma_0}) \subset H^{[\frac{m}{2}]}(\mathsf{E}_{|\Sigma_0})$, in particular $H^m_*(\mathsf{E}_{|\Sigma_0})$ embeds continuously into the space $\Gamma^p(\mathsf{E}_{|\Sigma_0})$ of C^p sections of $\mathsf{E}_{|\Sigma_0}$ as soon as $m > \frac{n}{2} + p$ by the Sobolev embedding theorem for compact manifolds with C^1 boundary, see e.g. [1, Ch. V]. For the sake of completeness, we recall part of Secchi's main result [66, Theorem 2.1] in our context: Fix any integers $m \geq 2[\frac{n}{2}] + 6$ and $1 \leq s \leq m$. Assume Σ_0 to be compact with smooth boundary and of nonzero and nonmaximal constant characteristic w.r.t. a symmetric hyperbolic system S on M. Let G_B to be a future admissible boundary condition for S along $\partial\mathsf{M}$. Given $\mathfrak{f} \in \bigcap_{j=0}^{s} H^j([0,\mathsf{T}], H^{s-j}_*(\Sigma_0))$ and $\mathfrak{h} \in H^s_*(\Omega)$, assume that the compatibility conditions

(4.3) are satisfied up to order $s-1$ and that $\mathfrak{h}_j \in H_*^{s-j}(\Sigma_0)$ for all $j = 0, \ldots, s-1$. Then there exists a unique $\Psi \in \bigcap_{j=0}^{s} \Gamma^j([0,\mathsf{T}], H_*^{s-j}(\Sigma_0))$ solution of the initial boundary value problem $\mathsf{S}\Psi = \mathfrak{f}$ on M_T, $\Psi_{|\Sigma_0} = \mathfrak{h}$ and $\Psi_{|\partial\mathsf{M}} \in \mathsf{B}|_{\mathsf{M}_\mathsf{T}}$. Note in particular that, if $\mathfrak{f}, \mathfrak{h}$ are smooth on M, then so must be Ψ because it lies in $\Gamma^j\left([0,T], \Gamma^p(\mathsf{E}_{|\Sigma_0})\right)$ for any j, p.

Theorem 4.7. *Let* M *be a globally hyperbolic manifold with timelike boundary and let* S *be a symmetric hyperbolic system of constant characteristic. If the data* $(\mathfrak{f}, \mathfrak{h})$ *are smooth and satisfy the compatibility conditions* (4.3) *and* (4.4)*, then the strong solution* Ψ *of the Cauchy problem* (1.1) *lies in* $\Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$.

Proof. First let $p \in \partial\mathsf{M} \cap \Sigma_\mathsf{T}$ for some and let $\varrho\colon [0,\mathsf{T}] \to \partial\mathsf{M}$ be a timelike curve with $\varrho(0) \in \Sigma_0$ and $p = \varrho(\mathsf{T})$. We fix $\varepsilon > 0$ such that we have Fermi coordinates on a "cube" $U_{\varrho(t)}$ around $\varrho(t)$ as in Section 4.2 for all $t \in [0,\mathsf{T}]$. This is always possible since the image of ϱ is compact and everything depends smoothly on the base point. For $\widetilde{U}_p := \bigcup_{t\in[0,\mathsf{T}]} U_{\varrho(t)}$ we know that the compatibility conditions (4.3) and (4.4) are fulfilled along Σ_0 by assumption. Thus, [66, Theorem 2.1] tells us that the strong solution Ψ lies in $\Gamma(\mathsf{E}_{|\widetilde{U}_p})$. If S is nowhere characteristic, Ψ is actually smooth on account of [64, Theorem 3.1]. For $p \in \mathsf{M} \setminus \partial\mathsf{M}$ we choose a timelike curve $\varrho\colon [0,\mathsf{T}] \to \mathsf{M} \setminus \partial\mathsf{M}$ with $\varrho(0) \in \Sigma_0$ and $p = \varrho(\mathsf{T})$ and proceed as before. It is even easier since we can just use geodesic normal coordinates in the Cauchy hypersurfaces around each $\rho(t)$. □

5 Global well-posedness

Up to now we have obtained a strong solution in any time strip M_T in Theorem 1.1 and showed that, if the Cauchy data $(\mathfrak{f}, \mathfrak{h})$ fulfill the compatibility condition (4.3), then the solution is actually smooth (*cf.* Theorem 4.7). We can now easily put everything together to obtain global well-posedness of the Cauchy problem for a symmetric hyperbolic system of constant characteristic.

Proof of Theorem 1.2. Fix $\mathfrak{h} \in \Gamma_c(\mathsf{E}_{|\Sigma_0})$. On account of Theorem 4.2, for any $T \in [0,\infty)$ there exists a weak solution Ψ_T to the Cauchy problem (1.1) in the time strip $\mathsf{M}_T := t^{-1}([0,T])$. By Theorem 4.7, we get in particular that Ψ_T is smooth in the time strip M_T. By uniqueness of the solution, we get $\Psi_{T_1}|_{t^{-1}[0,T_1]} = \Psi_{T_2}|_{t^{-1}[0,T_1]}$ for all $T_1 \leq T_2 \in [0,\infty)$. By combining a similar argument for negative time with Lemma 2.16, we get existence of solutions for negative times. Finally, the stability of the Cauchy problem follows by [4, Section 5], the fact that we have a boundary condition playing no role in the proof. □

A byproduct of the well-posedness of the Cauchy problem is the existence of Green operators:

Proposition 5.1. *A symmetric hyperbolic system of constant characteristic on a globally hyperbolic manifold with timelike boundary coupled with an admissible boundary condition is Green-hyperbolic, i.e., there exist linear maps, called advanced/retarded Green operator respectively,* $\mathsf{G}^\pm\colon \Gamma_c(\mathsf{E}) \to \Gamma_{sc,\mathsf{B}_\pm}(\mathsf{E})$ *satisfying*

(i) $\mathsf{S} \circ \mathsf{G}^\pm \mathfrak{f} = \mathfrak{f}$ *for all* $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ *and* $\mathsf{G}^\pm \circ \mathsf{S}\mathfrak{f} = \mathfrak{f}$ *for all* $\mathfrak{f} \in \Gamma_{c,\mathsf{B}_\pm}(\mathsf{E})$*;*

(ii) $\operatorname{supp}(\mathsf{G}^\pm \mathfrak{f}) \subset J^\pm(\operatorname{supp}\mathfrak{f})$ *for all* $\mathfrak{f} \in \Gamma_c(\mathsf{E})$,

where $J^\pm$ *denote the causal future (+) and past (−) and* $\Gamma_{\sharp,\mathsf{B}_\pm}(\mathsf{E}) \subset \Gamma_\sharp(\mathsf{E})$*,* $\sharp \in \{sc, c\}$ *denotes the space of smooth sections on* E *(with* $\sharp$ *support property) which fulfill the* $\mathsf{B}_\pm$*-boundary condition.*

Proof. Let $\mathfrak{f} \in \Gamma_c(\mathsf{E})$. We choose $t_0 \in \mathbb{R}$ such that $\operatorname{supp}\mathfrak{f} \subset J^+(\Sigma_{t_0})$. By Theorem 1.2, there exists a unique solution $\Psi = \Psi(\mathfrak{f})$ to the Cauchy problem

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi_{|\Sigma_{t_0}} = 0 \\ \Psi_{|\partial\mathsf{M}} \in \mathsf{B}_+. \end{cases}$$

We set $\mathsf{G}^+\mathfrak{f} := \Psi$ and notice that $\mathsf{S} \circ \mathsf{G}^+\mathfrak{f} = \mathsf{S}\Psi = \mathfrak{f}$. Note that by the finite speed of propagation, (cf. Proposition 3.3), $\mathsf{G}^+\mathfrak{f} \in \Gamma_{sc,\mathsf{B}_+}(\mathsf{E})$. Moreover, $\mathsf{G}^+ \circ \mathsf{S}\Psi = \mathsf{G}^+\mathfrak{f} = \Psi$ which shows (i). By Proposition 3.3, we obtain $\operatorname{supp} \mathsf{G}^+\mathfrak{f} \subset J^+(\operatorname{supp} f)$ and this concludes the proof of (ii) for G^+. The existence of the retarded Green operator G^- is proven analogously. □

6 Examples of symmetric hyperbolic systems

6.1 The Euler momentum equation

We briefly discuss an elementary example of a symmetric hyperbolic system where the notion of future and past admissible boundary conditions is essential. Most of what we present here is inspired from [64, Example p.305], [5, Exercise 3.9.22] and can be found in [38].

We consider $\mathsf{M} := \mathbb{R} \times \Sigma$ with metric $g := -dt^2 \oplus h$, where (Σ^n, h) is an arbitrary Riemannian manifold with nonempty boundary. Let $\mathsf{E} := \pi_2^*\mathsf{T}\Sigma \to \mathsf{M}$ be the tangent bundle of Σ pulled back onto M via the second canonical projection $\pi_2 \colon \mathsf{M} \to \Sigma$. Given any section u_0 of E, that is, any possibly time-dependent smooth vector field along Σ, we define the so-called linearized Euler operator

$$\mathsf{S} := \partial_t + \nabla_{u_0} + \nabla.u_0$$

acting on sections of E. Because of $\sigma_{\mathsf{S}}(\xi) = \xi(\partial_t + u_0) \cdot \mathrm{Id}$ for every $\xi \in \mathsf{T}^*\mathsf{M}$, the operator S is symmetric and is hyperbolic if and only if $h(u_0, u_0) \leq 1$ everywhere on M. Denoting by $\mathtt{n}$ the outward unit normal to $\partial\mathsf{M}$, we have $\sigma_{\mathsf{S}}(\mathtt{n}^\flat) = h(\mathtt{n}, u_0) \cdot \mathrm{Id} = \langle \mathtt{n}, u_0\rangle \cdot \mathrm{Id}$, so that S is of constant characteristic as soon as $\langle \mathtt{n}, u_0\rangle$ vanishes either identically or nowhere along $\partial\mathsf{M}$:

1. If $\langle \mathtt{n}, u_0\rangle > 0$ along $\partial\mathsf{M}$, then S is nowhere characteristic along $\partial\mathsf{M}$ and the only possible future and past admissible conditions for S along $\partial\mathsf{M}$ are $\mathsf{B}_+ = \mathsf{E}_{|\partial\mathsf{M}}$ and $\mathsf{B}_- = \{0\}$.
2. If $\langle \mathtt{n}, u_0\rangle = 0$ along $\partial\mathsf{M}$, then S is of constant characteristic and the only possible future and past admissible conditions for S along $\partial\mathsf{M}$ are $\mathsf{B}_\pm = \mathsf{E}_{|\partial\mathsf{M}}$.
3. If $\langle \mathtt{n}, u_0\rangle < 0$ along $\partial\mathsf{M}$, then S is nowhere characteristic along $\partial\mathsf{M}$ and the only possible future and past admissible conditions for S along $\partial\mathsf{M}$ are $\mathsf{B}_+ = \{0\}$ and $\mathsf{B}_- = \mathsf{E}_{|\partial\mathsf{M}}$.

From now on, we assume $\Sigma := \mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$ with standard Euclidean metric and u_0 to be the restriction of any nonzero parallel vector field from $\mathbb{R}^n$ to $\mathbb{R}^n_+$. Up to rescaling u_0, we may assume that $h(u_0, u_0) \leq 1$ on $\mathbb{R}^n_+$. We fix $\Sigma_0 := \{0\} \times \Sigma$ as a Cauchy hypersurface in M and $v_0 \in \Gamma(\mathsf{E}_{|\Sigma_0})$ as initial data. Consider the Cauchy problem for S:

$$\begin{cases} \mathsf{S}u & = 0 \quad \text{on } \mathsf{M} \\ u_{|\Sigma_0} & = v_0 \quad \text{on } \Sigma_0 \\ u_{|_{J^+(\Sigma_0)\cap\partial\mathsf{M}}} & \in \mathsf{B}_+ \quad \text{along } J^+(\Sigma_0) \cap \partial\mathsf{M} \\ u_{|_{J^-(\Sigma_0)\cap\partial\mathsf{M}}} & \in \mathsf{B}_- \quad \text{along } J^-(\Sigma_0) \cap \partial\mathsf{M} \end{cases} \tag{6.1}$$

The equation $\mathsf{S}u = 0$ on M with initial data $u_{|\Sigma_0} = v_0$ along Σ_0 has a unique solution u which can be explicitely written as

$$u(t, x) = v_0(x - tu_0)$$

for all $(t, x) \in \mathsf{M}$. Clearly, if $\langle \mathtt{n}, u_0\rangle > 0$ along $\partial\mathsf{M}$ (which is the case as soon as this inequality is satisfied at one point of $\partial\mathsf{M}$), then no boundary condition can be imposed for u along $J^+(\Sigma_0) \cap \partial\mathsf{M}$, whereas u must vanish along $J^-(\Sigma_0) \cap \partial\mathsf{M}$, otherwise there would exist infinitely many solutions to (6.1). This is precisely what the boundary conditions $\mathsf{B}_\pm$ prescribe in that case. Analogously, if $\langle \mathtt{n}, u_0\rangle < 0$ along $\partial\mathsf{M}$, then no boundary condition can be imposed for u along $J^-(\Sigma_0) \cap \partial\mathsf{M}$, whereas u must vanish along $J^+(\Sigma_0) \cap \partial\mathsf{M}$, otherwise the same violation of uniqueness for solutions to symmetric hyperbolic systems occurs. If $\langle \mathtt{n}, u_0\rangle = 0$ along $\partial\mathsf{M}$, then no boundary condition at all, whether in the past or the future of Σ_0, can be imposed, which is consistent with the fact that the curves $t \mapsto x - tu_0$ in that case run either entirely along $\partial\mathsf{M}$ or in $\mathsf{M} \setminus \partial\mathsf{M}$.

In all three cases, the compatibility conditions (4.3) and (4.4) for the solution u only mean that v_0 vanishes along $\partial\Sigma$ as well as all its time derivatives.

6.2 The classical Dirac operator

Let (M, g) be a globally hyperbolic manifold of dimension $n+1$ and assume to have a spin structure *i.e.* a twofold covering map from the $\mathrm{Spin}_0(1,n)$-principal bundle $\mathsf{P}_{\mathrm{Spin}_0}$ to the bundle of positively-oriented tangent frames $\mathsf{P}_{\mathrm{SO}^+}$ of M such that the following diagram is commutative:

$$\begin{array}{ccccc}
\mathsf{P}_{\mathrm{Spin}_0} \times \mathrm{Spin}_0(1,n) & \longrightarrow & \mathsf{P}_{\mathrm{Spin}} & & \\
\downarrow & & \downarrow & \searrow & \\
\mathsf{P}_{\mathrm{SO}^+} \times \mathrm{SO}(1,n) & \longrightarrow & \mathsf{P}_{\mathrm{SO}^+} & \longrightarrow & \mathsf{M}
\end{array}$$

The existence of spin structures is related to the topology of M. A sufficient (but not necessary) condition for the existence of a spin structure is the parallelizability of the manifold. Therefore, since any 3-dimensional orientable manifold is parallelizable, it follows by Theorem 2.3 that any 4-dimensional globally hyperbolic manifold admits a spin structure. Given a fixed spin structure, one can use the spinor representation to construct the spinor bundle, *i.e.* the complex vector bundle

$$\mathbb{S}\mathsf{M} := \mathrm{Spin}_0(1,n) \times_\rho \mathbb{C}^N$$

where $\rho : \mathrm{Spin}_0(1,n) \to \mathrm{Aut}(\mathbb{C}^N)$ is the complex $\mathrm{Spin}_0(1,n)$ representation and $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$. The spinor bundle comes together with the following structures:

- a natural $\mathrm{Spin}_0(1,n)$-invariant indefinite fiber metric

$$\prec \cdot \,|\, \cdot \succ_p : \mathbb{S}_p\mathsf{M} \times \mathbb{S}_p\mathsf{M} \to \mathbb{C};$$

- a *Clifford multiplication*, *i.e.* a fiber-preserving map

$$\gamma \colon \mathsf{TM} \to \mathrm{End}(\mathbb{S}\mathsf{M})$$

  which satisfies for all $p \in \mathsf{M}$, $u, v \in \mathsf{T}_p\mathsf{M}$ and $\psi, \phi \in \mathbb{S}_p\mathsf{M}$

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u,v)\mathrm{Id}_{\mathbb{S}_p\mathsf{M}}\,, \quad \prec \gamma(u)\psi \,|\, \phi \succ_p = \prec \psi \,|\, \gamma(u)\phi \succ_p \,.$$

Definition 6.1. The *(classical) Dirac operator* D is the operator defined as the composition of the metric connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}\mathsf{M}$, obtained as a lift of the Levi-Civita connection on TM, and the Clifford multiplication:

$$\mathsf{D} = \gamma \circ \nabla^{\mathbb{S}\mathsf{M}} \colon \Gamma(\mathbb{S}\mathsf{M}) \to \Gamma(\mathbb{S}\mathsf{M})\,.$$

In local coordinates and with a trivialization of the spinor bundle $\mathbb{S}\mathsf{M}$, the Dirac operator reads as

$$\mathsf{D}\psi = \sum_{\mu=0}^{n} \varepsilon_\mu \gamma(e_\mu) \nabla^{\mathbb{S}\mathsf{M}}_{e_\mu} \psi$$

where $\{e_\mu\}$ is a local Lorentzian-orthonormal frame of the tangent bundle TM and $\varepsilon_\mu = g(e_\mu, e_\mu) = \pm 1$.

Proposition 6.2. *The classical Dirac operator* D *on globally hyperbolic spin manifolds* M *with timelike boundary is a nowhere characteristic symmetric hyperbolic system.*

Proof. Our claim follows from [62, Proposition 2.15] and [57, Corollary 3.12]. □

Examples of admissible boundary conditions

The aim of this section is to test whether particular known boundary conditions for the Dirac operator are admissible in the sense of Definition 2.13. In particular, we shall show that the Lorentzian counterpart of the standard Riemannian boundary conditions are admissible, see e.g. [56, Section 1.5].

Lorentzian chirality boundary conditions. Given a so-called chirality operator $\mathcal{G}$ on $\mathbb{S}\mathsf{M}$, i.e. a parallel involutive antiunitary (with respect to $\prec\cdot\,|\,\cdot\succ$) endomorphism-field of $\mathbb{S}\mathsf{M}$ that anti-commutes with Clifford multiplication by vectors, one may define the so-called *chirality* boundary space which is defined as the range of the map

$$\pi_{\mathrm{CHI}} := \frac{1}{2}\left(\mathrm{Id} - \gamma(\mathtt{n})\mathcal{G}\right)$$

where $\gamma(\mathtt{n})$ denotes Clifford multiplication for the outward-pointing unit normal along $\partial\mathsf{M}$.
The map π_{CHI} is clearly a linear projection since it satisfies $\pi_{\mathrm{CHI}}^2 = \pi_{\mathrm{CHI}}$. Furthermore, the range of π_{CHI} is the pointwise eigenspace of $\gamma(\mathtt{n})\mathcal{G}$ to the eigenvalue -1 and $\mathcal{G}$ exchanges that eigenspace with the eigenspace to the eigenvalue 1 since $\{\mathcal{G}, \gamma(\mathtt{n})\mathcal{G}\} = 0$. Therefore, the range of π_{CHI} has dimension $2^{[\frac{n}{2}]-1}$, which is the number of nonnegative – actually positive – eigenvalues of the endomorphism $\sigma_{\mathsf{D}}(\mathtt{n}^\flat)$. Since $\mathcal{G}$ is skew-Hermitian with respect to the indefinite spin product $\prec\cdot\,|\,\cdot\succ$, the complex number $\prec\mathcal{G}\psi\,|\,\psi\succ$ must be imaginary for any $\psi\in\mathbb{S}\mathsf{M}_{|\partial\mathsf{M}}$, therefore we have, for any $\psi\in\mathbb{S}\mathsf{M}_{|\partial\mathsf{M}}$,

$$\begin{aligned}
\prec\sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi_{\mathrm{CHI}}\psi\,|\,\pi_{\mathrm{CHI}}\psi\succ &= \prec\gamma(\mathtt{n})\pi_{\mathrm{CHI}}\psi\,|\,\pi_{\mathrm{CHI}}\psi\succ \\
&= \prec\mathcal{G}\pi_{\mathrm{CHI}}\psi\,|\,\pi_{\mathrm{CHI}}\psi\succ,
\end{aligned}$$

whose right-hand side is simultaneously real and imaginary and hence must vanish. This proves the chirality condition to be admissible. Analogous arguments show that the range of $\pi_{\mathrm{CHI}} := \frac{1}{2}\left(\mathrm{Id} + \gamma(\mathtt{n})\mathcal{G}\right)$ is also an admissible boundary space.

Example 6.3. An important example of a chirality operator is given by

$$\mathcal{G} := i^{[\frac{n}{2}]}\gamma(e_0)\gamma(e_1)\ldots\gamma(e_n)\colon \mathbb{S}\mathsf{M} \longrightarrow \mathbb{S}\mathsf{M},$$

where $(e_0, e_1, \ldots, e_n)$ is any pointwise Lorentzian orthonormal basis of TM. Up to an imaginary scalar factor, $\mathcal{G}$ is the action of the volume form of (M, g). It is easy to see that $\mathcal{G}$ is involutive and parallel and that, if n is odd (i.e, M has even dimension), then $\mathcal{G}$ is skew-Hermitian (hence antiunitary) with respect to $\prec\cdot\,|\,\cdot\succ$ and anti-commutes with the Clifford action of any tangent vector. Therefore, if M has even dimension, then $\mathcal{G}$ is a chirality operator in the above sense.

Riemannian chirality boundary conditions. Let $\mathcal{G}$ be a chirality operator as before, but we now assume $\mathcal{G}$ to commute with $\gamma(\partial_t)$ and to be unitary (with respect to $\prec\cdot\,|\,\cdot\succ$). Consider the projector operator

$$\pi_{\mathrm{CHI}} := \frac{1}{2}\left(\mathrm{Id} + \frac{i}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\mathcal{G}\right).$$

Since the Riemannian Clifford multiplication on the spacelike slice Σ_t is related to the Lorentzian one by

$$\gamma_{\Sigma_t}(X) \simeq \frac{\imath}{\beta}\gamma(X)\gamma(\partial_t) \tag{6.2}$$

for all $X \in \mathsf{T}\Sigma_t$, we can interpret the range of π_{CHI} to be a Riemannian chirality boundary space. Contrary to the (Lorentzian) chirality boundary condition, the map π_{CHI} is an orthogonal projection: it clearly satisfies $\pi_{\text{CHI}}^2 = \pi_{\text{CHI}}$ and, for any $\psi, \varphi \in \mathbb{S}\mathsf{M}$,

$$\begin{aligned}
\prec \pi_{\text{CHI}}\psi \,|\, \varphi \succ &= \frac{1}{2} \prec \psi + \frac{\imath}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\mathcal{G}\psi \,|\, \varphi \succ \\
&= \frac{1}{2}\left(\prec \psi \,|\, \varphi \succ - \frac{1}{\beta} \prec \mathcal{G}\psi \,|\, \imath\gamma(\mathtt{n})\gamma(\partial_t)\varphi \succ \right) \\
&= \frac{1}{2} \prec \psi \,|\, \varphi + \frac{\imath}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\mathcal{G}\varphi \succ \\
&= \prec \psi \,|\, \pi_{\text{CHI}}\varphi \succ \ .
\end{aligned}$$

Moreover, the range of π_{CHI} is the pointwise eigenspace of $\frac{\imath}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\mathcal{G}$ to the eigenvalue 1 and $\mathcal{G}$ exchanges that eigenspace with the eigenspace to the eigenvalue -1 since $\{\mathcal{G}, \frac{i}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\mathcal{G}\} = 0$. Therefore, the range of π_{CHI} has dimension $2^{\left[\frac{n}{2}\right]-1}$, which is the number of nonnegative eigenvalues of the endomorphism $\sigma_{\mathsf{D}}(\mathtt{n}^\flat)$. As another consequence of the above computation, we have, for any $\psi \in \mathbb{S}\mathsf{M}$,

$$\gamma(\mathtt{n})\pi_{\text{CHI}}\psi = -\frac{\imath}{\beta}\gamma(\partial_t)\mathcal{G}\pi_{\text{CHI}}\psi,$$

where $\gamma(\partial_t)\mathcal{G}$ is Hermitian with respect to $\prec \cdot \,|\, \cdot \succ$ since $[\mathcal{G}, \partial_t] = 0$ by assumption. Now, for any $\psi \in \gamma(\mathbb{S}\mathsf{M})$, we obtain

$$\begin{aligned}
\prec \sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi_{\text{CHI}}\psi \,|\, \pi_{\text{CHI}}\psi \succ &= \prec \gamma(\mathtt{n})\pi_{\text{CHI}}\psi \,|\, \pi_{\text{CHI}}\psi \succ \\
&= -\frac{i}{\beta} \prec \gamma(\partial_t)\mathcal{G}\pi_{\text{CHI}}\psi \,|\, \pi_{\text{CHI}}\psi \succ,
\end{aligned}$$

and the right-hand side of the last identity is simultaneously real and imaginary, therefore vanishes. This proves the Riemannian chirality boundary condition to be admissible. Analogous arguments show that the range of $\pi_{\text{CHI}} := \frac{1}{2}\left(\text{Id} - \frac{i}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\mathcal{G}\right)$ is also an admissible boundary space.

Lorentzian MIT bag boundary conditions. Consider the so-called MIT bag boundary space, which is defined as the range of

$$\pi_{\text{MIT}} := \frac{1}{2}\left(\text{Id} - \imath\gamma(\mathtt{n})\right),$$

where $\gamma(\mathtt{n})$ is again the Lorentzian Clifford multiplication for the outward-pointing unit normal vector along $\partial\mathsf{M}$. It is clear it is a pointwise linear projection whose range is the pointwise eigenspace of $\imath\gamma(\mathtt{n})$ to the eigenvalue -1 and that is exchanged with the other eigenspace (to the eigenvalue 1) by the Clifford multiplication of any nonzero vector that is orthogonal to $\mathtt{n}$. Therefore, the range of π_{MIT} has dimension $2^{\left[\frac{n}{2}\right]-1}$, which is the number of nonnegative eigenvalues of the endomorphism $\sigma_{\mathsf{D}}(\mathtt{n}^\flat)$. Moreover, for any $\psi \in \mathbb{S}\mathsf{M}_{|\partial\mathsf{M}}$,

$$\begin{aligned}
\prec \sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi_{\text{MIT}}\psi \,|\, \pi_{\text{MIT}}\psi \succ_\beta &= \prec \gamma(\mathtt{n})\pi_{\text{MIT}}\psi \,|\, \pi_{\text{MIT}}\psi \succ \\
&= \imath \prec \pi_{\text{MIT}}\psi \,|\, \pi_{\text{MIT}}\psi \succ,
\end{aligned}$$

which is simultaneously real and imaginary, therefore vanishes. This proves the MIT bag boundary condition to be also admissible.

Analogous arguments show that the range of $\pi_{\text{MIT}} := \frac{1}{2}\left(\text{Id} + \imath\gamma(\mathtt{n})\right)$ is also an admissible boundary space.

Riemannian MIT boundary condition. We shall now present the Riemannian counterpart of the MIT boundary condition, replacing the Clifford multiplication on M by that along each Σ_t. Motivated by (6.2), consider the operator

$$\pi_{\mathrm{MIT}} := \frac{1}{2}\left(\mathrm{Id} - \frac{1}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\right).$$

As the (Lorentzian) MIT boundary condition, it is a projection whose range has dimension $2^{\left[\frac{n}{2}\right]-1}$. Moreover, since for any $\psi \in \mathbb{S}\mathsf{M}$ it holds

$$\gamma(\mathtt{n})\pi_{\mathrm{MIT}}\psi = \frac{1}{\beta}\gamma(\partial_t)\pi_{\mathrm{MIT}}\psi,$$

this implies

$$\begin{aligned} \prec \sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ &= \prec \gamma(\mathtt{n})\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ \\ &= \frac{1}{\beta} \prec \gamma(\partial_t)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ \geq 0\,. \end{aligned}$$

This proves the Riemannian MIT bag boundary space to be also admissible for the forward Cauchy problem. Notice that the range of $\pi_{\mathrm{MIT}} := \frac{1}{2}\left(\mathrm{Id} + \frac{1}{\beta}\gamma(\mathtt{n})\gamma(\partial_t)\right)$ is admissible for the backward Cauchy problem since we have

$$\prec \sigma_{\mathsf{D}}(\mathtt{n}^\flat)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ \;=\; -\frac{1}{\beta} \prec \gamma(\partial_t)\pi_{\mathrm{MIT}}\psi \,|\, \pi_{\mathrm{MIT}}\psi \succ \leq 0\,.$$

6.3 The geometric wave operator

Let V be a Hermitian vector bundle of finite rank and consider a normally hyperbolic operator $\mathsf{P}\colon \Gamma(\mathsf{V}) \to \Gamma(\mathsf{V})$, *i.e.* a 2^{nd}-order linear differential operator with principal symbol σ_{P} defined by

$$\sigma_{\mathsf{P}}(\xi) = -g(\xi,\xi)\cdot \mathrm{Id}_{\mathsf{V}}\,,$$

for every $\xi \in \mathsf{T}^*\mathsf{M}$. Then P can be turned into a symmetric hyperbolic system of a first order, see e.g. [5, Remark 3.7.11]. First, there exists a unique covariant derivative ∇ on V such that $\mathsf{P} = \nabla^*\nabla + c$ for some zero-order term c, see [8, Lemma 1.5.5]. By Theorem 2.3, the globally hyperbolic manifold M can be written as $(\mathbb{R}\times\Sigma, -\beta^2 dt^2 + h_t)$, where each $\{t\}\times\Sigma$ is a smooth spacelike Cauchy hypersurface of M, the function β is smooth and positive $\mathbb{R}\times\Sigma$ and $(h_t)_{t\in\mathbb{R}}$ is a smooth one-parameter-family of Riemannian metrics on Σ. Then computations show that

$$\nabla^*\nabla = \frac{1}{\beta^2}\nabla^2_{\partial_t} + \frac{1}{2\beta^2}\left(\mathrm{tr}_{h_t}(\partial_t h_t) - \frac{\partial_t\beta^2}{\beta^2}\right)\nabla_{\partial_t} + (\nabla^\Sigma)^*\nabla^\Sigma - \frac{1}{2\beta^2}\nabla^\Sigma_{\mathrm{grad}_{h_t}(\beta^2)},$$

where ∇^Σ is the restricted covariant derivative on Σ, that is, $\nabla^\Sigma_X u := \nabla_X u$ for all $X \in \mathsf{T}\Sigma$ and $u \in \Gamma(\mathsf{V})$. Therefore, P can be written under the form

$$\mathsf{P} = \frac{1}{\beta^2}\nabla^2_{\partial_t} + b_0\nabla_{\partial_t} + (\nabla^\Sigma)^*\nabla^\Sigma + \nabla^\Sigma_b + c,$$

where $b_0 := \frac{1}{2\beta^2}\left(\mathrm{tr}_{h_t}(\partial_t h_t) - \frac{\partial_t\beta^2}{\beta^2}\right) \in C^\infty(\mathbb{R}\times\Sigma,\mathbb{R})$, $b := -\frac{1}{2\beta^2}\mathrm{grad}_{h_t}(\beta^2) \in \Gamma(\pi_2^*\mathsf{T}\Sigma)$. This allows us to rewrite the Cauchy problem for P with boundary condition $\Pi_{\mathsf{B}}\colon \mathsf{V}\oplus(\mathsf{T}^*\Sigma\otimes\mathsf{V})\oplus\mathsf{V} \to \mathsf{B}$

$$\begin{cases} \mathsf{P}u = f \\ u_{|\Sigma_{t_0}} = h \\ \nabla_{\partial_t} u_{|\Sigma_{t_0}} = h' \\ (\nabla^{\mathsf{V}}_{\partial t}u, \nabla^\Sigma u, u)_{|\partial\mathsf{M}} \in \mathsf{B} \end{cases} \tag{6.3}$$

as a Cauchy problem for $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ with boundary condition $\Pi_{\mathsf{B}}\colon \mathsf{E} \to \mathsf{B}$,

$$\begin{cases} \mathsf{S}\Psi := (A_0\nabla^{\mathsf{V}}_{\partial_t} + A_\Sigma\nabla^\Sigma + C)\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_{t_0}} = \mathfrak{h} \\ \Psi|_{\partial\mathsf{M}} \in \mathsf{B} \end{cases} \tag{6.4}$$

where E is the Hermitian vector bundle $\mathsf{E} := \mathsf{V} \oplus (\mathsf{T}^*\Sigma \otimes \mathsf{V}) \oplus \mathsf{V}$, $B \in \Gamma(\mathrm{End}(\mathsf{E}))$ and

$$\Psi := \begin{pmatrix} \nabla^{\mathsf{V}}_{\partial_t} u \\ \nabla^\Sigma u \\ u \end{pmatrix}, \quad \mathfrak{f} := \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}, \quad A_0 := \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_\Sigma = \begin{pmatrix} 0 & -\mathrm{tr}_{h_t} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C := \begin{pmatrix} b_0 & b\lrcorner & c \\ 0 & \frac{1}{2}h_t^{-1}\partial_t h_t\lrcorner & R_{\partial_t,\cdot} \\ -1 & 0 & 0 \end{pmatrix}.$$

The Cauchy problem (6.4) should be read as follows: $\nabla_{\partial_t}\nabla^\Sigma u$ is defined by

$$\left(\nabla_{\partial_t}\nabla^\Sigma u\right)_X := \nabla_{\partial_t}\nabla^\Sigma_X u - \nabla^\Sigma_{(\nabla_{\partial_t}X)^\Sigma} u$$

for all $X \in \Gamma(\pi_2^*\mathsf{T}\Sigma)$. The term $\nabla^\Sigma\Psi$ is a section of $(\mathsf{T}^*\Sigma\otimes\mathsf{V})\oplus(\mathsf{T}^*\Sigma\otimes\mathsf{T}^*\Sigma\otimes\mathsf{V})\oplus(\mathsf{T}^*\Sigma\otimes\mathsf{V}) \to \mathsf{M}$, the trace coefficient contracting $\mathsf{T}^*\Sigma \otimes \mathsf{T}^*\Sigma$ of course. The coefficient $\frac{1}{2}h_t^{-1}\partial_t h_t\lrcorner$ is more or less the Weingarten map (or shape operator) put into the $\mathsf{T}\Sigma$ slot. The curvature tensor R is that of ∇ and is by convention given for all $X, Y \in \mathsf{TM}$ by $R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The only difference with Bär's expression for the first-order-operator, apart from swapping the first and the second components of Ψ, is the vanishing of the $(2,1)$-coefficient in the zero-order matrix (no coefficient $\pi^t(\cdot)$), which plays no role anyway for conditions (S) and (H) since those deal with the principal symbol.

Remark 6.4. Notice that, while any solution u of the Cauchy problem (6.3) gives a solution Ψ to the Cauchy problem (6.4), the contrary does not hold. Indeed, the space of initial data for Ψ is "too large" and some a suitable restriction has to be imposed. For further details we refer to [5, Remark 3.7.11].

We summarize the previous observation in the following proposition.

Proposition 6.5. *Any normally hyperbolic operator* P *on a globally hyperbolic manifold* M *with timelike boundary can be reduced to a symmetric hyperbolic system* S *of constant characteristic given as in* (6.4).

Proof. As in [5, Remark 3.7.11], Conditions (S) and (H) can be easily checked. Moreover, by choosing a Cauchy temporal function with gradient tangent to $\partial\mathsf{M}$, it is easy to see that S is of constant characteristic. Indeed, since

$$\sigma_{\mathsf{S}}(\mathtt{n}^\flat) = \begin{pmatrix} 0 & -\mathtt{n}^\flat\lrcorner & 0 \\ -\mathtt{n}^\flat\otimes & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the pointwise kernel of $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ is given by

$$\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat)) = \{0\} \oplus (\mathtt{n}^\flat)^\perp \otimes \mathsf{V} \oplus \mathsf{V},$$

which clearly has constant rank. □

Remark 6.6. Notice that $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ has pointwise three eigenvalues: 0 of multiplicity nk, where $n+1 = \dim(\mathsf{M})$ and $k = \mathrm{rk}_{\mathbb{R}}(\mathsf{V})$, 1 and -1, both of the same multiplicity k. Actually, for any $\varepsilon \in \{\pm 1\}$ and for any $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \mathsf{E}$, we have

$$\begin{aligned}
\sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi = -\varepsilon\Psi \quad &\Longleftrightarrow \quad (-\mathtt{n}^\flat \lrcorner \Psi_2, -\mathtt{n}^\flat \otimes \Psi_1, 0) = -\varepsilon(\Psi_1, \Psi_2, \Psi_3) \\
&\Longleftrightarrow \quad \begin{cases} \mathtt{n}^\flat \lrcorner \Psi_2 & = \varepsilon \Psi_1 \\ \mathtt{n}^\flat \otimes \Psi_1 & = \varepsilon \Psi_2 \\ \Psi_3 & = 0 \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} \mathtt{n}^\flat \otimes \Psi_1 & = \varepsilon \Psi_2 \\ \Psi_3 & = 0 \end{cases} \\
&\Longleftrightarrow \quad \Psi = (\Psi_1, \varepsilon \mathtt{n}^\flat \otimes \Psi_1, 0) \\
&\Longleftrightarrow \quad \Psi = \left(\left(\mathrm{Id} \oplus \varepsilon \mathtt{n}^\flat \otimes \right)(\Psi_1), 0 \right)
\end{aligned}$$

that is,

$$\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat) + \varepsilon) = \left(\mathrm{Id} \oplus \varepsilon \mathtt{n}^\flat \otimes \right)(\mathsf{V}) \oplus \{0\}.$$

As a consequence, since $\mathrm{Id} \oplus \varepsilon \mathtt{n}^\flat \otimes$ is injective, $\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat) + \varepsilon)$ has pointwise rank k. In particular,

$$\sum_{\lambda \geq 0} \dim(\ker(\sigma_{\mathsf{S}}(\mathtt{n}^\flat) - \lambda)) = (n+1)k.$$

Definition 6.7. Let P be a normally hyperbolic operator. We say that B' is an *admissible boundary space* for P if there exists an admissible boundary space B for S such that the Cauchy problems are equivalent.

Before showing some example of boundary conditions $\Pi_{\mathsf{B}'}$ for P which reduce to admissible boundary condition Π_{B} for S, let us state and prove the main result of this section:

Theorem 6.8. *Let M be a globally hyperbolic manifold with timelike boundary and denote with B' an admissible boundary space for a normally hyperbolic operator $\mathsf{P}\colon \Gamma(\mathsf{V}) \to \Gamma(\mathsf{V})$. Then the Cauchy problem for P is well-posed, namely for any data (f, h, h') satisfying the compatibility condition for any $k \geq 0$, there exists a unique smooth solution $u \in \Gamma(\mathsf{V})$ to the mixed initial-boundary value problem* (6.3) *which depends continuously on the data (f, h, h').*

Note that, when we require (f, h, h') to satisfy the compatibility condition (4.3) for any $k \geq 0$, we mean that the corresponding data $(\mathfrak{f}, \mathfrak{h}) = ((f, 0, 0), (h', \nabla^\Sigma h, h))$ for the first-order symmetric hyperbolic system S satisfies (4.3) for any $k \geq 0$. The proof is a straightforward consequence of Theorem 1.2.

Examples of admissible boundary conditions

The aim of this section is to test whether particular known boundary conditions for normally hyperbolic operators P are admissible in the sense of Definition 6.7.

Neumann-like boundary conditions. We look at a particular boundary condition, namely the condition

$$\nabla^\Sigma_{\mathtt{n}} u = 0 \tag{6.5}$$

along $\partial\mathsf{M}$. We could call it the *Neumann-like boundary condition.* In that case, for the corresponding symmetric hyperbolic systems S the boundary space B coincides with the kernel of the pointwise projection

$$G_{\mathsf{B}}\colon \mathsf{E}_{|\partial\mathsf{M}} \longrightarrow \mathsf{E}_{|\partial\mathsf{M}}, \qquad G_{\mathsf{B}} := \begin{pmatrix} 0 & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That kernel can be written explicitly down

$$\ker(G_{\mathsf{B}}) = \mathsf{V} \oplus \ (\mathtt{n}^\flat)^\perp \otimes \mathsf{V} \ \oplus \mathsf{V}$$

and direct computations shows that $\dim(\ker(G_{\mathsf{B}})) = (n+1)k$ pointwise. Furthermore, for any $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in \ker(G_{\mathsf{B}})$,

$$\begin{aligned} \langle \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi, \Psi \rangle &= \langle (-\mathtt{n}\lrcorner\Psi_2, -\mathtt{n}^\flat \otimes \Psi_1, 0), (\Psi_1, \Psi_2, \Psi_3) \rangle \\ &= -2\Re e(\langle \mathtt{n}\lrcorner\Psi_2, \Psi_1 \rangle) = 0 \end{aligned}$$

where we used $\mathtt{n}\lrcorner\Psi_2 = \nabla^\Sigma_{\mathtt{n}} u = 0$ since $\Psi \in \ker(G_{\mathsf{B}})$. This proves (6.5) to be admissible in the sense of Definition 6.7.

Transparent boundary conditions. The transparent boundary condition is defined as

$$\nabla^\Sigma_{\mathtt{n}} u = -b \nabla_{\partial_t} u \tag{6.6}$$

along $\partial\mathsf{M}$ for some real parameter b, see e.g. [42, Eq. (1)]. In that case, the bundle B coincides with the kernel of the pointwise projection

$$G_{\mathsf{B}} \colon \mathsf{E}_{|\partial\mathsf{M}} \longrightarrow \mathsf{E}_{|\partial\mathsf{M}}, \qquad G_{\mathsf{B}} := \begin{pmatrix} b & \mathtt{n}\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that is,

$$\begin{aligned} \mathsf{B} &= \ker(G_{\mathsf{B}}) \\ &= \left\{ \left(-\frac{1}{b}\mathtt{n}\lrcorner X_2, X_2, X_3 \right) \mid (X_2, X_3) \in \mathsf{T}^*\Sigma \otimes \mathsf{V} \ \oplus \ \mathsf{V} \right\} \\ &= \left(-\frac{1}{b}\mathtt{n}\lrcorner \cdot \oplus \mathrm{Id} \right) (\mathsf{T}^*\Sigma \otimes V) \ \oplus \ \mathsf{V}. \end{aligned}$$

In particular, $\mathrm{rk}_{\mathbb{R}}(\mathsf{B}) = (n+1)k$, as required. Moreover, for any $X = (X_1, X_2, X_3) \in \mathsf{B}$,

$$\langle \sigma_{\mathsf{S}}(\mathtt{n}^b)X, X \rangle = -2\Re e(\langle \mathtt{n}\lrcorner X_2, X_1 \rangle) = \frac{2}{b}|\mathtt{n}\lrcorner X_2|^2,$$

which is nonnegative as soon as $b \geq 0$. This shows (6.6) to be admissible for the forward Cauchy problem when $b \geq 0$, while if admissible for the backward Cauchy problem if $b \leq 0$.

An example of a non-admissible boundary condition.

Robin boundary condition for differential forms. In the particular situation where $\mathsf{V} = \Lambda^p\mathsf{T}^*\mathsf{M}$ is the bundle of differential forms on M for some $p \in \{0, 1, \ldots, n+1\}$, there is another boundary condition called the Robin boundary condition. It is defined, for any p-form ω on M by

$$\begin{cases} \iota^*(\mathtt{n}\lrcorner d\omega) &= \tau\iota^*\omega \\ \iota^*(\mathtt{n}\lrcorner\omega) &= 0 \end{cases},$$

where τ is a real parameter. Here d denotes the exterior differential as usual and $\iota \colon \partial\mathsf{M} \longrightarrow \mathsf{M}$ is the inclusion map. The case where $\tau = 0$ is considered (at least by some geometric analysts) as the "standard" generalization of the Neumann boundary condition for forms; it is usually called "absolute boundary condition" in the literature (there are also relative ones). For Robin boundary conditions – we let τ be any real parameter for the time being, so this includes the absolute boundary condition – the bundle B is the kernel of the pointwise projection

$$G_{\mathsf{B}} := \begin{pmatrix} -dt\wedge(\mathtt{n}\lrcorner\cdot) & \iota^*(\mathtt{n}\lrcorner\cdot) - \sum_{j=2}^n e_j^*\wedge\iota^*(\mathtt{n}\lrcorner e_j\lrcorner\cdot) & -\tau\iota^* \\ 0 & 0 & 0 \\ 0 & 0 & \iota^*(\mathtt{n}\lrcorner\cdot) \end{pmatrix},$$

where $(e_j)_{2\leq j\leq n}$ denotes any pointwise o.n.b. of $T(\partial\mathsf{M}\cap\Sigma)$. Next we make B a bit more precise.

It is already clear that

$$\ker(\iota^*(\mathtt{n}\lrcorner\cdot)) = \{\omega\in\mathsf{V}\,|\,\iota^*\omega=\omega\} = \Lambda^p\mathsf{T}^*\partial\mathsf{M},$$

whose pointwise rank is $\begin{pmatrix} n \\ p \end{pmatrix}$. To see what condition the first line in the above matrix G_{B} gives, we split any $\omega\in\mathsf{V}$ as follows:

$$\begin{aligned} \omega &= \mathtt{n}^*\wedge\omega^{(1)}+\omega^T \\ &= \mathtt{n}^*\wedge dt\wedge(\omega^{(1)})^{(t)} + n^*\wedge(\omega^{(1)})^{\partial\mathsf{M}\cap\Sigma}+\omega^T \\ &= \mathtt{n}^*\wedge dt\wedge(\omega^{(1)})^{(t)} + n^*\wedge(\omega^{(1)})^{\partial\mathsf{M}\cap\Sigma}+dt\wedge(\omega^T)^{(t)}+(\omega^T)^{\partial\mathsf{M}\cap\Sigma}, \end{aligned}$$

where $(\omega^{(1)})^{(t)}\in\Lambda^{p-2}\mathsf{T}^*(\partial\mathsf{M}\cap\Sigma)$, $(\omega^{(1)})^{\partial\mathsf{M}\cap\Sigma},(\omega^T)^{(1)}\in\Lambda^{p-1}\mathsf{T}^*(\partial\mathsf{M}\cap\Sigma)$ and $(\omega^T)^{\partial\mathsf{M}\cap\Sigma}\in\Lambda^p\mathsf{T}^*(\partial\mathsf{M}\cap\Sigma)$. For any $X=(X_1,X_2,X_3)\in\mathsf{B}$, we write $X_2=\sum_{j=1}^n e_j^*\otimes\omega_j$, where $(e_1=\nu,e_2,\ldots,e_n)$ is a pointwise o.n.b. of $\mathsf{T}\Sigma$ and $\omega_j\in\mathsf{V}$ for all $j\geq 1$. Then, setting $\Omega := -dt\wedge(\mathtt{n}\lrcorner X_1)+\iota^*(X_2(\mathtt{n}))-\sum_{j=2}^n e_j^*\wedge\iota^*(\mathtt{n}\lrcorner X_2(e_j))-\tau\iota^*X_3$, we compute and use $\iota^*X_3=X_3$ as well as $\iota^*(\mathtt{n}\lrcorner\omega_j)=\mathtt{n}\lrcorner\omega_j$:

$$\begin{aligned} \Omega &= -dt\wedge(\mathtt{n}\lrcorner X_1)+\iota^*\omega_1-\sum_{j=2}^n e_j^*\wedge(\mathtt{n}\lrcorner\omega_j)-\tau X_3 \\ &= -dt\wedge X_1^{(1)}+\omega_1^T-\sum_{j=2}^n e_j^*\wedge\omega_j^{(1)}-\tau X_3 \\ &= -dt\wedge\left(dt\wedge(X_1^{(1)})^{(t)}+(X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}\right)+dt\wedge(\omega_1^T)^{(t)}+(\omega_1^T)^{\partial\mathsf{M}\cap\Sigma} \\ &\quad -\sum_{j=2}^n e_j^*\wedge\left(dt\wedge(\omega_j^{(1)})^{(t)}+(\omega_j^{(1)})^{\partial\mathsf{M}\cap\Sigma}\right)-\tau dt\wedge(X_3)^{(t)}-\tau X_3^{\partial\mathsf{M}\cap\Sigma} \\ &= -dt\wedge(X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}+dt\wedge(\omega_1^T)^{(t)}+(\omega_1^T)^{\partial\mathsf{M}\cap\Sigma} \\ &\quad +dt\wedge\left(\sum_{j=2}^n e_j^*\wedge(\omega_j^{(1)})^{(t)}\right)-\sum_{j=2}^n e_j^*\wedge(\omega_j^{(1)})^{\partial\mathsf{M}\cap\Sigma}-\tau dt\wedge(X_3)^{(t)} \\ &\quad -\tau X_3^{\partial\mathsf{M}\cap\Sigma} \\ &= dt\wedge\left((\omega_1^T)^{(t)}-(X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}+\sum_{j=2}^n e_j^*\wedge(\omega_j^{(1)})^{(t)}-\tau(X_3)^{(t)}\right) \\ &\quad +(\omega_1^T)^{\partial\mathsf{M}\cap\Sigma}-\sum_{j=2}^n e_j^*\wedge(\omega_j^{(1)})^{\partial\mathsf{M}\cap\Sigma}-\tau X_3^{\partial\mathsf{M}\cap\Sigma}. \end{aligned}$$

Therefore, $\Omega=0$ if and only if

$$\begin{cases} (\omega_1^T)^{(t)}-(X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}+\sum_{j=2}^n e_j^*\wedge(\omega_j^{(1)})^{(t)}-\tau(X_3)^{(t)} &= 0 \\ (\omega_1^T)^{\partial\mathsf{M}\cap\Sigma}-\sum_{j=2}^n e_j^*\wedge(\omega_j^{(1)})^{\partial\mathsf{M}\cap\Sigma}-\tau X_3^{\partial\mathsf{M}\cap\Sigma} &= 0 \end{cases} \tag{6.7}$$

This first shows that both components $(\omega_1^T)^{(t)}$ and $(\omega_1^T)^{\partial\mathsf{M}\cap\Sigma}$ (so actually ω_1^T) depend linearly on other components of X, however all other components of X can be chosen arbitrarily. Thus, the space of all X_1-components has dimension $\left(\begin{array}{c} n+1 \\ p \end{array}\right)$; the space of all X_2-components has dimension $\left(\begin{array}{c} n-1 \\ p-2 \end{array}\right) + \left(\begin{array}{c} n-1 \\ p-1 \end{array}\right) + (n-1)\left(\begin{array}{c} n+1 \\ p \end{array}\right)$, the first two terms corresponding to the components $(\omega_1^{(1)})^{(t)} \in \Lambda^{p-2}\mathsf{T}^*(\partial\mathsf{M}\cap\Sigma)$ and $(\omega_1^{(1)})^{\partial\mathsf{M}\cap\Sigma} \in \Lambda^{p-1}\mathsf{T}^*(\partial\mathsf{M}\cap\Sigma)$ respectively (actually both just correspond to $\omega_1^{(1)}$ lying pointwise in $\Lambda^{p-1}\mathsf{T}^*\partial\mathsf{M}$) and the last one to the components $\omega_2,\ldots,\omega_n \in \Lambda^p\mathsf{T}^*\mathsf{M}$; and the space of all X_3-components has dimension $\left(\begin{array}{c} n \\ p \end{array}\right)$ since $X_3 \in \Lambda^p\mathsf{T}^*\partial\mathsf{M}$. All in all, B has rank

$$\left(\begin{array}{c} n+1 \\ p \end{array}\right) + \left(\begin{array}{c} n \\ p-1 \end{array}\right) + (n-1)\left(\begin{array}{c} n+1 \\ p \end{array}\right) + \left(\begin{array}{c} n \\ p \end{array}\right) = (n+1)\left(\begin{array}{c} n+1 \\ p \end{array}\right),$$

which is exactly the rank of $\ker(\sigma_{\mathsf{S}}(\mathtt{n}^b))$.
Now we look at the sign of the quadratic form $X \mapsto \langle\sigma_{\mathsf{S}}(\mathtt{n}^b)X, X\rangle$ on B. Given $X = (X_1, X_2, X_3) \in \mathsf{B}$, we can express as above

$$\begin{array}{rcl} \langle\sigma_{\mathsf{S}}(\mathtt{n}^b)X, X\rangle & = & -2\Re e(\langle \mathtt{n}\lrcorner X_2, X_1\rangle) \\ & = & -2\Re e(\langle\omega_1, X_1\rangle) \\ & = & -2\Re e(\langle\omega_1^{(1)}, X_1^{(1)}\rangle) - 2\Re e(\langle\omega_1^T, X_1^T\rangle) \\ & = & -2\Re e(\langle(\omega_1^{(1)})^{(t)}, (X_1^{(1)})^{(t)}\rangle) \\ & & -2\Re e(\langle(\omega_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}, (X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}\rangle) - 2\Re e(\langle\omega_1^T, X_1^T\rangle). \end{array}$$

But, as we mentioned above, the components $(X_1^{(1)})^{(t)}$ of X_1 and $(\omega_1^{(1)})^{(t)}$ of ω_1 (which is itself a component of X_2) can be chosen arbitrarily. Furthermore, none of the components $(\omega_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}, (X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}, \omega_1^T, X_1^T$ depend on $((X_1^{(1)})^{(t)}, (\omega_1^{(1)})^{(t)})$. Therefore whatever the value of the real number $\Re e(\langle(\omega_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}, (X_1^{(1)})^{\partial\mathsf{M}\cap\Sigma}\rangle) + \Re e(\langle\omega_1^T, X_1^T\rangle)$ is, and provided $p \geq 1$, we can always choose $(X_1^{(1)})^{(t)}$ and $(\omega_1^{(1)})^{(t)}$ such that

$$\langle\sigma_{\mathsf{S}}(\mathtt{n}^b)X, X\rangle < 0.$$

In case $p = 0$, $\omega = \omega^{\partial\mathsf{M}\cap\Sigma}$ (all other components vanish) and hence (6.7) is equivalent to $\omega_1 = \tau X_3$. Therefore $\langle\sigma_{\mathsf{S}}(\mathtt{n}^b)X, X\rangle = -2\Re e(\omega_1\overline{X_1})$ which vanishes if $\tau = 0$ (because then $\omega_1 = 0$) and whose sign can be arbitrary if $\tau \neq 0$, as we have already seen for condition (6.5) which coincides with the Robin boundary condition in that case. This proves the Robin boundary condition to be non-admissible unless $p = 0$ and $\tau = 0$.

If $p \geq 1$, there is actually an eigenvector of $\sigma_{\mathsf{S}}(\mathtt{n}^b)$ associated to the eigenvalue -1 that lies in B: choose $X = (X_1, \mathtt{n}^b \otimes X_1, 0)$ with $X_1 = \mathtt{n}^* \wedge dt \wedge (X_1^{(1)})^{(t)} + (\mathtt{n}^* + dt) \wedge (X_1^T)^{(t)}$, then $X \in \mathsf{B} \cap \ker(\sigma_{\mathsf{S}}(\mathtt{n}) + 1)$ and therefore

$$\langle\sigma_{\mathsf{S}}(\mathtt{n}^b)X, X\rangle = -|X|^2 < 0$$

as soon as $(X_1^{(1)})^{(t)}$ or $(X_1^T)^{(1)}$ is nonzero.

7 Examples of symmetric positive systems

7.1 Klein-Gordon operator

Let ∇ be a covariant derivative on a Hermitian vector bundle V of finite rank k over a globally hyperbolic manifold M with timelike boundary. The Klein-Gordon operator reads as $\mathsf{P} = \nabla^*\nabla +$

m^2, where m is the mass of the scalar field. It is by defintion a normally hyperbolic operator and hence its Cauchy problem can be written as in (6.3). Unlike in Section 6.3, we can rewrite the Cauchy problem for P in terms of the Cauchy problem for the symmetric positive system $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$, namely

$$\begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi_{|\Sigma_{t_0}} = \mathfrak{h} \\ \mathsf{G}_\mathsf{B}\Psi_{|\partial\mathsf{M}} = 0 \end{cases}$$

where G_B is a boundary condition, E is the Hermitian vector bundle $\mathsf{E} := \mathsf{V} \oplus \mathsf{T}^*\mathsf{M} \otimes \mathsf{V}$ and

$$\mathsf{S} = \begin{pmatrix} 0 & -\mathrm{tr} \\ -1 & 0 \end{pmatrix} \nabla + \begin{pmatrix} m^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \qquad \Psi = \begin{pmatrix} u \\ \nabla u \end{pmatrix}, \quad F = \begin{pmatrix} \mathfrak{f} \\ 0 \end{pmatrix}.$$

As in Section 6.3, some restriction on the space of initial data for Ψ has to be imposed in order to obtain a correspondence between the Cauchy problems.

It is not difficult to check that the principal symbol $\sigma_\mathsf{S}(\xi)$ is Hermitian for every $\xi \in \mathsf{T}^*\mathsf{M}$. Moreover, since the trace can be seen as a contraction of tensors, the principal symbol is parallel, *i.e.* $\nabla\sigma_\mathsf{S} = 0$ and we get

$$\Re e(\mathsf{S} + \mathsf{S}^\dagger) = \begin{pmatrix} 2m^2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Hence, S is a nowhere characteristic symmetric positive system. Indeed, since

$$\sigma_\mathsf{S}(\mathtt{n}^\flat) = \begin{pmatrix} 0 & -\mathtt{n}^\flat\lrcorner \\ -\mathtt{n}^\flat\otimes & 0 \end{pmatrix},$$

the pointwise kernel of $\sigma_\mathsf{S}(\mathtt{n}^\flat)$ is given by

$$\ker \sigma_\mathsf{S}(\mathtt{n}^\flat) = \{0\} \oplus \mathtt{n}^\perp \otimes V,$$

where $\mathtt{n}$ denotes again the normal vector to $\partial\mathsf{M}$. Notice that $\sigma_\mathsf{S}(\mathtt{n}^\flat)$ has pointwise two further eigenvalues 1 and -1 both with multiplicity k.

The net advantage of this reduction is that the Robin boundary conditions for P can be rewritten as an admissible boundary condition for S. Note that, if $P = \mathsf{D}^2$ is the squared Dirac operator on M assumed to be spin, then the Schrödinger-Lichnerowicz formula states that $P = \nabla^*\nabla + \frac{\mathrm{Scal}}{4}$, where Scal is the scalar curvature of (M, g). If Scal is bounded below by a positive constant on M, then by analogous arguments as those described above P can be turned into a first-order symmetric positive system and therefore the analysis we have developed for that category of operators can also be applied. This is particularly interesting when looking at certain boundary conditions.

Examples of admissible boundary conditions

Robin boundary condition. The Robin boundary conditions for the Klein-Gordon operator reads as

$$a\nabla_\mathtt{n} u - bu = 0$$

for some real constant parameters a, b. In that case, the bundle B coincides with the kernel of the pointwise projection

$$G_\mathsf{B} := \begin{pmatrix} -b & a\mathtt{n}\lrcorner \\ 0 & 0 \end{pmatrix}$$

and it has rank k has required. For any $\Psi = (\Psi_0, \Psi_1) \in \ker G_\mathsf{B}$ it holds $a\mathtt{n}\lrcorner\Psi_1 = b\Psi_0$. If $a, b \geq 0$ or $a, b \leq 0$, we get

$$\prec \sigma_\mathsf{S}(\mathtt{n}^\flat)\Psi \mid \Psi \succ = 2\Re e \prec \mathtt{n}\lrcorner\Psi_1 \mid \Psi_0 \succ \geq 0,$$

showing that, if $ab \geq 0$, then the Robin boundary conditions are admissible for the forward Cauchy problem. Note that those Robin boundary conditions should not be confused with the ones arising in elliptic systems such as [55, Theorem 6.31], where $ab < 0$ has to be assumed.

7.2 Diffusion-reaction system

As for Section 7.1, let ∇ be a metric connection on an Hermitian vector bundle V of rank k. Consider the diffusion-reaction operator

$$\mathsf{P} := \nabla_{\partial_t} - \operatorname{tr}(\nabla^\Sigma \nabla^\Sigma) + c$$

where c is a zero order term, dubbed *linear reaction term.* The notation here is the same as in Section 6.3. These systems are used to model a wide range of phenomena in physics, biology, social sciences, see e.g. ([25,36,43,44]) and the prototype example of a diffusion-reaction system is the heat equation, where c is set to zero. Note that this is not the usual way to handle such evolution equations but we emphasize those equations fit into our framework.

Let us rewrite the Cauchy problem for the diffusion-reaction operator in terms of the Cauchy problem for the first order symmetric system defined by

$$\mathsf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nabla_{\partial_t} + \begin{pmatrix} 0 & -\operatorname{tr} \\ -1 & 0 \end{pmatrix} \nabla^\Sigma + \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} .$$

This equivalence can be obtained by setting $\Psi = \begin{pmatrix} u \\ \nabla^\Sigma u \end{pmatrix}$. Differently from the case of the Klein-Gordon operator treated in Section 7.1, S is not a symmetric positive system if c not positive definite. However, we can use a similar trick as in Lemma 2.9, to obtain the Property (P) of Definition 2.4. To this end, let us assume c to be uniformely bounded from below and chose a positive λ such that $\lambda - c > 0$. Then the operator $\mathsf{K}_\lambda \colon \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}}) \to \Gamma(\mathsf{E}_{|\mathsf{M}_\mathsf{T}})$ defined by

$$\mathsf{K}_\lambda := \mathsf{S} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

K_λ is clearly a symmetric system. Futhermore, its Cauchy problem is equivalent to the one of S, namely

$$\begin{cases} \mathsf{K}_\lambda \widetilde{\Psi} = \widetilde{\mathfrak{f}} \\ \widetilde{\Psi}|_{\Sigma_0} = \widetilde{\mathfrak{h}} \\ \widetilde{\Psi} \in \mathsf{B} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \mathsf{S}\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h} \\ \Psi \in \mathsf{B}, \end{cases}$$

where $\widetilde{\mathfrak{f}} = e^{-\lambda t}\mathfrak{f}$, $\widetilde{\mathfrak{h}} = \mathfrak{h}$ and $\widetilde{\Psi} = e^{-\lambda t}\Psi$. Indeed, we have, for every $\phi \in \Gamma(E)$ and for every $t \in \mathbb{R}$,

$$\begin{aligned} \mathsf{K}_\lambda(e^{-\lambda t}\phi) &= \left(\mathsf{S} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)(e^{-\lambda t}\phi) \\ &= -\lambda e^{-\lambda t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \phi + e^{-\lambda t}\left(\mathsf{S} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\phi \\ &= e^{-\lambda t}\mathsf{S}\phi, \end{aligned}$$

Since $\lambda - c > 0$ by assumption and the principal symbol is parallel, then a straighforward computations shows that K_λ is a positive symmetric system. Of course, a restriction on the class of initial data for S, and consequently for K_λ has to be imposed to get an equivalence between the Cauchy problem for S and the one for P.

Examples of admissible boundary conditions

Robin boundary condition. The Robin boundary conditions for the diffusion-reaction system reads as

$$a\nabla_{\mathtt{n}} u - bu = 0$$

for some real parameters a, b. In that case, the bundle B coincides with the kernel of the pointwise projection

$$G_{\mathsf{B}} := \begin{pmatrix} -b & a\mathtt{n}\lrcorner \\ 0 & 0 \end{pmatrix}$$

and it has rank k has required. For any $\Psi = (\Psi_0, \Psi_1) \in \ker G_{\mathsf{B}}$ it holds $a\mathtt{n}\lrcorner\Psi_1 = b\Psi_0$. As in Section 7.1, if $ab \geq 0$, we get

$$\prec \sigma_{\mathsf{S}}(\mathtt{n}^\flat)\Psi \mid \Psi \succ = 2\Re e \prec \mathtt{n}\lrcorner\Psi_1 \mid \Psi_0 \succ \geq 0\,,$$

showing that the Robin boundary condition are admissible for the forward Cauchy problem under that assumption.

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