Analysis on Kähler and Lorentzian manifolds

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Chapter 1

Summary

1.1 Introduction

Though not obvious at first sight, this habilitation thesis deals with the interplay between *analysis* on and *geometry* of smooth manifolds. More precisely, we address the following questions:

1. What geometric properties can be extracted from particular differential operators on a given manifold?

2. How to use analysis to find “good” metrics on a given manifold?

3. How to “encode” manifolds carrying solutions of geometric partial differential equations in a physically pertinent way?

Analysis of partial differential equations and geometry of manifolds have each enjoyed a long and rich evolution for the last hundred and fifty years. However, it is only about fifty years ago that both disciplins started to interact in a systematic and efficient way. One of the oldest and most famous successes of this collaboration is probably the *Yamabe problem*, concerned with the *a priori* purely geometric question of finding metrics with constant scalar curvature in any conformal class of Riemannian metrics on compact manifolds. Along with minimal surface theory, the Yamabe problem gave birth to a new field of research nowadays called *geometric analysis*, which has proven extremely fruitful over the years with the resolution of the Calabi and the Poincaré conjectures, just to name a few.

A mathematical domain where the meeting of both analysis and geometry becomes particularly interesting is *spectral geometry*. Originally developed to study the spectrum of the scalar Laplace operator on Riemannian manifolds, it has exhibited many fine and unsuspected features in Riemannian geometry. On the one hand, the geometry of the manifold influences its spectrum: for instance, a compact Riemannian manifold (M^n, g) with Ricci curvature satisfying $\mathrm{Ric} \geq (n-1)k \cdot g$ for a positive $k \in \mathbb{R}$ has a gap of width at least nk between 0 and the first positive Laplace eigenvalue [43]. On the other hand, the spectrum tells a lot on the geometry of the underlying manifold: for instance, if the above gap *coincides* with nk (for $k > 0$), then actually (M^n, g) is a round sphere [43, 52].

More recently, another differential operator has attracted the attention of *a priori* unrelated communities: the *Dirac operator*. Introduced in the thirties by physicist Paul Dirac and long ignored by the mathematicians (with the notable exception of André Lichnerowicz, see e.g. [44]), this first order operator made a spectacular appearance on different stages of mathematics and physics from the end of the seventies on, with (among others) Gromov and Lawson's index-theoretical obstructions to metrics with positive scalar curvature or Edward Witten's proof of the positive mass theorem. In parallel, the spectral theory of the Dirac operator underwent a drastic evolution with e.g. the first sharp lower bound for the Dirac eigenvalues established by Thomas Friedrich [23] in terms of scalar curvature (see Section 1.3.1 below). One of the major gains from the spectral theory of the Dirac operator when compared to the scalar Laplacian is that many more geometries can be characterised with Dirac eigenvalues than with Laplace ones: the equality case in Thomas Friedrich's estimate is equivalent to the existence of so-called *real Killing spinors*, which are in turn characterised purely in terms of *holonomy* [59, 7]. Incidentally, real Killing spinors in dimension 6 turn out to be an essential ingredient in string theory.

Roughly ten years ago, spectral theory of the Dirac operator received a new powerful input with the investigation of geometry of *submanifolds*. For there is surprisingly a lot to read off the Dirac spectrum of submanifolds. As an example, the study of the limiting-case of a certain lower bound for Dirac eigenvalues of manifolds with boundary leads to a straightforward proof [37] of the (*a priori* unrelated) Alexandroff's theorem, which states that the only embedded compact hypersurfaces with constant mean curvature in Euclidean space are the round spheres. In another context, new index-theoretical obstructions to the existence of Lagrangian embeddings have been discovered [36].

It is along this line that we answer the first question at the beginning. More precisely, we investigate finer interactions between the spectrum of various Dirac-type operators and the geometry of the underlying manifold. This is the object of Sections 1.2 and 1.3 below, where we consider manifolds immersed in real or complex spaceforms. Section 1.4, which deals with a purely geometric issue, focusses on a partial differential equation arising in Dirac eigenvalue estimates on Kähler manifolds.

The framework for our second question deals with the search for "beautiful" metrics on pseudo-Riemannian manifolds. In Lorentzian signature, there has been a lot of effort in understanding vacuum *Einstein's equations*, which ask for the existence of a *Ricci-flat* metric on spacetime built out of initial data along a spacelike hypersurface. In Section 1.5, we weaken the Einstein condition and ask for constant scalar curvature in an arbitrary conformal class: this is the afore-mentioned Yamabe problem, but for Lorentzian metrics. We start by investigating the existence of such metrics on standard static spacetimes with the use of elementary analytical methods. We emphasize that there is still a lot to do to fully understand the general case and that probably more geometry is needed to go further on.

Our third and last question deals with different physical interpretations of solutions to geometric partial differential equations. It is not directly related to the first two ones, even if it is based on the existence and uniqueness of those solutions, which both deeply combine analysis and geometry. The fundamental concept developed in Section 1.6 is that of (locally covariant) *quantum field theory*, which is also a quickly developing topic.

The presentation is organized as follows. The different pieces of work are summarised in the sections below, whereas their original (and possibly published) version is contained in the next chapters.

Before turning to the core of the thesis, we introduce a bit of notations and concepts used throughout this chapter. The first central concept is that of Dirac-type operator. Loosely speaking, a Dirac-type operator on a pseudo-Riemannian manifold is a first order linear differential operator acting on the sections of some vector bundle and whose principal symbol satisfies the so-called *Clifford relations*:

$$X \cdot Y \cdot + Y \cdot X \cdot = -2g(X,Y),$$

for all tangent (co)vectors X,Y and where g is the metric on the underlying manifold. In the particular case where the manifold is *spin*, which is an orientability condition of second order, one can define the concept of *twisted* Dirac operator in a more precise way as follows. Given a spin structure on M, which is a non-trivial two-fold covering of the oriented frame bundle of the manifold, there is a Hermitian complex vector bundle $\Sigma M \to M$, called the *spinor bundle*, on which the tangent bundle of the manifold acts by *Clifford multiplication*, meaning that there exists a vector bundle homomorphism $T^*M \otimes \Sigma M \to \Sigma M$, $X^\flat \otimes \varphi \to X \cdot \varphi$, with $X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X,Y)\varphi$. This vector bundle carries a natural metric connection – often called the spinorial Levi-Civita connection – preserving Clifford multiplication. Given any Riemannian or Hermitian vector bundle with metric connection $E \to M$, the tensor product $\Sigma M \otimes E \to M$ can be formed which carries a natural Hermitian metric and metric connection denoted by $\nabla^{\Sigma M \otimes E}$. The Dirac operator D_M^E of M twisted by E (in short twisted Dirac operator) is then defined as the composition of Clifford multiplication and connection: given any section φ of $\Sigma M \otimes E$, we have

$$D_M^E \varphi := i^q \sum_{j=1}^{n} (\varepsilon_j e_j \cdot \otimes \mathrm{Id}) \nabla_{e_j}^{\Sigma M \otimes E} \varphi,$$

where $(e_j)_{1 \leq j \leq n}$ is a local orthonormal basis of TM and q is the index of the metric g. The operator D_M^E is a formally self-adjoint linear differential operator of first order on the Hermitian vector bundle $\Sigma M \otimes E \to M$. In the particular case where $E \to M$ is a trivial line bundle and the connection is the standard flat one, one obtains the so-called *spin* (also called fundamental) Dirac operator D_M, acting on sections of ΣM.

In the setting of Lorentzian manifolds, we use the following (standard) notations for the following notions. Given a Lorentzian metric g on a manifold M, we call a tangent (co)vector X *timelike* if $g(X,X) < 0$, *lightlike* if $g(X,X) = 0$ and $X \neq 0$, and *spacelike* otherwise. A *causal* vector is a vector which is either time- or lightlike. All those concepts carry out to curves and vector fields. A *time orientation* on M – if it exists – is fixed by a (smooth) timelike vector field X on M: at each point x of M, the causal future (resp. past) in the tangent space T_xM is the set of causal vectors lying in the same (resp. opposite) connected component of the set of causal vectors as X_x. Time-oriented Lorentzian manifolds are often referred to as *spacetimes*. On a spacetime, the causal (resp. chronological) future of a subset $A \subset M$ can be defined as the subset $J_+^M(A)$ (resp. $I_+^M(A)$) of all points in M that can be joined by a future-directed causal (resp. timelike) curve from (some point in) A. Correspondingly, there is the notion of causal (resp.

chronological) past $J_-^M(A)$ (resp. $I_-^M(A)$) of A in M. One should pay attention that those subsets not only depend on A but also on M.

1.2 Dirac operators on homogeneous spaces

The results presented in this section are based on the article [29], see Chapter 2 below.

1.2.1 Motivation

This work deals with *sharp* extrinsic Dirac eigenvalue estimates for hypersurfaces in real spaceforms. Namely, let $M^n \overset{\iota}{\hookrightarrow} \widetilde{M}^{n+1}(\kappa)$ be an isometric immersion of an oriented closed Riemannian manifold (M^n, g) into the simply-connected spaceform $\widetilde{M}^{n+1}(\kappa)$ of constant sectional curvature $\kappa \in \mathbb{R}$. Then the existence of a global smooth unit normal ν on M^n compatible with orientations of M^n and $\widetilde{M}^{n+1}(\kappa)$ makes it possible to restrict the (canonical) spin structure of $\widetilde{M}^{n+1}(\kappa)$ to a spin structure on M^n. If $H := -\frac{1}{n}\mathrm{tr}(\widetilde{\nabla}\nu)$ is the mean curvature of the immersion ι w.r.t. ν on M^n, then the smallest eigenvalue $\lambda_1(D_M^2)$ of the squared Dirac operator D_M^2 of (M^n, g) (for the restricted spin structure) is known to satisfy

$$\lambda_1(D_M^2) \leq \begin{cases} \frac{n^2}{4\mathrm{Vol}(M,g)} \int_M (H^2+\kappa) dv_g & \text{if } \kappa \geq 0 \quad [9, \text{Thm. } 4.1] \\ \\ \frac{n^2}{4}\left(\max\limits_M (H^2) + \kappa\right) & \text{if } \kappa < 0 \quad [25, \text{Thm. } 2.2]\,\&\,[26, \text{Thm. } 1]. \end{cases}$$

In other words, the smallest eigenvalue of the intrinsically defined Dirac operator can be *a priori* controlled by a relatively weak extrinsic geometric invariant, namely the mean curvature of the immersion. Those estimates rely on a clever application of the min-max principle. Surprisingly enough, determining for which immersed hypersurface the inequality above is an equality – this is the *limiting-case* of the inequality – turns out to be a difficult question. It is elementary to show that, if the equality holds, then the mean curvature H must be constant on M. A direct computation shows that round hyperspheres fulfil the equality whatever κ is. For $\kappa = 0$, it was shown only recently using a variational formula for Dirac eigenvalues that, apart from round hyperspheres, no hypersurface can satisfy the limiting-case [38, Thm. 1]. In case $\kappa = 1$, there is a one-parameter-family of so-called generalized Clifford tori in the sphere also enjoying this property [27]. But besides those two families, no other example was known. This led to the following question:

Question 1: *How large is the family of (constant mean curvature) hypersurfaces M^n in the round sphere $\mathbb{S}^{n+1}$ for which $\lambda_1(D_M^2) = \frac{n^2}{4}(H^2+1)$ holds?*

It is reasonable to start looking at *homogeneous* hypersurfaces, whose Dirac spectrum can be hoped to be computed using representation theoretical tools. In [29], we considered the space $M^3 := {}^{\mathrm{SU}_2}/_{\mathrm{Q}_8}$, which is the simplest homogeneous hypersurface in the sphere which is neither a hypersphere nor a generalized Clifford torus, see e.g. [A6] for the classification of homogeneous hypersurfaces in spaceforms. Here $\mathrm{Q}_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ denotes the finite group of quaternions. The space M^3 admits a three-parameter-family of homogeneous Riemannian metrics, a one-parameter-subfamily of which arises as induced from (a one-parameter-family of) embeddings into $\mathbb{S}^4$. Here one should pay attention to the fact that M^3 carries no less than four different spin structures – each corresponding to a group homomorphism $\mathrm{Q}_8 \to \{\pm 1\}$ – but that only one of those is induced from the above embeddings into $\mathbb{S}^4$, see [29, Sec. 1 & 2] for details.

1.2.2 Main results

Considering any of the homogeneous metrics and spin structures mentioned above, Frobenius reciprocity allows to split (under the G-action) the Hilbert space of L^2-spinor fields on M^3 into a Hilbert direct sum of finite-dimensional subspaces, each of which is preserved by the Dirac operator, see e.g. [6, Thm. 2 & Prop. 1]. Furthermore, a formula involving solely representation-theoretical data gives the explicit form the endomorphism induced by the Dirac operator on each of those subspaces. Therefore the determination of the Dirac spectrum reduces to that of the eigenvalues of *matrices*.

The finite-dimensional Dirac operators for arbitrary homogeneous metrics and spin structures are computed in [29, Thm. 0.1] and we shall not reproduce this result here since the general form of the matrices we obtain is particularly involved. Let us mention however that, even if we can choose adequate bases such as to obtain upper triangular matrices, we cannot compute their spectrum explicitly in general. Still there is a two-parameter-subfamily of metrics – precisely those induced by the so-called *Berger metrics* on $\mathbb{S}^3 \cong \mathrm{SU}_2$, up to scaling by a positive constant – where the eigenvalues can be explicitly expressed. This is the first main result we present here:

Theorem 1.2.1 ([29, Cor. 0.2]) *The compact* 3*-dimensional manifold* $M^3 = {}^{\mathrm{SU}_2}/_{\mathrm{Q}_8}$ *carries a two-parameter family of homogeneous Riemannian metrics (indexed by* $a_1, a_2 \in \mathbb{R}^\times$*) such that its Dirac spectrum can be computed for any of its four spin structures (indexed by* ε_j*,* $j = 0, \ldots, 3$*). More precisely, the spectrum of the operator* $D_M + \frac{2a_1^2+a_2^2}{2a_1}\mathrm{Id}$ *for the metric induced by* a_1, a_2 *and the spin structure given by* ε_j *consists of the following family of eigenvalues:*

0. *for* $j = 0$*,*

$$\begin{aligned}
&\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \\
&\qquad\qquad | \, k \in \{0, \ldots, \frac{n-5}{2}\} \text{ even}, a_1 + (n+1)a_2 \Big\} \\
\bigcup \quad &\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \\
&\qquad\qquad | \, k \in \{1, \ldots, \frac{n-5}{2}\} \text{ odd}, a_1 - (n+1)a_2, -na_1 \Big\},
\end{aligned}$$

each eigenvalue having multiplicity $n+1$ *for the corresponding n.*

1. *for $j=1$,*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$
$$\,|\, k\in\{0,\ldots,\frac{n-5}{2}\}\text{ even}, a_1-(n+1)a_2\Big\}$$
$$\bigcup\quad\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$
$$\,|\, k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, a_1+(n+1)a_2, -na_1\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

2. *for $j=2$ and $j=3$,*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$
$$\,|\, k\in\{1,\ldots,\frac{n-3}{2}\}\text{ odd}, -na_1\Big\}$$
$$\bigcup\quad\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$
$$\,|\, k\in\{0,\ldots,\frac{n-3}{2}\}\text{ even}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

Each upper bound (e.g. $\frac{n-5}{2}$) for the possible values of k in Theorem 1.2.1 must be understood as follows: if for a given n it is negative then the corresponding eigenvalues do not appear. For example if M carries the ε_0-spin structure and $n=1$ then $D_n+\frac{2a_1^2+a_2^2}{2a_1}\mathrm{Id}$ has only one eigenvalue, namely a_1+2a_2 (with multiplicity 2). Similarly, if $j=2,3$ and $n=1$, then only $-a_1$ appears with multiplicity 2.

In the particular case of *round* metrics on M^3, which corresponds to $a_1=a_2$, our results coincide with those already proven by Christian Bär [8, Thm. 2] using another method.

Coming back to Question 1, it turns out that the family of Berger metrics on M^3 does not coincide with that of metrics induced from $\mathbb{S}^4$ but is transverse to it. Both families intersect in exactly six points which correspond to the *minimal* embeddings among the family. For those embeddings, the space M^3 satisfies the limiting-case from Question 1:

Theorem 1.2.2 ([29, Cor. 0.3]) *Let $M^3={}^{\mathrm{SU}_2}/_{\mathrm{Q}_8}$ carry a homogeneous Riemannian metric induced by a minimal embedding into $\mathbb{S}^4$. Then, for the induced spin structure (which is the ε_0-one), we have $\lambda_1(D_M^2)=\frac{9}{4}$.*

1.2.3 Perspectives

Theorem 1.2.2 tends to indicate that the family of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ for which $\lambda_1(D_M^2) = \frac{n^2}{4}(H^2+1)$ holds could contain all homogeneous hypersurfaces. This would show an interesting analogy with the scalar Laplace operator, whose first non-zero eigenvalue coincides with the dimension for minimally embedded homogeneous hypersurfaces in the round sphere [41, 49]. Further along this line, one could even expect $\lambda_1(D_M^2) = \frac{n^2}{4}(H^2+1)$ to hold for *isoparametric* hypersurfaces, i.e., with constant principal curvatures, in $\mathbb{S}^{n+1}$, the analogous result holding true for the scalar Laplace operator in the minimal setting [48, 56, 57].

1.3 Dirac operators on Kähler submanifolds

The results presented in this section are based on the article [30], see Chapter 3 below.

1.3.1 Motivation

This project also deals with sharp eigenvalue estimates, but in the very different context of submanifolds of Kähler manifolds. It takes its roots in the following very general question: how sensitive is the Dirac spectrum to the presence of supplementary geometric structures on the underlying manifold? It has been known for a long time (see e.g. [35]) that the existence of a non-zero – and non-trivial – parallel form on a given closed Riemannian spin manifold forbides that of non-zero so-called *real Killing spinors*, which characterise the limiting-case of Thomas Friedrich's inequality [23]

$$\lambda_1(D_M^2) \geq \frac{n}{4(n-1)} \min_M(S)$$

for the first eigenvalue of the squared Dirac operator D_M^2 in terms of the scalar curvature S of the manifold (M^n, g). In fact, the Dirac eigenvalues of manifolds with parallel forms lie well above Friedrich's lower bound, see [1, Thm. 1.1] for the simplest situation where the form is of degree 1.

In the case of Kähler (spin) manifolds, which admit a parallel 2-form, the existence of a sharp lower bound for the Dirac spectrum has been proven by Klaus-Dieter Kirchberg in [39]. Interestingly enough, the case where his inequality is an equality can also be characterised via spinor fields on the manifold, that are called *real Kählerian Killing spinors* in case the complex dimension is odd, see Section 1.4.1 below for a formal definition. Those complete Kähler spin manifolds with non-zero real Kählerian Killing spinors have been classified by Andrei Moroianu [45, Thm. A], building on earlier partial achievements by (among others) Klaus-Dieter Kirchberg and Oussama Hijazi, see [45] for references. They can all be described as twistor spaces of quaternionic-Kähler manifolds with positive scalar curvature. The complex projective space $\mathbb{C}\mathrm{P}^n$ of odd complex dimension is such an example (and is the only example in complex dimension $n \equiv 1\ (4)$).

How does the *Kähler* structure of a given spin manifold now influence the spectrum of geometric operators on its *submanifolds*? This question echoes those of Section 1.2.1, with the notable difference that up to now little has been done in the Kähler setting. Prior to [30], geometric bounds for the spectrum of Lagrangian submanifolds had been determined in [28]. For Kähler (i.e., complex) submanifolds, Georges Habib and I got interested into the following:

Question 2: *How can the spectrum of Dirac operators of Kähler submanifolds of* $\mathbb{C}\mathrm{P}^n$ *be controlled in terms of intrinsic or extrinsic geometric data?*

There is a subtle point about the concept of Dirac operator in this context, since for a given submanifold of codimension at least 2 the Dirac-type operator which lies closest to the extrinsic data provided by the immersion lives on the sections of a *twisted* spinor bundle. More precisely, the spinor bundle of the submanifold must be twisted by the spinor bundle of its *normal bundle*. In particular, the twisted Dirac operator does not in general – unless that normal bundle is trivial and flat – coincide with the intrinsic Dirac operator of the submanifold.

1.3.2 Main results

We focus on closed Kähler submanifolds M^{2d} of complex projective spaces $\mathbb{C}\mathrm{P}^n$ of odd complex dimension – which are exactly those that are spin, see e.g. [42, Ex. II.2.4]. We must assume that M itself is spin since there is no naturally induced spin structure on M – as a matter of fact, a complex submanifold must not be spin (think of e.g. $\mathbb{C}\mathrm{P}^d \hookrightarrow \mathbb{C}\mathrm{P}^n$ with d even). Fixing the standard complex structure and Fubini-Study metric of constant holomorphic sectional curvature 4 on $\mathbb{C}\mathrm{P}^n$, we denote by g and J the induced Riemannian and complex structure on M respectively. In view of Question 2, we ask for lower and upper bounds for the smallest eigenvalue of the Dirac operator $D_M^{\Sigma N}$ of M twisted with the spinor bundle of the normal bundle $NM = T^\perp M \to M$ of the immersion. To see that this operator is well-defined, let us mention that the normal bundle carries a spin structure induced by those of M and $\mathbb{C}\mathrm{P}^n$ and hence has an associated spinor bundle.

Upper bounds *a priori* for the smallest eigenvalues of $D_M^{\Sigma N}$ can be derived in terms of extrinsic data, i.e., those built out of the second fundamental form of the immersion. As it turns out, they only depend on the *dimension* of the submanifold: there are $2\begin{pmatrix} n \\ \frac{n+1}{2} \end{pmatrix}$ eigenvalues λ of $(D_M^{\Sigma N})^2$ satisfying [25, Thm. 4.2] (reproduced in [30, Thm. 2.2])

$$\lambda \leq \begin{cases} (d+1)^2 & \text{if } d \text{ is odd} \\ \\ d(d+2) & \text{if } d \text{ is even.} \end{cases} \tag{1.1}$$

The proof of (1.1) relies on the use of real Kählerian Killing spinors as test spinors in the min-max principle. The application of that principle furthermore provides a technical necessary condition for (1.1) to be an equality for the smallest eigenvalue of $(D_M^{\Sigma N})^2$, however that condition is not sufficient to deduce clear features about the geometry of M and its immersion, see [30, Thm. 2.2] and discussion below.

To get a better control of the eigenvalues of $(D_M^{\Sigma N})^2$, one can compare the upper bound of (1.1) with *a priori* lower bounds in terms of curvature quantities, considering $(D_M^{\Sigma N})^2$ as an arbitrary twisted Dirac operator on a compact Kähler manifold:

Theorem 1.3.1 ([30, Cor. 3.2]) *Let (M^{2d}, g, J) be a closed Kähler spin manifold and $E \to M$ be a Hermitian vector bundle with metric connection. Denote by D_M^E :*

$\Gamma(M, \Sigma M \otimes E) \circlearrowleft$ *the associated twisted Dirac operator. Then for any eigenvalue* λ *of* $(D_M^E)^2$,

$$\lambda \geq \begin{cases} \frac{d+1}{4d}(\min_M(S) + \kappa_1) & \textit{if } d \textit{ is odd} \\ \\ \frac{d}{4(d-1)}(\min_M(S) + \kappa_1) & \textit{if } d \textit{ is even}, \end{cases}$$

where S *is the scalar curvature of the manifold,* κ_1 *denotes the smallest eigenvalue of the (pointwise) self-adjoint operator* $\psi \mapsto 2\sum_{i,j=1}^{2d}(e_i \cdot e_j \cdot \mathrm{Id} \otimes R^E_{e_i,e_j})\psi$ *and* R^E *is the curvature operator of the connection on* $E \to M$.

Those estimates generalise those proven in [39] since we allow arbitrary twisting bundles. Moreover, it can also be seen as the Kählerian analogue to the corresponding Riemannian estimate, see e.g. [31, Prop. 4.1].

Unfortunately, the combination of (1.1) and Theorem 1.3.1 does not provide any characterisation of the limiting-case in (1.1): even in the simplest case where $M = \mathbb{C}\mathrm{P}^d$ (with d odd) is standardly embedded into $\mathbb{C}\mathrm{P}^n$, the presence of normal curvature does not allow to conclude that $\lambda_1((D_M^{\Sigma N})^2) = (d+1)^2$. And in fact an explicit computation of the spectrum of the twisted Dirac operator $D_M^{\Sigma N}$ on $\mathbb{C}\mathrm{P}^d$ (first carried out in [15] and independently in [22]) shows that, according to the values of d and n, equality in (1.1) may or may not hold. For instance, if $d = 1$, then $\lambda_1((D_M^{\Sigma N})^2) = (d+1)^2 = 4$ for $n = 3, 5, 7$, however $\lambda_1((D_M^{\Sigma N})^2) < 4$ for all odd $n \geq 9$, see [30, Prop. 4.8].

1.3.3 Perspectives

There is still a lot to be done to understand which kind of geometric information is contained in the equality case of (3.7) and, more generally, what geometry is contained in the restriction of particular spinor fields from Kähler manifolds onto their submanifolds. Let us mention that, after [30] was published, an approach within the framework of so-called spin^c structures, which are better fitted to the setting of Kähler geometry, has been tackled in [33].

1.4 Imaginary Kählerian Killing spinors

The results presented in this section are based on the article [32], see Chapter 4 below.

1.4.1 Motivation

This project deals with a first-order linear partial differential equation originating in a Dirac eigenvalue estimate on Kähler manifolds. As mentioned in Section 1.3.1 above, there exists a sharp lower bound for the eigenvalues of the Dirac operator on compact Kähler spin manifolds in terms of their scalar curvature due to Klaus-Dieter Kirchberg [39]. In case the complex dimension of the manifold is odd, the equality case of that estimate is characterised by the existence of non-zero so-called *real Kählerian Killing spinors*. Let us briefly give the precise definition. Let M^{2n} be a Kähler manifold with metric g and compatible complex structure J. If M is spin, then its spinor bundle $\Sigma M \to M$ carries a Clifford multiplication $(X, \varphi) \mapsto X \cdot \varphi$ from the tangent bundle as well as a compatible metric connection ∇ which can be seen as its Levi-Civita spinorial connection. Given a constant $\alpha \in \mathbb{C}^\times$, an *$\alpha$-Kählerian Killing spinor* on M is a pair (ψ,ϕ) of sections of $\Sigma M \to M$ satisfying the following coupled system of linear equations for all $X \in TM$:

$$\begin{cases} \nabla_X \psi &= -\frac{\alpha}{2}(X + iJ(X)) \cdot \phi \\ \nabla_X \phi &= -\frac{\alpha}{2}(X - iJ(X)) \cdot \psi. \end{cases} \tag{1.2}$$

For α real (resp. purely imaginary) the pair (ψ,ϕ) is called real (resp. imaginary) Kählerian Killing spinor.

It is important to note that the system (1.2) is actually *overdetermined*: the existence of non-zero α-Kählerian Killing spinors imposes strong restrictions on the geometry of the underlying manifold. Among others, they have to be Einstein with scalar curvature $4n(n+1)\alpha^2$ and must have odd complex dimension n. In case $\alpha \in \mathbb{R}^\times$, Andrei Moroianu completely classified [45] those complete Kähler spin manifolds with non-zero real Kählerian Killing spinors. They are all compact and can be described as twistor spaces of quaternionic-Kähler manifolds with positive scalar curvature. This includes all complex projective spaces, which are furthermore the only such manifolds in complex dimension $n \equiv 1\ (4)$.

In case $\alpha \in i\mathbb{R}^\times$, complex hyperbolic spaces are known to carry non-zero imaginary Kählerian Killing spinors, see e.g. [40]. In complex dimension 3, there is no further example of complete Kähler spin manifold with non-zero imaginary Kählerian Killing spinors since the sectional holomorphic curvature can be shown to be constant negative [40, Thm. 16]. Nevertheless, the classification in higher complex dimensions remained open. This made Uwe Semmelmann and me address the following:

Question 3: *Can Kähler spin manifolds with non-zero imaginary Kählerian Killing spinors be classified?*

The angle of attack we took consists in picking a non-zero imaginary Kählerian Killing spinor (ψ,ϕ) and looking at the *level sets* of the (positive) smooth function $|\psi|^2 + |\phi|^2$ on M. The reason for this is that, because of (1.2), this function is expected to have "few" critical values, fact which should allow a "simple" description of the underlying M. In the Riemannian setting (concerned with *imaginary Killing spinors*, that we are not going to describe in detail here), this approach showed successful and led Helga Baum to a full classification [14].

1.4.2 Main results

We focus on the system (1.2) with $\alpha \in i\mathbb{R}^\times$ on an arbitrary Kähler spin manifold M^{2n} of odd complex dimension n. Up to scaling the metric and changing α into $-\alpha$, we may assume $\alpha = i$. Fixing a non-zero i-Kählerian Killing spinor (ψ,ϕ), a computation of the second derivatives of the function $f := |\psi|^2 + |\phi|^2$ already sheds light on the structure of M: the function f has at most one critical value, which is then a minimum and in that case the set of minima is a connected totally geodesic Kähler submanifold of M [32, Prop. 2.3].

In case all values of f are regular, the manifold M can be split into the product of a level hypersurface of f with the real line, at least when the metric is complete. To see what kind of metric and complex structure we could face on such a product, we considered the family of so-called *doubly warped products*, which are the Kähler analogues of Riemannian warped products. First introduced in his diploma thesis [3] by Patrick Baier to compute the Dirac spectrum of the complex hyperbolic space, they can be described as follows. Let $(\check{M}^{2n-1},\check{g},\check{\xi})$ be a so-called *Sasaki manifold*; Sasaki manifolds can be characterised as those Riemannian manifolds whose metric cone is Kähler. Sasaki manifolds carry a *transverse Kähler structure* on the distribution which is the pointwise orthogonal complement to the Reeb vector field $\check{\xi}$. Let $I\subset\mathbb{R}$ be an open interval and $\rho,\sigma: I\to\mathbb{R}_+^\times$ be smooth positive functions on I. Then the *doubly warped product* of M by I with warping functions (ρ,σ) is the manifold

$$(M^{2n},g) := (\check{M}\times I, \rho(t)^2(\sigma(t)^2\check{g}_{\check{\xi}}\oplus\check{g}_{\check{\xi}^\perp})\oplus dt^2),$$

where $\check{g}_{\check{\xi}}$ and $\check{g}_{\check{\xi}^\perp}$ denote the restrictions of the metric $\check{g}$ onto the subbundles $\mathbb{R}\cdot\check{\xi}$ and $\check{\xi}^\perp$ of $T\check{M}$ respectively. Note that M^{2n} is in general not Kähler. However, if $\frac{\rho'}{\sigma}$ is constant on I, then (M^{2n},g) can be endowed with a Kähler structure that more or less extends the transverse Kähler structure of $(\check{M}^{2n-1},\check{g},\check{\xi})$, see [32, Lemma 3.4].

The equations (1.2) can be completely translated onto arbitrary spin doubly warped products [32, Lemma 3.8]. As could be expected, the existence of a non-zero solution to (1.2) restricts strongly the possibilities for the function ρ and the Sasaki structure on $(\check{M}^{2n-1},\check{g},\check{\xi})$:

Theorem 1.4.1 ([32, Thm. 3.9]) *Let $(M^{2n},g) := (\check{M}\times I, \rho(t)^2(\rho'(t)^2\check{g}_{\check{\xi}}\oplus\check{g}_{\check{\xi}^\perp})\oplus dt^2)$ be the Kähler doubly warped product of a Sasaki manifold $(\check{M}^{2n-1},\check{g},\check{\xi})$ with an open interval and warping functions (ρ,ρ') (the relation $\rho'=\sigma$ is fulfilled because M^{2n} is Kähler). Assume M^{2n} carries a non-zero i-Kählerian Killing spinor. Then we have the following:*

i) Up to standard normalisations, the function ρ is one of the functions exp, cosh *or* sinh *and $(\check{M}^{2n-1},\check{g},\check{\xi})$ must admit* transversally parallel spinors.

ii) Conversely, any non-zero transversally parallel spinor lying in some particular eigenspace of the Clifford action of the transverse Kähler form on $(\check{M}^{2n-1},\check{g},\check{\xi})$ induces a non-zero i-Kählerian Killing spinor on the doubly warped product of $\check{M}$ by $\mathbb{R}$ with warping functions $\rho=\sigma=e^t$.

Even if there is presently no classification of Sasaki manifolds with transversally parallel spinors (though there may be some hope from the recently appeared paper [34] where foliations with particular transverse holonomies are classified), Theorem 1.4.1 allows the construction of infinitely many examples of Kähler spin manifolds with imaginary Kählerian Killing spinors and with non-constant holomorphic sectional curvature. For every simply-connected Hodge hyperkähler manifold of complex dimension $4k$ carries a $\mathbb{U}_1$-bundle whose total space $\check{M}$ is Sasaki and has a transversally parallel spinor lying in the right eigenspace [32, Lemma 3.11]; moreover, the holomorphic sectional curvature of the doubly warped product $(\check{M}\times\mathbb{R}, e^{2t}(e^{2t}\check{g}_{\check{\xi}}\oplus\check{g}_{\check{\xi}^\perp})\oplus dt^2)$ is not constant since hyperkähler manifolds are not flat, see [32, Lemma 3.12]. We also reobtain the complex hyperbolic space as a doubly warped product in three different ways, see [32, Rem. 3.10] and [32, Thm. 3.18].

The question remained open whether all Kähler spin manifolds with Kählerian Killing spinors are doubly warped products or not. Under a supplementary technical assumption, we could show that the doubly warped product structure can be recovered:

Theorem 1.4.2 ([32, Thm. 4.1]) *Let (M^{2n}, g, J) be any complete Kähler spin manifold with a non-zero i-Kählerian Killing spinor (ψ, ϕ). Assume $|\psi| = |\phi|$ as well as the existence of a (real) smooth vector field W and of a non-identically vanishing continuous complex-valued function μ such that $W \cdot \psi = \mu\phi$ on M.*
Then (M^{2n}, g, J) is holomorphically isometric to a doubly warped product of a Sasaki spin manifold $\check{M}$ by $\mathbb{R}$, with warping functions $\rho = \sigma = e^t$, and (ψ, ϕ) comes from a transversally parallel spinor on $\check{M}$.

1.4.3 Perspectives

There is still work to be done to fully understand the structure of Kähler spin manifolds with imaginary Kählerian Killing spinors. Alongside, it would be interesting to be able to characterise large families of Kähler manifolds *à la Obata*, i.e., with the help of functions satisfying some kind of partial differential equation involving their Hessian. Partial results have been obtained for complex spaceforms in [53] and [54].

1.5 The Lorentzian Yamabe problem

The results presented in this section have not been published yet and are based on Chapter 5 below.

1.5.1 Motivation

This project is motivated by the search for "best" pseudo-Riemannian metrics on manifolds. This could be understood in many different ways and we choose here the curvature point of view: we look for those pseudo-Riemannian metrics with "most constant" curvature. The latter still admits various interpretations and we make this a bit more precise by focussing on the weakest curvature invariant, namely *scalar curvature*. It is now well-known that constant scalar curvature metrics exist in any *Riemannian conformal class* on any closed manifold: this is the celebrated *Yamabe problem*, first formulated (and thought to be solved) by Hidehiko Yamabe in 1960 [60] and finally solved in the eighties by Richard Schoen [55] after essential contributions by – among others – Neil Trudinger [58] and Thierry Aubin [2].

By comparison, very little has been done in the pseudo-Riemannian setting. For instance, which function on a given manifold can be the scalar curvature of some pseudo-Riemannian (non-Riemannian) metric? The study of scalar curvature functions for pseudo-Riemannian metrics really started with Marc Nardmann's PhD thesis [50], with the focus on *compact* manifolds, where a clever use of auxiliary Riemannian metric allows to reformulate the problem as an *elliptic* equation, for which standard techniques can be applied.

In the Lorentzian setting, there is a category of manifolds which are better-fitted than compact ones for both physical and analytical purposes. They are called *globally hyperbolic*. They can be roughly thought of as the product M of some arbitrary

Riemannian manifold Σ by an interval (standing for the "time-axis") and with a kind of warped-product metric. The essential feature is that Σ must be a so-called *Cauchy hypersurface* in M, meaning that every "event" is caught once and only once by an observer along Σ; in mathematical terms, every inextendible timelike curve in the Lorentzian manifold must cross Σ exactly once. In particular, globally hyperbolic Lorentzian manifolds have no closed timelike or even causal curves. Many reasonable physical models for our 4-dimensional universe are globally hyperbolic. On the other hand, globally hyperbolic Lorentzian manifolds are best suited for *hyperbolic* equations, for which a Cauchy-problem-ansatz with initial data along the hypersurface Σ allows to discuss *global* existence and uniqueness of solutions. This is by the way the origin of the term *globally hyperbolic*, which has nothing to do with Riemannian hyperbolicity.

The main question we address in this project is the following:

Question 4: *Does there exist a metric with constant scalar curvature in any conformal class of globally hyperbolic metrics?*

We mainly look at globally hyperbolic manifolds with *closed* Cauchy hypersurface, which are for analytical investigations the simplest ones. The equation we have to solve (see (5.1) and (5.2)) is a semilinear wave equation, for which standard techniques from the analysis of so-called symmetric hyperbolic systems show the *local* existence of smooth solutions. Thus the only issue is about *global* existence as well as uniqueness of solutions.

1.5.2 Main results

In dimension 2, the existence of global smooth solutions to *linear* wave equations on globally hyperbolic manifolds already shows that globally hyperbolic surfaces are always *conformally flat*, see Theorem 5.1.4 below. What goes unnoticed in this case and leads to the main concern in higher dimensions is the fact that the *sign* of the solution of (5.1) need not be cared about in dimension 2. In dimension $n \geq 3$, the equation to be solved is the following on the globally hyperbolic manifold (M^n, g):

$$\square\varphi + \frac{n-2}{4(n-1)} S_g \varphi = \frac{n-2}{4(n-1)} S_{\overline{g}} \varphi^{\frac{n+2}{n-2}},$$

where S_g is the scalar curvature of g and $S_{\overline{g}} \in \mathbb{R}$ is the constant scalar curvature associated to the conformal metric $\overline{g} := \varphi^{\frac{4}{n-2}} \cdot g$ we look for. Solving that equation means finding a *smooth positive* function φ on M satisfying it. Fixing a smooth Cauchy hypersurface Σ that is spacelike (i.e., on which g restricts as a Riemannian metric) in M, we must show that the solution of the Cauchy problem associated to that equation and with initial data along Σ extends for all time – or at least find some conditions for which it does.

As mentioned just above, the main difficulty consists in controlling the sign of the solution, because no maximum principle works outside the elliptic or parabolic world. To understand what can happen, we start by considering product Lorentzian manifolds, i.e., products $I \times \Sigma$ of an interval I with a Riemannian manifold Σ, carrying the corresponding Lorentzian product metric. Those Lorentzian manifolds are called *standard* *static* in the literature. In that case, the existence of a solution can be simply tackled by separating variables (see Theorem 5.2.6 below):

Theorem 1.5.1 *Let an $n(\geq 3)$-dimensional Lorentzian manifold (M^n, g) be conformally equivalent to the product $(I \times \Sigma, -dt^2 \oplus g_\Sigma)$, where $I \subset \mathbb{R}$ is an open interval and (Σ^{n-1}, g_Σ) is a closed Riemannian manifold. Let $\mu_1 \in \mathbb{R}$ be the smallest eigenvalue of the linear operator $L_{g_\Sigma} := \Delta_\Sigma + \frac{n-2}{4(n-1)} S_{g_\Sigma}$, where Δ_Σ and S_{g_Σ} denote the scalar Laplace operator and the scalar curvature of (Σ^{n-1}, g_Σ) respectively.*
Then there exists in the conformal class of g a metric with constant scalar curvature μ_1 on M^n.

The proof of Theorem 1.5.1 relies on the fact that the equation can be reduced to a *subcritical* non-linear Yamabe-type equation on the compact Riemannian manifold Σ, in particular it does not involve any study of sign-change since the solutions we obtain do not depend on time. Still one has to pay attention to the fact that the constant conformal scalar curvature we obtain depends on the sign of the eigenvalue μ_1, which itself depends on the geometry of (Σ^{n-1}, g_Σ). To test the limits of the study, one could ask for the stronger *prescription* of the conformal scalar curvature: *given any $S_{\overline{g}} \in \mathbb{R}$*, does there always a conformal metric with that scalar curvature? It turns out that the answer depends on the sign of μ_1. If e.g. $\mu_1 \leq 0$, then there always exists a conformal metric with vanishing scalar curvature on a standard static spacetime; by contrast, if $\mu_1 > 0$, then such a metric only exists for short times, i.e., if the interval I is sufficiently short. In the latter case, we even obtain the optimal length of I for global existence, see Theorem 5.2.9 below. As an application, the de Sitter spacetime has a conformal metric with vanishing scalar curvature if and only if its dimension is $2, 3$ or 4, see Corollary 5.2.10 below.

Unlike the Riemannian setting, where conformal metrics with constant negative scalar curvature are unique up to homothety on closed manifolds, uniqueness of solutions seems never to hold. In some sense, smooth positive solutions to the Lorentzian Yamabe equation look stable: one may perturb, along a given Cauchy hypersurface, the initial conditions of a solution a bit and still obtain a smooth positive solution. Although we cannot make for now any general statement, there are examples for all three cases $S_{\overline{g}} \in \{-1, 0, 1\}$ of globally hyperbolic spacetimes with infinitely many non-homothetic metrics with constant scalar curvature $S_{\overline{g}}$. We refer to Section 5.2.2 for the discussion of uniqueness in the standard static situation.

1.5.3 Perspectives

The general setting of arbitrary globally hyperbolic spacetimes with closed Cauchy hypersurface remains open. As we mention in Section 5.3 below, the issue is not so much about existence of (weak) solutions to the Lorentzian Yamabe equation, which should follow from “standard” techniques for semi-linear wave equations, taking into account that the exponent $\frac{2n}{n-2}$ is *subcritical* for some Sobolev embedding on the Cauchy hypersurface; it is about how to control the *sign* of solutions. It is for the moment unclear which kind of criterion, either analytical or geometric in nature, could help in this respect.

Apart from stronger curvature quantities such as sectional curvature (see e.g. [4]), a natural concept to be discussed around this project is that of “best metric”. For on globally hyperbolic spacetimes the properties of the *time function* inducing the smooth splitting by Cauchy hypersurfaces is crucial in many respects. For example, the existence of a time function with "large" gradient on any globally hyperbolic spacetime makes it isometrically embeddable in some Minkowski space of sufficiently high dimension [47]. One may also try to control the second fundamental form of the Cauchy hypersurfaces, see e.g. [46]. Other important issues deal with the existence at all of codimension one foliations [51] and with foliations by constant scalar or mean curvature Cauchy hypersurfaces, a topic which is still in progress, see e.g. [24, 5].

1.6 Quantization on Lorentzian manifolds

The results presented in this section are based on the article [11], with a shorter version published in [12], see Chapter 6 below.

1.6.1 Motivation

In physics, *quantization* can be thought of as a bridge between "classical" general relativity and "non-classical" quantum mechanics, which deals with physics at very small scales. There are several approaches to quantization (see the excellent introduction [17] or the recently published book [21]) and one of the most intuitive and mathematically easiest to formulate is probably the *locally covariant* one: given a fixed background spacetime M and a (linear) *differential operator* P on M, one associates to any region of M some kind of *algebra* built out of the solutions to the equation $Pu = 0$ – called *fields* – on that region. This algebra should be interpreted as the algebra of *observables* in that region. Stated like this, there is of course still a lot of (mathematical) freedom, however physical considerations lead to a certain family of *axioms* that must be satisfied: for instance, two "independent" regions of the spacetime must give rise to two "commuting" or "anti-commuting" algebras, in a sense that must be made precise; if a region is contained in another, then the corresponding algebra must be "contained" in the other. The latter reflects the fact that algebras have to be associated in a *covariant* manner. Based on pioneering work such as [19] and first described in a general and consistent framework by Romeo Brunetti, Klaus Fredenhagen and Rainer Verch [18], this approach is called *locally covariant quantum field theory*.

For linear *wave* operators on arbitrary *globally hyperbolic* spacetimes (see Section 1.5.1 above for a brief definition), a *bosonic* locally covariant quantum field theory can be successfully carried out by means so-called CCR *representations*, where "CCR" stands for "Canonical Commutation Relations". The idea, presented in [13, Ch. 4], consists in associating to each open subset of a given globally hyperbolic spacetime a symplectic vector space built directly out of the solutions to the wave equation; there is a natural and covariant way of doing this. Then standard representation theory of symplectic vector spaces allows to associate – also in a covariant way – C^* algebras to symplectic vector spaces by means of CCR *representations*, see e.g. [16]. What we formally obtain at the end is a *functor* from a "classical" category whose objects are spacetimes together with wave operators to a "non-classical" category whose objects are particular C^*-algebras called CCR *algebras*. Although we shall not introduce CCR algebras in detail, let us mention that they show the following essential and relatively intuitive feature called *quantum causality* (see Theorem 1.6.1 below): any two causally independent domains in a spacetime give rise to *commuting* CCR (sub)algebras; or, in more concrete terms, independent events give rise to independent observables.

Still plenty of differential operators are not of wave type. Although numerous papers have been dedicated to covariant quantization for particular operators (see references in [11]), there had been no attempt to develop a "general" field quantization, that could be applied to the most general differential operators. This led Christian Bär and me to address the following:

Question 5: *How large is the family of differential operators for which locally covariant quantization can be carried out?*

Here one has to pay attention to the fact that, even for a given operator, possibly different types of algebras may come out and be physically meaningful. Therefore the kind of algebra we aim at obtaining must be made precise.

1.6.2 Main results

We first focussed on bosonic locally covariant quantization, which is in terms of CCR algebras. An essential step in the construction of symplectic vector spaces out of solutions to a linear wave equation consists in extracting the *fundamental solutions* for the wave operator under consideration – which are known to exist, see e.g. [13, Ch. 3]. Actually, the existence of such fundamental solutions, or equivalently, of Green's operators, *suffices* for that, because the symplectic structure only depends on those (and the formal self-adjointness of the wave operator). This remark led us to define the very general category of *Green-hyperbolic* operators, which are linear differential operators, acting on sections of a (real or complex) vector bundle over a spacetime, and admitting *Green's operators* on any globally hyperbolic open subset of the spacetime. Recall that an advanced (resp. retarded) Green's operator for a differential operator $P:\Gamma(M,S)\circlearrowleft$ on a vector bundle $S\to M$ can be defined as a linear map $G_+:\Gamma_c(M,S)\to\Gamma(M,S)$ (resp. $G_-:\Gamma_c(M,S)\to\Gamma(M,S)$) with $P\circ G_\pm=\mathrm{Id}$, $G_\pm\circ P=\mathrm{Id}$ on $\Gamma_c(M,S)$ and with the support condition $\mathrm{supp}(G_\pm(\varphi))\subset J_\pm^M(\mathrm{supp}(\varphi))$ for all $\varphi\in\Gamma_c(M,S)$. There are whole families of Green-hyperbolic operators, including all wave or Dirac-type operators as well as physically relevant operators such as the Proca or the Rarita-Schwinger operator, see [11, Sec. 2.3-2.6]. Let us mention however that the family of Green-hyperbolic operators is strictly larger than that of hyperbolic ones since for instance the direct sum of two Green-hyperbolic operators is again Green-hyperbolic [11, Lemma 2.29], nevertheless not hyperbolic in general.

To formalize quantization in a mathematically rigorous manner, proper *categories* have first to be defined. The "source" category, denoted by $\mathsf{GlobHypGreen}$, has triples (M,S,P) as objects, consisting of a globally hyperbolic spacetime M, a (real) pseudo-Riemannian vector bundle $S\to M$ and a formally self-adjoint *Green-hyperbolic* operator P acting on sections of S. Its morphisms are pairs (f,F) consisting of a time-orientation preserving embedding f satisfying some causality condition together with a vector bundle (pointwise) isometry F preserving the operators, see [11, Def. 3.1]. The "target" category, denoted by $\mathsf{C^*Alg}$, has C^*-algebras with unit as objects and unit-preserving injective C^*-homomorphisms as morphisms. Our first main result shows the existence of a *functor* from the former category to the latter, which enjoys important physical properties:

Theorem 1.6.1 ([11, Thm. 3.10]) *With the above notations, there is a covariant functor* $\mathfrak{A}_{\mathsf{bos}} : \mathsf{GlobHypGreen} \longrightarrow \mathsf{C^*Alg}$ *which is a bosonic locally covariant quantum field theory, i.e., the following axioms are fulfilled:*

i) (Quantum causality) Let (M_j, S_j, P_j) *be objects in* $\mathsf{GlobHypGreen}$, $j = 1,2,3$, *and* (f_j, F_j) *morphisms from* (M_j, S_j, P_j) *to* (M_3, S_3, P_3), $j = 1,2$, *such that* $f_1(M_1)$ *and* $f_2(M_2)$ *are causally disjoint in* M_3. *Then the subalgebras* $\mathfrak{A}_{\mathsf{bos}}(f_1,F_1)(\mathfrak{A}_{\mathsf{bos}}(M_1,S_1,P_1))$ *and* $\mathfrak{A}_{\mathsf{bos}}(f_2,F_2)(\mathfrak{A}_{\mathsf{bos}}(M_2,S_2,P_2))$ *of* $\mathfrak{A}_{\mathsf{bos}}(M_3,S_3,P_3)$ *commute.*

ii) (Time slice axiom) Let (M_j, S_j, P_j) *be objects in* $\mathsf{GlobHypGreen}$, $j = 1,2$, *and* (f, F) *a morphism from* (M_1, S_1, P_1) *to* (M_2, S_2, P_2) *such that there is a Cauchy hypersurface* $\Sigma \subset M_1$ *for which* $f(\Sigma)$ *is a Cauchy hypersurface of* M_2. *Then*

$$\mathfrak{A}_{\mathsf{bos}}(f,F) : \mathfrak{A}_{\mathsf{bos}}(M_1,S_1,P_1) \to \mathfrak{A}_{\mathsf{bos}}(M_2,S_2,P_2)$$

is an isomorphism.

The *time slice axiom* roughly states that, if all events in two different domains of spacetime can be caught from a common region of space, then the observables from the two domains must coincide.

Another locally covariant quantum field theory has been developed which is better fitted for Dirac-type operators, namely *fermionic* quantum field theory. First discussed by Jonathan Dimock [20] for the classical Dirac operator on flat 4-dimensional Minkowski spacetime, it is carried out in terms of CAR algebras, where CAR stands for “Canonical Anticommutation Relations”. This time a *Hilbert space* is associated to the solutions of the equation under consideration; again, standard representation theory provides CAR algebras from Hilbert spaces, see [16]. There is no difficulty in adapting Jonathan Dimock’s construction to arbitrary twisted Dirac operators on arbitrary spacetimes. Our main improvement of Jonathan Dimock’s work [20] consists in enlarging the category of operators for which this can be performed by noticing that the Hilbert space structure only depends on the *principal symbol* of the operator – provided the order of the operator is one. Namely, given any formally self-adjoint first order differential operator P acting on the sections of a vector bundle S over a globally hyperbolic spacetime M, one may fix a smooth spacelike Cauchy hypersurface Σ in M with unit normal ν. Denote by σ_P the principal symbol of P. Then an elementary integration by parts combined with Gauß’ divergence theorem provides that, given any two solutions ψ, ϕ to $Pu = 0$ with spacelike compact support, the integral $\int_\Sigma \langle \sigma_P(\nu^\flat)\psi, \phi\rangle d\sigma$ does *not* depend on Σ. In particular, the map $(\psi, \phi) \mapsto \int_\Sigma \langle i\sigma_P(\nu^\flat)\psi, \phi\rangle d\sigma$ defines a non-degenerate inner product on spacelike compact solutions to $Pu = 0$ and gives rise to a Hilbert-space-structure as soon as it is *positive definite*. This led us to define the general category $\mathsf{GlobHypDef}$. Its objects are triples (M, S, P) consisting of a spacetime M together with a complex vector bundle S, carrying a non-degenerate (but non-necessarily positive definite) Hermitian inner product $\langle \cdot , \cdot \rangle$, and a formally self-adjoint first order *Green-hyperbolic* linear differential operator P acting on the sections of S such that the pointwise inner product $(\psi, \phi) \mapsto \langle i\sigma_P(\nu^\flat)\psi, \phi\rangle$ is (positive or negative) *definite* for any future-directed timelike vector ν; its morphisms are the same as those of the category $\mathsf{GlobHypGreen}$ above. For example, all twisted Dirac (but not all Dirac-type) operators fall in this category. It was shown very recently [10] that actually any formally self-adjoint first-order linear differential operator giving rise to a definite pointwise Hermitian inner product as above is automatically Green-hyperbolic, since it is symmetric hyperbolic up to a multiplicative constant. As target category, we consider CAR-algebras, which are still C^* algebras but which also have a natural $\mathbb{Z}_2$-graduation. In particular, one may talk about *super-commuting* subalgebras, meaning that the odd parts of the algebras anti-commute while the even parts commute with everyone. The CAR quantization procedure applies on $\mathsf{GlobHypDef}$ and we obtain the following:

Theorem 1.6.2 ([11, Thm. 3.20]) *With the above notations, there is a covariant functor $\mathfrak{A}_{\mathrm{ferm}} : \mathsf{GlobHypDef} \longrightarrow \mathsf{C^*Alg}$ which is a fermionic locally covariant quantum field theory, i.e., the following axioms are fulfilled:*

i) (Quantum causality) Let (M_j, S_j, P_j) be objects in $\mathsf{GlobHypDef}$, $j = 1,2,3$, and (f_j, F_j) morphisms from (M_j, S_j, P_j) to (M_3, S_3, P_3), $j = 1,2$, such that $f_1(M_1)$ and $f_2(M_2)$ are causally disjoint in M_3. Then the subalgebras $\mathfrak{A}_{\mathrm{ferm}}(f_1, F_1)(\mathfrak{A}_{\mathrm{ferm}}(M_1, S_1, P_1))$ and $\mathfrak{A}_{\mathrm{ferm}}(f_2, F_2)(\mathfrak{A}_{\mathrm{ferm}}(M_2, S_2, P_2))$ of $\mathfrak{A}_{\mathrm{ferm}}(M_3, S_3, P_3)$ super-commute.

ii) (Time slice axiom) Let (M_j, S_j, P_j) be objects in $\mathsf{GlobHypDef}$, $j = 1,2$, and (f, F) a morphism from (M_1, S_1, P_1) to (M_2, S_2, P_2) such that there is a Cauchy hypersurface $\Sigma \subset M_1$ for which $f(\Sigma)$ is a Cauchy hypersurface of M_2. Then

$$\mathfrak{A}_{\mathrm{ferm}}(f, F) : \mathfrak{A}_{\mathrm{ferm}}(M_1, S_1, P_1) \to \mathfrak{A}_{\mathrm{ferm}}(M_2, S_2, P_2)$$

is an isomorphism.

Although both Theorems 1.6.1 and 1.6.2 show how to obtain observables, they do not give any “concrete”, i.e., numerical interpretation. This is done by introducing *states*, which are (*positive, normed*) linear forms on the target C^* algebras. In the last part of [11], we show how states which are “sufficiently regular” in a certain sense give rise to so-called *quantum fields*, which are operator-algebra-valued distributions on the underlying spacetime and which solve the equation under consideration. We refer to [11, Sec. 4] for the details of this very technical construction.

1.6.3 Perspectives

Despite our very general ansatz to construct CCR algebras of observables out of classical fields, there are still whole families of physically relevant operators – among which all non-linear ones – where it does not apply. Another interesting issue deals with particular conditions on the states involved in the contruction of the quantum field such as the *Hadamard* condition. This condition lacks investigation in a unified framework.

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Chapter 2

The spectrum of the Dirac operator on $\mathrm{SU}_2/\mathrm{Q}_8$

This chapter coincides (up to minor changes such as enumeration of pages, sections, theorems, references etc.) with the published article [29].

Nicolas Ginoux

Abstract. We compute the fundamental Dirac operator for the three-parameter-family of homogeneous Riemannian metrics and the four different spin structures on $\mathrm{SU}_2/\mathrm{Q}_8$, where Q_8 denotes the group of quaternions. We deduce its spectrum for the Berger metrics and show the sharpness of Christian Bär's upper bound for the smallest Dirac eigenvalue in the particular case where $\mathrm{SU}_2/\mathrm{Q}_8$ is a homogeneous minimal hypersurface of S^4.

Mathematics Subject Classification: 53C27, 53C30, 58C40

Keywords: Spin geometry, homogeneous manifolds, spectral theory

2.1 Introduction

Throughout this paper and unless explicitly mentioned we denote by M the quotient of SU_2 by the right-action of the group of quaternions Q_8, i.e., the group with 8 elements defined by $\{\pm\mathrm{I}_2, \pm A_1, \pm A_2, \pm A_3\}$ with $A_1 := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, $A_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $A_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The manifold M is a 3-dimensional compact connected spin homogeneous space and at the same time the simplest example of homogeneous hypersurface in the round sphere with 3 different principal curvatures, see e.g. [A6] and end of Section 2.3.
Using classical techniques (see e.g. [A2]) we first compute the Dirac operator of M for any homogeneous metric and any spin structure:

Theorem 2.1.1

i) *The manifold M carries a 3-parameter family of homogeneous Riemannian metrics which are given by the orthonormal bases $\{X_1 := a_1A_1, X_2 := a_2A_2, X_3 := a_3A_3\}$ of $\mathfrak{su}(2)$, where $a_1,a_2,a_3 \in \mathbb{R}^*$. Conversely, every homogeneous metric on M is of that form.*

ii) *The isotropy representation α of M is given in the basis (X_1,X_2,X_3) of $\mathfrak{su}(2)$ by*

$$\begin{array}{ll} \alpha(\pm\mathrm{I}_2) = \mathrm{I}_3 & \alpha(\pm A_1) = \mathrm{diag}(1,-1,-1) \\ \alpha(\pm A_2) = \mathrm{diag}(-1,1,-1) & \alpha(\pm A_3) = \mathrm{diag}(-1,-1,1). \end{array}$$

In particular the manifold M is orientable.

iii) *The manifold M is spin and carries exactly 4 spin structures, each one corresponding to one of the following group homomorphisms $\mathrm{Q}_8 \xrightarrow{\varepsilon_j} \{-1,1\}$: $\varepsilon_0 \equiv 1$ and $\mathrm{Ker}(\varepsilon_j) = \{\pm\mathrm{I}_2, \pm A_j\}$ for $j \in \{1,2,3\}$.*

iv) *The finite dimensional Dirac operator D_n corresponding to the irreducible representation of SU_2 on the space V_n of homogeneous polynomials of degree n in two variables is non-trivial only if n is odd. In that situation*

$$D_n = D_n' - \frac{a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2}{2a_1a_2a_3}\mathrm{Id}$$

where D_n' is described by a $\frac{n+1}{2} \times \frac{n+1}{2}$ tridiagonal matrix. More precisely, there exists a basis $(v_0,\ldots,v_{\frac{n-1}{2}})$ in which D_n' can be expressed as

0) *in case M carries the spin structure given by ε_0,*

$$\begin{aligned} D_n'(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2+(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2-(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(a_1 + \frac{n+1}{2}(a_2+a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2-a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 1$ (4) and

$$\begin{aligned} D_n'(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2-(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2+(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(a_1 - \frac{n+1}{2}(a_2+a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2-a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 3$ (4).

1) *in case M carries the spin structure given by ε_1,*

$$\begin{aligned} D_n'(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2+(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2-(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(a_1 - \frac{n+1}{2}(a_2+a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2-a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 1\ (4)$ *and*

$$\begin{aligned} D_n'(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2-(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2+(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(a_1 + \frac{n+1}{2}(a_2+a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2-a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 3\ (4)$.

2) *in case* M *carries the spin structure given by* ε_2,

$$\begin{aligned} D_n'(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2-(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2+(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(-a_1 + \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 1\ (4)$ *and*

$$\begin{aligned} D_n'(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2+(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2-(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(-a_1 - \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 3\ (4)$.

3) *in case* M *carries the spin structure given by* ε_3,

$$\begin{aligned} D_n'(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2-(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2+(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(-a_1 - \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 1\ (4)$ *and*

$$\begin{aligned} D_n'(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2+(-1)^k a_3)v_{k+1} \\ &\quad +(n-k+1)(a_2-(-1)^k a_3)v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D_n'(v_{\frac{n-1}{2}}) &= \Big(-a_1 + \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 3\ (4)$.

We deduce the spectrum of the Dirac operator D of M for the so-called Berger metrics, which form a 2-parameter subfamily of homogeneous metrics:

Corollary 2.1.2 *With the notations of* Theorem 2.1.1, *assume furthermore that* $a_2 = a_3$. *Then the spectrum of the operator* $D + \frac{2a_1^2+a_2^2}{2a_1}\mathrm{Id}$ *on* M *for the metric induced by* a_1, a_2 *and the spin structure given by* ε_j *(*$j \in \{0,1,2,3\}$*) consists of the following family of eigenvalues:*

0. *for* $j=0$,

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\;|\,k\in\{0,\ldots,\frac{n-5}{2}\}\text{ even}, a_1+(n+1)a_2\Big\}$$
$$\bigcup\;\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\;|\,k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, a_1-(n+1)a_2, -na_1\Big\},$$

each eigenvalue having multiplicity $n+1$ *for the corresponding* n*.*

1. *for* $j=1$,

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\;|\,k\in\{0,\ldots,\frac{n-5}{2}\}\text{ even}, a_1-(n+1)a_2\Big\}$$
$$\bigcup\;\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\;|\,k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, a_1+(n+1)a_2, -na_1\Big\},$$

each eigenvalue having multiplicity $n+1$ *for the corresponding* n*.*

2. *for* $j=2$ *and* $j=3$,

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\;|\,k\in\{1,\ldots,\frac{n-3}{2}\}\text{ odd}, -na_1\Big\}$$
$$\bigcup\;\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}}\Big\{a_1\pm\sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\;|\,k\in\{0,\ldots,\frac{n-3}{2}\}\text{ even}\Big\},$$

each eigenvalue having multiplicity $n+1$ *for the corresponding* n*.*

In the case where $a_1=a_2=a_3$, i.e., M is a space-form with positive curvature, we reobtain the Dirac spectrum computed by Christian Bär in [A3, Thm. 2], see Corollary 2.4.2.

On the other hand, considering M as embedded homogeneous hypersurface in the 4-dimensional round sphere S^4 one could ask if the following inequality due to Christian

Bär [A5, Cor. 4.3] is an equality:

$$\lambda_1(D^2) \leq \frac{9}{4}(\mathscr{H}^2+1), \tag{2.1}$$

where $\lambda_1(D^2)$ is the smallest eigenvalue of the Dirac Laplacian on M (for the induced metric and spin structure) and $\mathscr{H}$ is the mean curvature of M in S^4. This question takes its origin in the study of the equality case in Christian Bär's estimate [A5, Cor. 4.3] for the smallest eigenvalue $\lambda_1(D^2)$ of the Dirac Laplacian. If this inequality is an equality, then the mean curvature of the hypersurface has to be constant, nevertheless the reverse statement has up to now neither been proved nor been contradicted. We give a partial answer to that question for M:

Corollary 2.1.3 *With the notations of* Theorem 2.1.1, *assume furthermore that M carries a homogeneous metric coming from a minimal embedding in S^4 and the spin structure described by ε_0. Then* (2.1) *is an equality.*

The paper is organized as follows. In the first section we describe the metrics and spin structures on M and thus prove Theorem 2.1.1 $i) - iii)$. In the second one we compute the Dirac operator of M (Theorem 2.1.1 *iv*)) and the eigenvalue of D_1 (Corollary 2.3.9), which in the case where M is a hypersurface of S^4 turns out to coincide with the upper bound in (2.1), see Corollary 2.3.11. In the third section we prove Corollary 2.1.2 and derive the Dirac spectrum of M in case its metric either is of constant sectional curvature or comes from a minimal embedding in S^4, see Corollary 2.4.2. We deduce in Corollary 2.4.3 the existence of non-zero real Killing spinors in the first case and Corollary 2.1.3 in the other one.

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2.2 Metrics and spin structures on *M*

The Lie-algebra of Q_8 being trivial the adjoint representation α of the homogeneous space M is nothing but the restriction of the adjoint map $\mathrm{SU}_2 \longrightarrow \mathrm{Aut}(\mathfrak{su}(2))$ to Q_8, where $\mathfrak{su}(2)$ denotes the Lie-algebra of SU_2. We define the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{su}(2)$ by declaring the following basis to be orthonormal:

$$\begin{aligned} X_1 &:= a_1A_1 \\ X_2 &:= a_2A_2 \\ X_3 &:= a_3A_3, \end{aligned}$$

where $a_1, a_2, a_3 \in \mathbb{R}^*$ are fixed parameters. The map α is given in the basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ by

$$\begin{aligned} \alpha(\pm \mathrm{I}_2) &= \mathrm{I}_3 \\ \alpha(\pm A_1) &= \mathrm{diag}(1,-1,-1) \\ \alpha(\pm A_2) &= \mathrm{diag}(-1,1,-1) \\ \alpha(\pm A_3) &= \mathrm{diag}(-1,-1,1), \end{aligned}$$

therefore it obviously preserves $\langle\cdot,\cdot\rangle$ which hence induces a homogeneous metric on M. Using the form of α in the basis (A_1,A_2,A_3) computed above it is easy to prove that every homogeneous metric on M comes from such a scalar product on $\mathfrak{su}(2)$, i.e., it admits $\{a_1A_1,a_2A_2,a_3A_3\}$ as orthonormal basis for suitable $a_1,a_2,a_3\in\mathbb{R}^*$. Note also that α preserves the orientation of $\mathfrak{su}(2)$, so that if we choose (X_1,X_2,X_3) as positively-oriented orthonormal basis of $\mathfrak{su}(2)$ then α is expressed in that basis by a map $\mathrm{Q}_8\xrightarrow{\alpha}\mathrm{SO}_3$.

We now examine the spin structures on M considering the metric and the orientation given by (X_1,X_2,X_3). From [A2, Lemma 3] the manifold M is spin if and only if its isotropy representation α lifts to Spin_3 through the non-trivial two-fold covering $\mathrm{Spin}_3\xrightarrow{\xi}\mathrm{SO}_3$, and in that case spin structures on M are in one-to-one correspondence with those lifts, each one of those being uniquely determined by a group homomorphism $\mathrm{Q}_8\xrightarrow{\varepsilon}\{-1,1\}$. Here Q_8 already lies in $\mathrm{SU}_2\cong\mathrm{Spin}_3$ so that M is obviously spin. Denoting by $\widehat{\alpha}$ the inclusion $\mathrm{Q}_8\subset\mathrm{SU}_2$, every spin structure on M is uniquely described by a map $\widetilde{\alpha}:\mathrm{Q}_8\longrightarrow\mathrm{SU}_2$ of the form $\widetilde{\alpha}(h)=\varepsilon(h)\widehat{\alpha}(h)$ for every $h\in\mathrm{Q}_8$, where $\varepsilon:\mathrm{Q}_8\longrightarrow\{-1,1\}$ is a group homomorphism. But there are exactly 4 such homomorphisms: the trivial one $\varepsilon_0\equiv 1$ and the ε_j's, $j=1,2,3$, with $\mathrm{Ker}(\varepsilon_j)=\{\pm\mathrm{I}_2,\pm A_j\}$. This proves Theorem 2.1.1 $i)-iii)$.
In the following we shall call the spin structure corresponding to $\varepsilon_j\cdot\widehat{\alpha}$ the ε_j-spin structure on M.

2.3 The Dirac operator on M

Let us denote by $\mathrm{Spin}_n\xrightarrow{\delta_n}\mathrm{Aut}(\Sigma_n)$ the spinor representation in dimension n. We recall the following theorem allowing the representation-theoretical computation of the fundamental Dirac operator on a homogeneous space, see e.g. [A2, Thm. 2 & Prop. 1]:

Theorem 2.3.1 *Let $M:=G/H$ be an n-dimensional Riemannian homogeneous spin manifold with G compact and simply-connected. Let $\mathfrak{p}$ be a supplementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$. Fix a p.o.n.b $(X_1,\ldots,X_n)$ of $\mathfrak{p}$ and let $\alpha:H\longrightarrow\mathrm{SO}_n$ be the isotropy representation of M expressed in the basis $(X_1,\ldots,X_n)$. Let $\widetilde{\alpha}:H\longrightarrow\mathrm{Spin}_n$ be the lift of α to Spin_n induced by the given spin structure of M and $\Sigma_{\widetilde{\alpha}}M\longrightarrow M$ be the spinor bundle of M associated with $\widetilde{\alpha}$. Let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of G (in the following we shall always identify an element of $\widehat{G}$ with one of its representants).*

$i)$ The space $L^2(M,\Sigma_{\widetilde{\alpha}}M)$ splits under the unitary left action of G into a direct Hilbert sum

$$\overline{\bigoplus_{\gamma\in\widehat{G}}V_\gamma\otimes\mathrm{Hom}_H(V_\gamma,\Sigma_n)}\tag{2.2}$$

where V_γ is the space of the representation γ (i.e., $\gamma:G\longrightarrow\mathrm{U}(V_\gamma)$) and

$$\mathrm{Hom}_H(V_\gamma,\Sigma_n):=\Big\{f\in\mathrm{Hom}(V_\gamma,\Sigma_n)\ s.t.\ \forall h\in H,\ f\circ\gamma(h)=(\delta_n\circ\widetilde{\alpha})(h)\circ f\Big\}.$$

ii) *The Dirac operator D of M preserves each summand of* (2.2)*; more precisely, if $(e_1,\ldots,e_n)$ denotes the canonical basis of $\mathbb{R}^n$, then for every $\gamma \in \widehat{G}$, the restriction of D to $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ is given by $\mathrm{Id} \otimes D_\gamma$, where, for every $A \in \mathrm{Hom}_H(V_\gamma, \Sigma_n)$,*

$$D_\gamma(A) := -\sum_{k=1}^{n} e_k \cdot A \circ T_e\gamma(X_k) + \Big(\sum_{i=1}^{n} \beta_i e_i + \sum_{i<j<k} \alpha_{ijk} e_i \cdot e_j \cdot e_k\Big) \cdot A, \qquad (2.3)$$

and

$$\begin{aligned} \beta_i &:= \frac{1}{2}\sum_{j=1}^{n} \langle [X_j, X_i]_{\mathfrak{p}}, X_j\rangle \\ \alpha_{ijk} &:= \frac{1}{4}\left(\langle [X_i, X_j]_{\mathfrak{p}}, X_k\rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i\rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j\rangle\right) \end{aligned}$$

(here and henceforth $X_{\mathfrak{p}}$ will denote the image of $X \in \mathfrak{g}$ under the projection $\mathfrak{g} \longrightarrow \mathfrak{p}$ with kernel $\mathfrak{h}$).

The following statement will be useful for taking the symmetries of M into account, see Examples 2.3.4 below.

Lemma 2.3.2 *Under the hypotheses of* Theorem 2.3.1 *let $\langle\cdot,\cdot\rangle'$ be a further homogeneous metric on M and $f : G \longrightarrow G$ be a Lie-group-homomorphism such that $f(H) \subset H$ and $f_* := [T_e f]$ is an orientation-preserving isometry $(T_{[e]}M, \langle\cdot,\cdot\rangle) \longrightarrow (T_{[e]}M, \langle\cdot,\cdot\rangle')$. Then the pull-back spin structure $f^*\mathrm{Spin}_{\widetilde{\alpha}}(TM)$ is described by*

$$\begin{aligned} H &\longrightarrow \mathrm{Spin}_n \\ h &\longmapsto \widehat{f}^{-1} \cdot \widetilde{\alpha} \circ f(h) \cdot \widehat{f} \end{aligned}$$

where $\widehat{f} \in \mathrm{Spin}_n$ satisfies $\xi(\widehat{f}) = f_*$.

Proof: The proof relies on the identity $f_* \circ \mathrm{Ad}(g) = \mathrm{Ad}(f(g)) \circ f_*$ for every $g \in G$, which implies in particular

$$\alpha(h) = f_*^{-1} \circ \alpha(f(h)) \circ f_*$$

for every $h \in H$. □

Notes 2.3.3

1. Of course the homomorphism describing the pull-back spin structure in Lemma 2.3.2 is well-defined since $\widehat{f}$ is uniquely determined up to a sign.

2. One should pay attention that Lemma 2.3.2 can only be applied once p.o.n.b. $(X_1,\ldots,X_n)$ and $(X_1',\ldots,X_n')$ of $\mathfrak{p}$ w.r.t. $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ respectively have been chosen. Then all the objects above should be expressed in those bases, see Examples 2.3.4 below.

Examples 2.3.4 Consider again $M := \mathrm{SU}_2/\mathrm{Q}_8$, fix $a_1, a_2, a_3 \in \mathbb{R}^*$ and as above set $X_k := a_k A_k$ for $k \in \{1,2,3\}$. We write $(M, \langle\cdot,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_j)$ for M endowed with the metric and the orientation given by (X_1, X_2, X_3) and the ε_j-spin structure ($j \in \{0,1,2,3\}$).

1. Set $X_1' := X_1$, $X_2' := -X_2$ and $X_3' := -X_3$. Let $f(A_1) := A_1$, $f(A_2) := -A_2$ and $f(A_3) := -A_3$. Setting $f(I_2) := I_2$ and extending f linearly one obtains a Lie-group-homomorphism $\mathrm{SU}_2 \to \mathrm{SU}_2$ inducing an orientation-preserving isometry $(M, \langle\cdot,\cdot\rangle_{a_1,a_2,a_3}) \longrightarrow (M, \langle\cdot,\cdot\rangle_{a_1,-a_2,-a_3})$. The matrix of $f_* = f$ in the bases (X_1, X_2, X_3) and (X_1', X_2', X_3') respectively is the identity so that $\widehat{f} = 1$ can be chosen. Applying Lemma 2.3.2 the pull-back of the ε_j-spin structure by f is then described by
$$\mathrm{Q}_8 \longrightarrow \mathrm{SU}_2, \qquad h \longmapsto \varepsilon_j(h) f(h)$$
(remember that $-\mathrm{I}_2 \in \mathrm{Ker}(\varepsilon_j)$), i.e., the pull-back of the ε_0- (resp. ε_2-) spin structure is the ε_1- (resp. ε_3-) one. In other words, changing the sign of both a_2 and a_3 changes neither the metric nor the orientation, however it permutes the ε_0- (resp. ε_2-) spin structure with the ε_1- (resp. ε_3-) one. In particular the Dirac operator on e.g. $(M, \langle\cdot,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_0)$ coincides with that of $(M, \langle\cdot,\cdot\rangle_{a_1,-a_2,-a_3}, \varepsilon_1)$.

2. Let σ be a permutation of $\{0,1,2,3\}$ with $\sigma(0) = 0$ and set $X_k' := a_{\sigma(k)} A_k$ for $k \in \{1,2,3\}$. Let $f(A_1) := A_{\sigma^{-1}(1)}$, $f(A_2) := A_{\sigma^{-1}(2)}$ and $f(A_3) := \varepsilon(\sigma) A_{\sigma^{-1}(3)}$ where $\varepsilon(\sigma) \in \{-1,1\}$ is the signature of σ. Setting in the same way as just above $f(I_2) := I_2$ and extending f linearly one obtains a Lie-group-homomorphism $\mathrm{SU}_2 \to \mathrm{SU}_2$ inducing an orientation-preserving isometry $(M, \langle\cdot,\cdot\rangle_{a_1,a_2,a_3}) \longrightarrow (M, \langle\cdot,\cdot\rangle_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}})$. This time the matrix of $f_* = f$ in the bases (X_1, X_2, X_3) and (X_1', X_2', X_3') respectively is not the identity, however it coincides with the matrix of f in the basis (A_1, A_2, A_3) so that, per definition of the universal 2-fold covering map,
$$\widehat{f}^{-1} \cdot f(h) \cdot \widehat{f} = h$$
for any lift $\widehat{f}$ of f to SU_2 and every $h \in \mathrm{Q}_8$. The pull-back through f of the ε_j-spin structure is therefore the $(\varepsilon_j \circ f)$-one, that is, the $\varepsilon_{\sigma(j)}$-one. In other words, permuting the coefficients a_1, a_2, a_3 induces an orientation-preserving isometry permuting the spin structure in the reverse way, the ε_0-one staying unchanged under that transformation. In particular the Dirac operator on $(M, \langle\cdot,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_j)$ coincides with that of
$(M, \langle\cdot,\cdot\rangle_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}}, \varepsilon_{\sigma^{-1}(j)})$.

3. It is well-known that, for any fixed metric and spin structure on M, the Dirac operators for the two different orientations are just opposite from one another (this is always the case in odd dimensions). For example, if one turns a_1 into $-a_1$ and lets a_2 and a_3 unchanged, then the Dirac operator on e.g. $(M, \langle\cdot,\cdot\rangle_{-a_1,a_2,a_3}, \varepsilon_0)$ coincides with minus that of $(M, \langle\cdot,\cdot\rangle_{a_1,-a_2,-a_3}, \varepsilon_0)$, i.e., with minus that of $(M, \langle\cdot,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_1)$.

Note that Examples 2.3.4 essentially exhausts all possible isometric transformations of M since the only Lie-group-automorphisms f of SU_2 preserving Q_8 are characterized by $f(A_k) = \varepsilon(k)A_{\sigma(k)}$ for some permutation σ of $\{1,2,3\}$ and $\varepsilon(k) \in \{-1,1\}$.

We come now to the computation of the Dirac operator on $M = \mathrm{SU}_2/\mathrm{Q}_8$. We begin with the part of the Dirac operator that does not depend on the representation γ of SU_2. Note also that this part only depends on the metric chosen on M and not on its spin structure.

Proposition 2.3.5 *For the metric on M given by a_1, a_2, a_3 we have $\beta_j = 0$ for every $j \in \{1,2,3\}$ and $\alpha_{123} = \frac{a_1^2a_2^2+a_2^2a_3^2+a_1^2a_3^2}{2a_1a_2a_3}$. In particular*

$$\sum_{j=1}^{3} \beta_j e_j \cdot + \alpha_{123} e_1 \cdot e_2 \cdot e_3 \cdot = -\frac{a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2}{2a_1a_2a_3}\mathrm{Id}.$$

Proof: We compute the Lie-brackets $[X_j, X_k]$ for all $1 \leq j < k \leq 3$. Since $A_1A_2 = -A_2A_1 = A_3$ we have

$$\begin{aligned}
[X_1, X_2] &= a_1a_2[A_1, A_2] \\
&= 2a_1a_2A_3 \\
&= \frac{2a_1a_2}{a_3}X_3,
\end{aligned}$$

and analogously $[X_2, X_3] = \frac{2a_2a_3}{a_1}X_1$, $[X_3, X_1] = \frac{2a_1a_3}{a_2}X_2$. We straightforward deduce that $\beta_1 = \beta_2 = \beta_3 = 0$. Furthermore,

$$\begin{aligned}
\alpha_{123} &= \frac{1}{4}\left(\langle [X_1, X_2], X_3\rangle + \langle [X_2, X_3], X_1\rangle + \langle [X_3, X_1], X_2\rangle\right) \\
&= \frac{1}{4}\left(\frac{2a_1a_2}{a_3} + \frac{2a_2a_3}{a_1} + \frac{2a_1a_3}{a_2}\right) \\
&= \frac{a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2}{2a_1a_2a_3}.
\end{aligned}$$

It remains to notice that, by convention, the complex volume form $i^{[\frac{3+1}{2}]} e_1 \cdot e_2 \cdot e_3 = -e_1 \cdot e_2 \cdot e_3$ acts by the identity on Σ_3. This concludes the proof. □

We next determine the space of equivariant homomorphisms for each $\gamma \in \widehat{\mathrm{SU}_2}$ and each ε_j-spin structure on M. First recall that the irreducible unitary representations of SU_2 are given by its natural action on the $n+1$-dimensional vector spaces of all n-graded homogeneous complex polynomials in two variables: set, for any $n \in \mathbb{N}$ (we include $n = 0$)

$$V_n := \{P \in \mathbb{C}[z_1, z_2], \quad P = 0 \text{ or } P \text{ homogeneous and } d^\circ P = n\}.$$

Then SU_2 acts on V_n through

$$\begin{aligned}
\pi_n : \mathrm{SU}_2 &\longrightarrow \mathrm{Aut}(V_n) \\
A &\longmapsto (\pi_n(A) : P \mapsto P \circ R_A),
\end{aligned}$$

where $P \circ R_A(z) := P(zA)$ for every $z = (z_1\ \ z_2) \in \mathbb{C}^2$. From now on we shall always work with the following basis of V_n:

$$(P_k(z_1, z_2) := z_1^{n-k} z_2^k, \ \ 0 \leq k \leq n).$$

Identifying Spin_3 to SU_2 the spinor representation $\mathrm{Spin}_3 \xrightarrow{\delta_3} \mathrm{Aut}(\Sigma_3)$ is equivalent to the standard representation $\mathrm{SU}_2 \longrightarrow \mathrm{Aut}(\mathbb{C}^2)$. For every lift $\varepsilon_j \cdot \widehat{\alpha}$ of the isotropy representation α of M the space of equivariant homomorphisms for π_n and for the ε_j-spin structure - that we shall denote by $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ - is then given by

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2) = \left\{ f \in \mathrm{Hom}(V_n,\mathbb{C}^2) \text{ s.t. } f \circ \pi_n(h) = \varepsilon_j(h) h \circ f \quad \forall h \in \mathrm{Q}_8 \right\}.$$

We fix the following basis $(F_0,\ldots,F_n,G_0,\ldots,G_n)$ of $\mathrm{Hom}(V_n,\mathbb{C}^2)$ (which is that of [A2, p.73]): set, for every $k \in \{0,\ldots,n\}$,

$$F_k(P_l) := \begin{cases} (1\ 0) & \text{if } l = k \text{ and } k \text{ even} \\ (0\ 1) & \text{if } l = k \text{ and } k \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_k(P_l) := \begin{cases} (0\ 1) & \text{if } l = k \text{ and } k \text{ even} \\ (1\ 0) & \text{if } l = k \text{ and } k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

W.r.t. the bases $(P_0,\ldots,P_n)$ and $((1\ 0),(0\ 1))$ of V_n and $\mathbb{C}^2$ respectively the elements F_k and G_k are described by matrices of the form:

$$F_k = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}$$

if k is even and

$$F_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

if k is odd, where the “1” always stands in the $(k+1)^{\text{st}}$ column.

Lemma 2.3.6 *Let M carry the ε_j-spin structure for $j \in \{0,1,2,3\}$. Then* $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2) = \{0\}$ *if n is even. Moreover*

0. *for $j = 0$ we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_0}(V_n,\mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k + F_{n-k}) & \text{if } n \equiv 1\ (4) \\[2ex] \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k - G_{n-k}) & \text{if } n \equiv 3\ (4). \end{cases}$$

1. *for $j = 1$ we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_1}(V_n,\mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k - F_{n-k}) & \text{if } n \equiv 1\ (4) \\[2ex] \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k + G_{n-k}) & \text{if } n \equiv 3\ (4). \end{cases}$$

2. *for $j = 2$ we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_2}(V_n,\mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k + G_{n-k}) & \text{if } n \equiv 1\ (4) \\[2ex] \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k - F_{n-k}) & \text{if } n \equiv 3\ (4). \end{cases}$$

3. *for* $j = 3$ *we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_3}(V_n,\mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k - G_{n-k}) & \text{if } n \equiv 1\ (4) \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k + F_{n-k}) & \text{if } n \equiv 3\ (4). \end{cases}$$

Proof: Since $-\mathrm{I}_2 \in \mathrm{Ker}(\varepsilon_j)$ any element $f \in \mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ must satisfy $f \circ \pi_n(-\mathrm{I}_2) = -f$, with $\pi_n(-\mathrm{I}_2) = (-1)^n \mathrm{Id}_{V_n}$, so that the condition reads

$$(-1)^n f = -f,$$

which requires $f = 0$ as soon as n is even.
From now on, we assume that n is odd. We compute $\pi_n(A_j)$ for $j = 1,2$ (remember that A_1 and A_2 generate Q_8): for every $k \in \{0,\dots,n\}$ and $z \in \mathbb{C}^2$,

$$\begin{aligned} \{\pi_n(A_1)\}(P_k)(z) &= P_k\left((z_1\ z_2)\cdot\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) \\ &= P_k(-iz_1, iz_2) \\ &= (-iz_1)^{n-k}(iz_2)^k \\ &= (-1)^{n-k} i^n z_1^{n-k} z_2^k, \end{aligned}$$

i.e., $\{\pi_n(A_1)\}(P_k) = (-1)^{n-k} i^n P_k$. Analogously,

$$\begin{aligned} \{\pi_n(A_2)\}(P_k)(z) &= P_k\left((z_1\ z_2)\cdot\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \\ &= P_k(iz_2, iz_1) \\ &= (iz_2)^{n-k}(iz_1)^k, \end{aligned}$$

i.e., $\{\pi_n(A_2)\}(P_k) = i^n P_{n-k}$. The conditions $f \circ \pi_n(A_l) = \varepsilon_j(A_l) A_l \circ f$ for $l = 1,2$ then read

$$\left| \begin{aligned} f(P_k) &= (-1)^{k+\frac{n-1}{2}} i\varepsilon_j(A_1)(A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n+1}{2}} i\varepsilon_j(A_2)(A_2 \circ f)(P_k) \end{aligned} \right. \tag{2.4}$$

for every $k \in \{0,1,\dots,n\}$. From now on we denote by $\begin{pmatrix} f_{1k} \\ f_{2k} \end{pmatrix} := f(P_k) \in \mathbb{C}^2$. We examine each case separately.
• *Case* $j = 0$: In that case the conditions (2.4) are equivalent to

$$\left| \begin{aligned} f(P_k) &= (-1)^{k+\frac{n-1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n+1}{2}} i(A_2 \circ f)(P_k), \end{aligned} \right.$$

that is,

$$\left| \begin{aligned} f_{1k} &= (-1)^{k+\frac{n-1}{2}} f_{1k} \\ f_{2k} &= (-1)^{k+\frac{n+1}{2}} f_{2k} \\ f_{1n-k} &= (-1)^{\frac{n-1}{2}} f_{2k} \\ f_{2n-k} &= (-1)^{\frac{n-1}{2}} f_{1k}. \end{aligned} \right.$$

If $n \equiv 1\ (4)$ then those identities become

$$\left|\begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array}\right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 + F_n) + f_{21}(F_1 + F_{n-1}) + \ldots + f_{1\frac{n-1}{2}}(F_{\frac{n-1}{2}} + F_{\frac{n+1}{2}})$$

and the result in that case.
If $n \equiv 3\ (4)$ then those identities become

$$\left|\begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array}\right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 - G_n) + f_{11}(G_1 - G_{n-1}) + \ldots + f_{1\frac{n-1}{2}}(G_{\frac{n-1}{2}} - G_{\frac{n+1}{2}})$$

and the result in that case.
• *Case* $j = 1$: In that case the conditions (2.4) are equivalent to

$$\left|\begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n-1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n-1}{2}} i(A_2 \circ f)(P_k), \end{array}\right.$$

that is,

$$\left|\begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n-1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n+1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n+1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n+1}{2}} f_{1k}. \end{array}\right.$$

If $n \equiv 1\ (4)$ then those identities become

$$\left|\begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array}\right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 - F_n) + f_{21}(F_1 - F_{n-1}) + \ldots + f_{1\frac{n-1}{2}}(F_{\frac{n-1}{2}} - F_{\frac{n+1}{2}})$$

and the result in that case.
If $n \equiv 3\ (4)$ then those identities become

$$\left|\begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array}\right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 + G_n) + f_{11}(G_1 + G_{n-1}) + \ldots + f_{1\frac{n-1}{2}}(G_{\frac{n-1}{2}} + G_{\frac{n+1}{2}})$$

and the result in that case.

• *Case* $j = 2$: In that case the conditions (2.4) are equivalent to

$$\left| \begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n+1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n+1}{2}} i(A_2 \circ f)(P_k), \end{array} \right.$$

that is,

$$\left| \begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n+1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n-1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n-1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n-1}{2}} f_{1k}. \end{array} \right.$$

If $n \equiv 1\ (4)$ then those identities become

$$\left| \begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 + G_n) + f_{11}(G_1 + G_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(G_{\frac{n-1}{2}} + G_{\frac{n+1}{2}})$$

and the result in that case.

If $n \equiv 3\ (4)$ then those identities become

$$\left| \begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 - F_n) + f_{21}(F_1 - F_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(F_{\frac{n-1}{2}} - F_{\frac{n+1}{2}})$$

and the result in that case.

• *Case* $j = 3$: In that case the conditions (2.4) are equivalent to

$$\left| \begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n+1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n-1}{2}} i(A_2 \circ f)(P_k), \end{array} \right.$$

that is,

$$\left| \begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n+1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n-1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n+1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n+1}{2}} f_{1k}. \end{array} \right.$$

If $n \equiv 1\ (4)$ then those identities become

$$\left| \begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 - G_n) + f_{11}(G_1 - G_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(G_{\frac{n-1}{2}} - G_{\frac{n+1}{2}})$$

and the result in that case.
If $n \equiv 3\ (4)$ then those identities become

$$\left| \begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \le k \le \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 + F_n) + f_{21}(F_1 + F_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(F_{\frac{n-1}{2}} + F_{\frac{n+1}{2}})$$

and the result in that case. This concludes the proof. $\square$

It remains to compute the map $T_{\mathrm{I}_2}\pi_n$ for every (odd) n.

Lemma 2.3.7 *The endomorphisms $T_{\mathrm{I}_2}\pi_n(X_j)$, $1 \le j \le 3$, are given in the basis $(P_0, \ldots, P_n)$ of V_n by:*

$$\begin{array}{rcl} \{T_{\mathrm{I}_2}\pi_n(X_1)\}(P_k) & = & -ia_1(n-2k)P_k \\ \{T_{\mathrm{I}_2}\pi_n(X_2)\}(P_k) & = & ia_2\left((n-k)P_{k+1} + kP_{k-1}\right) \\ \{T_{\mathrm{I}_2}\pi_n(X_3)\}(P_k) & = & a_3\left(-(n-k)P_{k+1} + kP_{k-1}\right) \end{array}$$

for every $k \in \{0, \ldots, n\}$, with the convention $P_{-1} = P_{n+1} = 0$.

Proof: For every $X \in \mathfrak{su}_2$, $P \in V_n$ and $z \in \mathbb{C}^2$, we have

$$\begin{array}{rcl} \left(\{T_{\mathrm{I}_2}\pi_n(X)\}(P)\right)(z) & = & \frac{d}{dt}|_{t=0}\left(P \circ R_{\exp(tX)}\right)(z) \\ & = & \frac{d}{dt}|_{t=0}\left(P \circ R_{\exp(tX)}(z)\right) \\ & = & \frac{d}{dt}|_{t=0}\left(P(z\exp(tX))\right) \\ & = & d_zP(zX) \\ & = & \frac{\partial P}{\partial z_1}(z)\,(zX)_1 + \frac{\partial P}{\partial z_2}(z)\,(zX)_2\,. \end{array}$$

Since $zA_1 = (-iz_1\ \ iz_2)$, $zA_2 = (iz_2\ \ iz_1)$ and $zA_3 = (-z_2\ \ z_1)$ we have, for every $k \in \{0,\dots,n\}$

$$\begin{aligned}
\{T_{\mathrm{I}_2}\pi_n(X_1)\}(P_k) &= a_1\{T_{\mathrm{I}_2}\pi_n(A_1)\}(P_k)\\
&= a_1\left(-iz_1\frac{\partial P_k}{\partial z_1}(z)+iz_2\frac{\partial P_k}{\partial z_2}(z)\right)\\
&= -ia_1\left((n-k)z_1z_1^{n-k-1}z_2^k-kz_2z_1^{n-k}z_2^{k-1}\right)\\
&= -ia_1\left((n-k)z_1^{n-k}z_2^k-kz_1^{n-k}z_2^k\right)\\
&= -ia_1(n-2k)P_k.
\end{aligned}$$

For X_2 we have

$$\begin{aligned}
\{T_{\mathrm{I}_2}\pi_n(X_2)\}(P_k) &= a_2\{T_{\mathrm{I}_2}\pi_n(A_2)\}(P_k)\\
&= a_2\left(iz_2\frac{\partial P_k}{\partial z_1}(z)+iz_1\frac{\partial P_k}{\partial z_2}(z)\right)\\
&= ia_2\left((n-k)z_1^{n-k-1}z_2^{k+1}+kz_1^{n-k+1}z_2^{k-1}\right)\\
&= ia_2\left((n-k)P_{k+1}+kP_{k-1}\right),
\end{aligned}$$

and for X_3 we obtain

$$\begin{aligned}
\{T_{\mathrm{I}_2}\pi_n(X_3)\}(P_k) &= a_3\{T_{\mathrm{I}_2}\pi_n(A_3)\}(P_k)\\
&= a_3\left(-z_2\frac{\partial P_k}{\partial z_1}(z)+z_1\frac{\partial P_k}{\partial z_2}(z)\right)\\
&= a_3\left(-(n-k)z_1^{n-k-1}z_2^{k+1}+kz_1^{n-k+1}z_2^{k-1}\right)\\
&= a_3\left(-(n-k)P_{k+1}+kP_{k-1}\right).
\end{aligned}$$

Note that the above expressions for $\{T_{\mathrm{I}_2}\pi_n(X_2)\}(P_k)$ and $\{T_{\mathrm{I}_2}\pi_n(X_3)\}(P_k)$ are also valid for $k=0$ or $k=n$ with the convention $P_{-1}=P_{n+1}=0$. The result follows. $\square$

We now compute the component D_n of the Dirac operator of M acting on $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$, see (2.3). We adopt henceforth the following convention: $F_k := G_k := 0$ as soon as $k \notin \{0,\dots,n\}$.
The fix part of D_n has already been computed in Proposition 2.3.5, so that only the endomorphism D_n' of $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ given by

$$D_n'A = -\sum_{j=1}^{3} e_j\cdot A\circ T_{\mathrm{I}_2}\pi_n(X_j)$$

for every $A \in \mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$, remains to be made explicit.
First note that the Clifford product by e_j can be identified with the matrix multiplication by A_j for $j \in \{1,2,3\}$.
Furthermore, it is straightforward to show using Lemma 2.3.7 that, for every $k \in \{0,1,\dots,n\}$,

$$\begin{aligned}
F_k\circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)F_k\\
F_k\circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\left((n-k+1)G_{k-1}+(k+1)G_{k+1}\right)\\
F_k\circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\left(-(n-k+1)G_{k-1}+(k+1)G_{k+1}\right).
\end{aligned}$$

Those identities still hold for $k=0$ or n using our convention above on the F_k's and G_k's. To obtain the corresponding identities on the G_k's one just has to exchange the roles of F_l and G_l for every l:

$$\begin{aligned}
G_k \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)G_k \\
G_k \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\left((n-k+1)F_{k-1} + (k+1)F_{k+1}\right) \\
G_k \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\left(-(n-k+1)F_{k-1} + (k+1)F_{k+1}\right).
\end{aligned}$$

We deduce the following set of identities:

$$\begin{aligned}
(F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)(F_k \mp F_{n-k}) \\
(F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\Big((k+1)(G_{k+1} \pm G_{n-k-1}) \\
&\qquad + (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
(F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\Big((k+1)(G_{k+1} \mp G_{n-k-1}) \\
&\qquad - (n-k+1)(G_{k-1} \mp G_{n-k+1})\Big) \\
(G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)(G_k \mp G_{n-k}) \\
(G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\Big((k+1)(F_{k+1} \pm F_{n-k-1}) \\
&\qquad + (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
(G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\Big((k+1)(F_{k+1} \mp F_{n-k-1}) \\
&\qquad - (n-k+1)(F_{k-1} \mp F_{n-k+1})\Big).
\end{aligned} \tag{2.5}$$

On the other hand, it is also a short calculation to show

$$\begin{aligned}
A_1 \cdot (F_k \pm F_{n-k}) &= (-1)^{k+1} i(F_k \mp F_{n-k}) \\
A_2 \cdot (F_k \pm F_{n-k}) &= i(G_k \pm G_{n-k}) \\
A_3 \cdot (F_k \pm F_{n-k}) &= (-1)^{k+1}(G_k \mp G_{n-k}) \\
A_1 \cdot (G_k \pm G_{n-k}) &= (-1)^{k} i(G_k \mp G_{n-k}) \\
A_2 \cdot (G_k \pm G_{n-k}) &= i(F_k \pm F_{n-k}) \\
A_3 \cdot (G_k \pm G_{n-k}) &= (-1)^{k}(F_k \mp F_{n-k}).
\end{aligned} \tag{2.6}$$

Bringing (2.5) and (2.6) together we deduce that

$$
\begin{aligned}
& D_n'(F_k \pm F_{n-k}) = -\sum_{j=1}^{3} e_j \cdot (F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_j) \\
= \quad & -\sum_{j=1}^{3} A_j \cdot (F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_j) \\
\overset{(2.5)}{=} \quad & ia_1(n-2k)A_1 \cdot (F_k \mp F_{n-k}) \\
& -ia_2A_2 \cdot \Big((k+1)(G_{k+1} \pm G_{n-k-1}) + (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
& -a_3A_3 \cdot \Big((k+1)(G_{k+1} \mp G_{n-k-1}) - (n-k+1)(G_{k-1} \mp G_{n-k+1})\Big) \\
\overset{(2.6)}{=} \quad & (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\
& +a_2\Big((k+1)(F_{k+1} \pm F_{n-k-1}) + (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
& +(-1)^k a_3\Big((k+1)(F_{k+1} \pm F_{n-k-1}) - (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
= \quad & (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\
& +(k+1)(a_2+(-1)^k a_3)(F_{k+1} \pm F_{n-k-1}) \\
& +(n-k+1)(a_2-(-1)^k a_3)(F_{k-1} \pm F_{n-k+1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& D_n'(G_k \pm G_{n-k}) = -\sum_{j=1}^{3} A_j \cdot (G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_j) \\
\overset{(2.5)}{=} \quad & ia_1(n-2k)A_1 \cdot (G_k \mp G_{n-k}) \\
& -ia_2A_2 \cdot \Big((k+1)(F_{k+1} \pm F_{n-k-1}) + (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
& -a_3A_3 \cdot \Big((k+1)(F_{k+1} \mp F_{n-k-1}) - (n-k+1)(F_{k-1} \mp F_{n-k+1})\Big) \\
\overset{(2.6)}{=} \quad & -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\
& +a_2\Big((k+1)(G_{k+1} \pm G_{n-k-1}) + (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
& -(-1)^k a_3\Big((k+1)(G_{k+1} \pm G_{n-k-1}) - (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
= \quad & -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\
& +(k+1)(a_2-(-1)^k a_3)(G_{k+1} \pm G_{n-k-1}) \\
& +(n-k+1)(a_2+(-1)^k a_3)(G_{k-1} \pm G_{n-k+1}).
\end{aligned}
$$

Note that, for $k=\frac{n-1}{2}$, $F_{k+1}\pm F_{n-k-1}=\pm(F_k\pm F_{n-k})$ and the same holds for the G_k's, so that

$$\begin{aligned}
& D_n'(F_{\frac{n-1}{2}}\pm F_{\frac{n+1}{2}}) \\
= \;& (-1)^{\frac{n-1}{2}}a_1(F_{\frac{n-1}{2}}\pm F_{\frac{n+1}{2}}) \\
& +\frac{n+1}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n+1}{2}}\pm F_{\frac{n-1}{2}}) \\
& +\frac{n+3}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n-3}{2}}\pm F_{\frac{n+3}{2}}) \\
= \;& \Big((-1)^{\frac{n-1}{2}}a_1\pm\frac{n+1}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)\Big)(F_{\frac{n-1}{2}}\pm F_{\frac{n+1}{2}}) \\
& +\frac{n+3}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n-3}{2}}\pm F_{\frac{n+3}{2}})
\end{aligned}$$

and in the same way

$$\begin{aligned}
& D_n'(G_{\frac{n-1}{2}}\pm G_{\frac{n+1}{2}}) \\
= \;& -(-1)^{\frac{n-1}{2}}a_1(G_{\frac{n-1}{2}}\pm G_{\frac{n+1}{2}}) \\
& +\frac{n+1}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n+1}{2}}\pm G_{\frac{n-1}{2}}) \\
& +\frac{n+3}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n-3}{2}}\pm G_{\frac{n+3}{2}}) \\
= \;& \Big(-(-1)^{\frac{n-1}{2}}a_1\pm\frac{n+1}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)\Big)(G_{\frac{n-1}{2}}\pm G_{\frac{n+1}{2}}) \\
& +\frac{n+3}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n-3}{2}}\pm G_{\frac{n+3}{2}}).
\end{aligned}$$

Denoting by $(v_0,\ldots,v_{\frac{n-1}{2}})$ the basis of $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ computed in Lemma 2.3.6 we conclude the proof of Theorem 2.1.1 *iv*).

Note 2.3.8 From Theorem 2.1.1 *iv*) the matrix representing the operator D_n in the basis $(v_0,\ldots,v_{\frac{n-1}{2}})$ is not symmetric. Beware however that this basis does not take A_1,A_2,A_3 into account the same way and turns out not to be orthonormal.

We now make the eigenvalue of D_1 explicit:

Corollary 2.3.9 *Fix $j\in\{0,1,2,3\}$ and let $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\{-1,1\}$ be defined by $\varepsilon_l:=-(-1)^{\delta_{j0}+\delta_{jl}}$ for $l\in\{1,2,3\}$. Then under the assumptions of* Theorem 2.1.1 *the following number is an eigenvalue of the Dirac operator of M for the spin structure given by ε_j and the metric induced by a_1,a_2,a_3:*

$$\frac{-(\varepsilon_2a_2-\varepsilon_3a_3)^2a_1^2+2a_2a_3(\varepsilon_2a_2+\varepsilon_3a_3)a_1-a_2^2a_3^2}{2a_1a_2a_3}.$$

If in particular $\varepsilon_2\varepsilon_3a_2a_3>0$ then there exists $a_1\in\mathbb{R}^*$ such that for the corresponding metric the Dirac operator of M has a non-zero kernel.

Proof: For $n=1$ the operator D'_n can be expressed from Theorem 2.1.1 as

$$D'_1 = (\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3)\mathrm{Id}$$

for the ε_l's defined above (beware that they depend on j). Therefore the corresponding Dirac operator D_n is given by

$$\begin{aligned} D_1 &= \Big(\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3 - \frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2a_1 a_2 a_3}\Big)\mathrm{Id} \\ &= \frac{-(\varepsilon_2 a_2 - \varepsilon_3 a_3)^2 a_1^2 + 2a_2 a_3(\varepsilon_2 a_2 + \varepsilon_3 a_3)a_1 - a_2^2 a_3^2}{2a_1 a_2 a_3}\mathrm{Id}, \end{aligned}$$

from which the first statement follows.
An elementary computation shows that, if $\varepsilon_2\varepsilon_3 a_2 a_3 > 0$, then the numerator of the eigenvalue vanishes for

$$a_1 = \frac{a_2 a_3(\varepsilon_2 a_2 + \varepsilon_3 a_3) \pm 2(\varepsilon_2\varepsilon_3 a_2 a_3)^{\frac{3}{2}}}{(\varepsilon_2 a_2 - \varepsilon_3 a_3)^2}$$

in the case $\varepsilon_2 a_2 \neq \varepsilon_3 a_3$ and

$$a_1 = \frac{\varepsilon_2 a_3}{4}$$

if $\varepsilon_2 a_2 = \varepsilon_3 a_3$. Note that none of those numbers can vanish because of $a_2 a_3 \neq 0$. This concludes the proof. □

Notes 2.3.10

1. It follows from Corollary 2.3.9 that, for any given spin structure on M, there exists a 2-parameter-family of Riemannian metrics for which M admits non-zero harmonic spinors. This is not a surprise since the existence of such metrics already follows from a purely theoretical result by Christian Bär [A4]. However we can make some of those metrics explicit here.

2. There may exist non-zero harmonic spinors for other metrics on M and possibly without needing the condition $\varepsilon_2\varepsilon_3 a_2 a_3 > 0$ from Corollary 2.3.9, since we have up to now only studied the eigenvalue corresponding to one particular representation.

3. In the same way the eigenvalue computed in Corollary 2.3.9 is not necessarily the smallest one in absolute value. Choose for example the ε_0-spin structure, $a_2 = a_3 < 0$ and $a_1 \in]-\frac{a_2}{8}, -\frac{a_2}{2}[$. Then $\frac{4a_1a_2 - a_2^2}{2a_1}$ and $-\frac{8a_1a_2+a_2^2}{2a_1}$ are eigenvalues of the Dirac operator of M, the first one corresponding to $n=1$ (i.e., to the one computed in Corollary 2.3.9) and the second one to $n=3$, see Corollary 2.1.2. However one has from the assumptions on a_1, a_2, a_3 that $|-\frac{8a_1a_2+a_2^2}{2a_1}| < |\frac{4a_1a_2-a_2^2}{2a_1}|$.

We end this section with an important remark which actually constitutes the main motivation for this work. The manifold M can be seen as hypersurface of the 4-dimensional round sphere S^4 (with sectional curvature 1): consider the manifold $\{A \in \mathrm{M}_{3\times 3}(\mathbb{R}),\ {}^tA = A,\ \mathrm{tr}(A) = 0 \text{ and } \mathrm{tr}(A^2) = 2\} \cong S^4$ with metric $(A,B) \longmapsto \langle A,B\rangle := \frac{1}{2}\mathrm{tr}(AB)$. Let $B := \mathrm{diag}(\lambda, -\lambda-\mu, \mu) \in S^4$ where $\lambda, \mu \in \mathbb{R}$ satisfy $\lambda + 2\mu \neq 0$, $\lambda \neq \mu$, $\mu + 2\lambda \neq 0$ and $\lambda^2 + (\lambda+\mu)^2 + \mu^2 = 2$. Set

$$N := \{\pi(P)\cdot B\cdot \pi(P)^{-1},\ P \in \mathrm{SU}_2\} \subset S^4,$$

where $\mathrm{SU}_2 \xrightarrow{\pi} \mathrm{SO}_3$ is the universal 2-fold covering map. Then it is an elementary exercise to show that N is a hypersurface of S^4 which is diffeomorphic to $\mathrm{SU}_2/\mathrm{Q}_8$, that the homogeneous metric induced by the inclusion map $N \subset S^4$ is given by $a_1 := -\frac{1}{2(\lambda+2\mu)}, a_2 := \frac{1}{2(\mu-\lambda)}, a_3 := \frac{1}{2(\mu+2\lambda)}$ and that choosing $\nu_B := \frac{1}{\sqrt{3}}\mathrm{diag}(2\mu+\lambda, \lambda-\mu, -2\lambda-\mu) \in T_BS^4$ as unit normal vector field the induced spin structure on N is the ε_0-one. Here beware that the metrics obtained form a one-parameter strict subfamily of that of all homogeneous metrics on M.
Furthermore, the Weingarten endomorphism-field of N w.r.t. ν_B - seen as endomorphism of $\mathfrak{su}(2)$ - is given in the basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ by

$$\mathrm{Mat}(\mathscr{A}) = \sqrt{3}\cdot \mathrm{diag}(\frac{\lambda}{2\mu+\lambda}, \frac{\mu+\lambda}{\mu-\lambda}, -\frac{\mu}{2\lambda+\mu}).$$

In particular, the mean curvature $\mathscr{H} := \frac{1}{3}\mathrm{tr}(\mathscr{A})$ of N in S^4 w.r.t. ν_B is

$$\mathscr{H} = \frac{3\sqrt{3}\cdot\lambda\mu(\lambda+\mu)}{(2\mu+\lambda)(\mu-\lambda)(2\lambda+\mu)}.$$

Corollary 2.3.11 *Under the hypotheses of* Theorem 2.1.1 *assume furthermore that M sits in S^4, i.e., that $a_1 = -\frac{1}{2(\lambda+2\mu)}, a_2 = \frac{1}{2(\mu-\lambda)}, a_3 = \frac{1}{2(\mu+2\lambda)}$ for some $\lambda, \mu \in \mathbb{R}$ satisfying $\lambda + 2\mu \neq 0$, $\lambda \neq \mu$, $\mu + 2\lambda \neq 0$ and $\lambda^2 + (\lambda+\mu)^2 + \mu^2 = 2$. Then $\frac{9}{4}(\mathscr{H}^2+1)$ is an eigenvalue of the Dirac Laplacian of M for the induced (ε_0-)spin structure.*

Proof: The result follows straightforward from Corollary 2.3.9 in the case $j = 0$ and from an elementary computation giving

$$\begin{aligned}\frac{9}{4}(\mathscr{H}^2+1) &= \frac{9}{(\lambda+2\mu)^2(\mu-\lambda)^2(\mu+2\lambda)^2}\\ &= \left(\frac{-(a_2-a_3)^2a_1^2 + 2a_2a_3(a_2+a_3)a_1 - a_2^2a_3^2}{2a_1a_2a_3}\right)^2.\end{aligned}$$

$\square$

Corollary 2.3.11 confirms what had been already noticed since Christian Bär's work [A5] on extrinsic upper eigenvalue bounds for the lower part of the Dirac spectrum: for any compact orientable hypersurface $\overline{M}^m$ with constant mean curvature $\mathscr{H}$ (and carrying the induced metric and spin structure) in the $(m+1)$-dimensional round sphere the number $\frac{m^2}{4}(\mathscr{H}^2+1)$ is an eigenvalue of its Dirac Laplacian. However the question still remains open whether this eigenvalue should be the smallest one or not.

2.4 Computation of the spectrum of the Dirac operator on M for particular metrics

Although the matrices representing the Dirac operator D of M have a "simple" shape (they are tridiagonal, see Theorem 2.1.1), their spectrum is still hard to compute explicitly since there does not exist any general formula giving the eigenvalues of such matrices. It is however possible to compute them for particular values of the parameters $a_1, a_2, a_3 \in \mathbb{R}^*$, i.e., for particular metrics on M. In Corollary 2.1.2 we do it for the so-called Berger metrics on M (compare with [A2, p.71] where the author chooses $a_2 = 1 = -a_3$ and $a_1 = -\frac{1}{T}$ with $T > 0$).
Namely, if we assume that $a_2 = a_3$ then the identities for $D'_n(F_k \pm F_{n-k})$ and $D'_n(G_k \pm G_{n-k})$ become

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\ &\quad +(k+1)(1+(-1)^k)a_2(F_{k+1} \pm F_{n-k-1}) \\ &\quad +(n-k+1)(1-(-1)^k)a_2(F_{k-1} \pm F_{n-k+1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\ &\quad +(k+1)(1-(-1)^k)a_2(G_{k+1} \pm G_{n-k-1}) \\ &\quad +(n-k+1)(1+(-1)^k)a_2(G_{k-1} \pm G_{n-k+1}) \end{aligned}$$

for every $k \in \{0, \dots, \frac{n-1}{2}\}$. In particular, if k is even, then

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= a_1(n-2k)(F_k \pm F_{n-k}) \\ &\quad +2(k+1)a_2(F_{k+1} \pm F_{n-k-1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= -a_1(n-2k)(G_k \pm G_{n-k}) \\ &\quad +2(n-k+1)a_2(G_{k-1} \pm G_{n-k+1}). \end{aligned}$$

If k is odd then

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= -a_1(n-2k)(F_k \pm F_{n-k}) \\ &\quad +2(n-k+1)a_2(F_{k-1} \pm F_{n-k+1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= a_1(n-2k)(G_k \pm G_{n-k}) \\ &\quad +2(k+1)a_2(G_{k+1} \pm G_{n-k-1}). \end{aligned}$$

We now consider each case separately. Remember that from Theorem 2.3.1 the Dirac operator D restricted to $V_n \otimes \mathrm{Hom}_{Q_8,\varepsilon_j}(V_n, \mathbb{C}^2)$ is given by $\mathrm{Id} \otimes D_n$ where $D_n = D'_n - (\frac{a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2}{2a_1 a_2 a_3})\mathrm{Id}$. In particular the multiplicity of each eigenvalue of D_n should be counted $n+1$ times for the spectrum of D.
• *Case* $j = 0$:

* If $n \equiv 1\ (4)$: It follows from the identities just above and from Lemma 2.3.6 that the matrix of D'_n consists of $\frac{n-1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{0, \ldots, \frac{n-5}{2}\}$ is even and of the isolated eigenvalue $a_1 + (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$). The eigenvalues of each such 2×2-matrix are simple and given by

$$a_1 \pm \sqrt{((n-2k)(n-2(k+1))+1)a_1^2 + 4(n-k)(k+1)a_2^2}$$

with $((n-2k)(n-2(k+1))+1) = (n-2k-1)^2$.

* If $n \equiv 3\ (4)$: It follows from the identities just above and from Lemma 2.3.6 that the matrix of D'_n consists of $\frac{n-3}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{1, \ldots, \frac{n-5}{2}\}$ is odd and of the isolated eigenvalues $-na_1$ (corresponding to $k = 0$) and $a_1 - (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$).

This shows 0.

• *Case* $j = 1$:

* If $n \equiv 1\ (4)$: It follows from the identities just above and from Lemma 2.3.6 that the matrix of D'_n consists of $\frac{n-1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{0, \ldots, \frac{n-5}{2}\}$ is even and of the isolated eigenvalue $a_1 - (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$). The eigenvalues of each such 2×2-matrix have already been computed in the case $j = 0$ above.

* If $n \equiv 3\ (4)$: It follows from the identities just above and from Lemma 2.3.6 that the matrix of D'_n consists of $\frac{n-3}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{1, \ldots, \frac{n-5}{2}\}$ is odd and of the isolated eigenvalues $-na_1$ (corresponding to $k = 0$) and $a_1 + (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$).

This shows 1.

• *Case* $j = 2$ *or* $j = 3$: Since $a_2 = a_3$ the Dirac spectra for the ε_2- and ε_3- spin structures coincide, see Examples 2.3.4.2 with $\sigma = (2\ 3)$.

* If $n \equiv 1\ (4)$: It follows from the identities just above and from Lemma 2.3.6 that the matrix of D'_n consists of $\frac{n-1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{1, \ldots, \frac{n-3}{2}\}$ is odd and of the isolated eigenvalue $-na_1$ (corresponding to $k = 0$).

* If $n \equiv 3\ (4)$: It follows from the identities just above and from Lemma 2.3.6 that the matrix of D'_n consists of $\frac{n+1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{0, \ldots, \frac{n-3}{2}\}$ is even.

This shows 2. and concludes the proof of Corollary 2.1.2.

Note 2.4.1 Of course one should understand each upper bound (e.g. $\frac{n-5}{2}$) for the possible values of k in Corollary 2.1.2 as follows: if for a given n it is negative then the corresponding eigenvalues do not appear. For example if M carries the ε_0-spin structure and $n = 1$ then $D_n + \frac{2a_1^2+a_2^2}{2a_1}\mathrm{Id}$ has only one eigenvalue, namely $a_1 + 2a_2$ (with multiplicity 2). Similarly, if $j = 2, 3$ and $n = 1$, then only $-a_1$ appears with multiplicity 2.

One could in a similar way compute the spectrum of the Dirac operator for $a_2 = -a_3$, in which case the spectra would coincide for the ε_0- and the ε_1-spin structure on M (use Examples 2.3.4).

We end this section with deriving from Corollary 2.1.2 the spectrum of the Dirac operator on M for any of the 4 spin structures and the following metrics: for one of the metrics with constant sectional curvature and for one of the 6 metrics induced by minimal isometric embeddings into S^4 (i.e., for $(\lambda = 0, \mu = \pm 1)$, $(\lambda = \pm 1, \mu = 0)$ or $(\lambda, \mu) = \pm(1, -1)$, see end of Section 2.3). In the first case the spectrum has already been computed by Christian Bär in [A3, Thm. 2] and it can be easily checked that his results coincide with ours.

Corollary 2.4.2 *Under the hypotheses of* Theorem 2.1.1, *assume furthermore that*

i) $a_1 = a_2 = a_3 = 1$. *Then the spectrum of the Dirac operator of M w.r.t. the ε_0-spin structure consists of the family*

$$\left| \begin{array}{ll} \frac{3}{2} + 4k & \textit{with multiplicity } 2(k+1)(2k+1) \\ \frac{3}{2} + 4k + 2 & \textit{with multiplicity } 4k(k+1) \\ -\frac{3}{2} - 4k - 1 & \textit{with multiplicity } 2k(2k+1) \\ -\frac{3}{2} - 4k - 3 & \textit{with multiplicity } 4(k+1)(k+2) \end{array} \right.$$

where k runs over $\mathbb{N}$ and w.r.t. any of the other spin structures ε_j of the family

$$\left|\begin{array}{ll} \frac{3}{2}+4k & \textit{with multiplicity } 2k(2k+1) \\ \frac{3}{2}+4k+2 & \textit{with multiplicity } 4(k+1)^2 \\ -\frac{3}{2}-4k-1 & \textit{with multiplicity } 2(k+1)(2k+1) \\ -\frac{3}{2}-4k-3 & \textit{with multiplicity } 4(k+1)^2 \end{array}\right.$$

where k runs over $\mathbb{N}$.

ii) $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$. *Then the spectrum of the Dirac operator of M*

* *w.r.t. the ε_0-spin structure is given by*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{\frac{1}{2} \pm \frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)} \;\Big|\; k \in \{0,\ldots,\frac{n-5}{2}\} \text{ even}, \frac{n}{2}+1\Big\}$$
$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{\frac{1}{2} \pm \frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)} \;\Big|\; k \in \{1,\ldots,\frac{n-5}{2}\} \text{ odd}, -\frac{n}{2}, \frac{n+3}{4}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

* *w.r.t. the ε_1-spin structure is given by*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{\frac{1}{2} \pm \frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)} \;\Big|\; k \in \{0,\ldots,\frac{n-5}{2}\} \text{ even}, -\frac{n}{2}\Big\}$$
$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{\frac{1}{2} \pm \frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)} \;\Big|\; k \in \{1,\ldots,\frac{n-5}{2}\} \text{ odd}, \frac{n}{2}+1, \frac{n+3}{4}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

* *w.r.t. the ε_2- or ε_3-spin structure is given by*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$|\, k\in\{1,\ldots,\frac{n-3}{2}\}\text{ odd}, \frac{n+3}{4}\Big\}$$
$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$|\, k\in\{0,\ldots,\frac{n-3}{2}\}\text{ even}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

Proof: In case $a_1=a_2=a_3=1$ one has on the one hand

$$(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2=(n+1)^2$$

for every possible k and on the other hand $\frac{2a_1^2+a_2^2}{2a_1}=\frac{3}{2}$. The result in $i)$ straightforward follows using Corollary 2.1.2 and Examples 2.3.4.
Assuming now $a_1=-\frac{1}{4}$ and $a_2=a_3=\frac{1}{2}$, one has

$$\begin{aligned} a_1 &\pm \sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}\\ &=-\frac{1}{4}\pm\frac{\sqrt{(n-2k-1)^2+16(n-k)(k+1)}}{4} \end{aligned}$$

and $\frac{2a_1^2+a_2^2}{2a_1}=-\frac{3}{4}$. This concludes the proof. $\square$

One can deduce from Corollary 2.4.2 and Examples 2.3.4 the spectrum of the Dirac operator of M for any spin structure and any metric induced by (a_1,a_1,a_1) with $a_1\in\mathbb{R}^*$ or any metric induced by a minimal embedding into S^4: in the first case rescale by a_1, in the second one exchange the roles of a_1,a_2,a_3 and possibly multiply all of them by -1.
For the next corollary recall that, for a given $\beta\in\mathbb{C}$, a β-Killing spinor on a spin manifold N is a smooth section ψ of the spinor bundle of N such that $\nabla_X\psi=\beta X\cdot\psi$ for every $X\in TN$.

Corollary 2.4.3 *Under the hypotheses of* Theorem 2.1.1 *the following holds:*

i) *If $a_1=a_2=a_3=1$ then the ε_0-spin structure is the only one for which M admits a non-zero space of Killing spinors, which is then 2-dimensional and associated to the constant $\beta=-\frac{1}{2}$. In particular $\frac{3}{2}$ is in absolute value the smallest eigenvalue of the Dirac operator of M for the ε_0-spin structure.*

ii) *If $a_1=-\frac{1}{4}, a_2=a_3=\frac{1}{2}$ and M carries the ε_0-spin structure then $\frac{3}{2}$ is in absolute value the smallest eigenvalue of the Dirac operator of M. In particular inequality* (2.1) *is an equality on M for the induced metric and spin structure.*

Proof: If $a_1 = a_2 = a_3 = 1$ then on the one hand the metric induced on M has constant sectional curvature 1; on the other hand Corollary 2.4.2 $i)$ implies that the smallest eigenvalue in absolute value of the Dirac operator of M is $\frac{3}{2}$ with multiplicity 2 w.r.t. the ε_0-spin structure and $-\frac{5}{2}$ with multiplicity 2 w.r.t. any of the other spin structures (both obtained for $n = 1$, i.e., they are the eigenvalues computed in Corollary 2.3.9). Now M carries a non-trivial Killing spinor if and only if the smallest eigenvalue of its Dirac Laplacian coincides with T. Friedrich's lower bound $\frac{3}{4(3-1)} \inf_M(\mathrm{Scal}_M)$ in terms of the scalar curvature of M, see [A7]. Here $\frac{3}{4(3-1)}\mathrm{Scal}_M = \frac{9}{4}$ so that M carries a 2-dimensional space of non-zero Killing spinors only for the ε_0-spin structure; in that case the corresponding constant β should obviously be $-\frac{1}{2}$. This shows $i)$.
If $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$ and M carries the ε_0-spin structure then from Corollary 2.4.2 $ii)$ the eigenvalues corresponding to $n = 1$ and $n = 3$ are $\frac{3}{2}$ and $-\frac{3}{2}, \frac{3}{2}$ with multiplicities 2, 4 and 4 respectively. Next we show that all eigenvalues corresponding to $n \geq 5$ are greater than $\frac{3}{2}$ in absolute value. Since this is obviously the case for $\frac{n}{2}+1$, $-\frac{n}{2}$ and $\frac{n+3}{4}$ we just have to deal with the eigenvalues $\frac{1}{2} \pm \frac{1}{4}\sqrt{(n-2k-1)^2 + 16(n-k)(k+1)}$, of which absolute value is greater than $\frac{3}{2}$ if and only if

$$(n-2k-1)^2 + 16(n-k)(k+1) - 64 > 0 \tag{2.7}$$

for every $k \in \{0, \ldots, \frac{n-5}{2}\}$. The l.h.s. of (2.7) is a trinomial in k with negative dominant coefficient and of which roots are given by $\frac{n-1}{2} \pm \sqrt{\frac{(n-3)(n+5)}{3}}$. If $n \geq 5$ then $\frac{n-1}{2} - \sqrt{\frac{(n-3)(n+5)}{3}} < 0 < \frac{n-1}{2} < \frac{n-1}{2} + \sqrt{\frac{(n-3)(n+5)}{3}}$, which shows that (2.7) is satisfied. Hence $\frac{3}{2}$ is in absolute value the smallest eigenvalue of the Dirac operator. Apply Corollary 2.3.11 to the case $\lambda = 0$ and $\mu = 1$ to conclude. $\square$

That M admits a 2-dimensional space of Killing spinors w.r.t. its ε_0-spin structure and any normal metric is also not a surprise, see [A1, Cor. 5.2.5 (1b)]. Moreover, following the symmetry arguments already used above (see Examples 2.3.4) Corollary 2.4.3 $ii)$ actually holds for any of the metrics induced by a minimal embedding into S^4. This proves Corollary 2.1.3.

Corollary 2.1.3 provides a further example (after geodesic spheres [A5] and generalized Clifford tori [A8]) of homogeneous hypersurface of the round sphere for which Christian Bär's inequality [A5, Cor. 4.3] is an equality for the smallest Dirac eigenvalue. Here it should furthermore be noticed that, still under the assumptions of Corollary 2.1.3, the multiplicity of the smallest eigenvalue of the Dirac Laplacian on M is greater than the corresponding one on the 3-dimensional round sphere. This shows an analogy with the generalized Clifford tori tested in [A8], on which the multiplicity of the smallest eigenvalue of the Dirac Laplacian is also greater than or equal to the corresponding one on the round sphere of same dimension.

We conjecture that the inequality in [A5, Cor. 4.3] for the smallest Dirac eigenvalue is an equality for every homogeneous hypersurface in the round sphere. We refer to [A9] for further work in this direction.

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Chapter 3

The spectrum of the twisted Dirac operator on Kähler submanifolds of the complex projective space

This chapter coincides (up to minor changes such as enumeration of pages, sections, theorems, references etc.) with the published article [30].

Nicolas Ginoux and Georges Habib

Abstract. We establish an upper estimate for the small eigenvalues of the twisted Dirac operator on Kähler submanifolds in Kähler manifolds carrying Kählerian Killing spinors. We then compute the spectrum of the twisted Dirac operator of the canonical embedding $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^n$ in order to test the sharpness of the upper bounds.

3.1 Introduction

One of the basic tools to get upper bounds for the eigenvalues of the twisted Dirac operator on spin submanifolds is the min-max principle. The idea consists in computing in terms of geometric quantities the so-called Rayleigh-quotient applied to some test section coming from the ambient manifold. In [B1], C. Bär established with the help of the min-max principle upper eigenvalue estimates for submanifolds in $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$, estimate which is sharp in the first two cases. In the same spirit, the first-named author studied in his PhD thesis [B6] different situations where the ambient manifold admits natural test-spinors carrying geometric information.

In this paper, we consider a closed spin Kähler submanifold M of a Kähler spin manifold $\widetilde{M}$ and derive upper bounds for the small eigenvalues of the corresponding twisted Dirac operator in case $\widetilde{M}$ carries so-called Kählerian Killing spinors (see (3.3) for a definition). Interestingly enough, the upper bound turns out to depend only on the complex dimension of M (Theorem 3.2.2). Whether this estimate is sharp is a much more involved question. A first approach consists in finding lower bounds for the spectrum and to compare them with the upper ones. In Section 3.3, we prove a Kirchberg-type lower bound for the eigenvalues of any twisted Dirac operator on a closed Kähler manifold (Corollary 3.3.2). Here the curvature of the twisting bundle has to be involved. Even for the canonical embedding $\mathbb{C}P^d \to \mathbb{C}P^n$, the presence of that normal curvature does not allow to state the equality between the lower bound and the upper one, see Proposition 3.3.3. The next approach consists in computing explicitly the spectrum of the twisted Dirac operator, at least for particular embeddings. In Section 3.4, we determine the eigenvalues (with multiplicities) of the twisted Dirac operator of the canonical embedding $\mathbb{C}P^d \to \mathbb{C}P^n$, using earlier results by M. Ben Halima [B3]. We first remark that the spinor bundle of the normal bundle splits into a direct sum of powers of the tautological bundle (Corollary 3.4.4). We deduce the spectrum of the twisted Dirac operator in Theorem 3.4.8, where we also include the multiplicities with the help of Weyl's character formula. We conclude that, for $d < \frac{n+1}{2}$, the twisted Dirac operator admits 0 as a lowest eigenvalue and $(n+1)(2d+1-n)$ for $d \geq \frac{n+1}{2}$ (see Proposition 3.4.9). This implies that, for $d = 1$, the upper estimate is optimal for $n = 3, 5, 7$, however it is no more optimal for $n \geq 9$.

This work is partially based on and extends the first-named author's PhD thesis [B6, Ch. 4].

3.2 Upper bounds for the submanifold Dirac operator of a Kähler submanifold

In this section, we prove *a priori* upper bounds for the smallest eigenvalues of some twisted Dirac operator on complex submanifolds in Kähler manifolds admitting so-called Kählerian Killing spinors.

Let M^{2d} be an immersed almost-complex submanifold in a Kähler manifold $(\widetilde{M}^{2n}, g, J)$ ("almost-complex" means that $J(TM) = TM$). Then for the induced metric and almost-complex structure the manifold (M^{2d}, g, J) is Kähler, in particular its immersion is minimal in $(\widetilde{M}^{2n}, g, J)$. We denote by $\widetilde{\Omega}$, Ω and Ω_N the Kähler form of $(\widetilde{M}^{2n}, g, J)$, (M^{2d}, g, J) and of the normal bundle $NM \longrightarrow M$ of the immersion respectively (in our convention, $\Omega(X,Y) = g(J(X),Y)$ for all X, Y).
Assuming both (M^{2d}, g, J) and $(\widetilde{M}^{2n}, g, J)$ to be spin, the bundle NM carries an induced spin structure such that the restricted (complex) spinor bundle $\Sigma\widetilde{M}_{|_M}$ of $\widetilde{M}$ can be identified with $\Sigma M \otimes \Sigma N$, where ΣM and ΣN are the spinor bundles of M and NM respectively. Denote by "$\underset{M}{\cdot}$", "$\underset{N}{\cdot}$" and "$\cdot$"the Clifford multiplications of M, NM and $\widetilde{M}$ respectively. By a suitable choice of invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ (with associated norm $|\cdot|$) on $\Sigma\widetilde{M}$ the identification above can be made unitary. Moreover, it can be assumed to respect the following rules: given any $X \in TM$ and $\nu \in NM$, one has

$$\left| \begin{array}{ll} X \cdot \varphi & = \{X \underset{M}{\cdot} \otimes (\mathrm{Id}_{\Sigma^+ N} - \mathrm{Id}_{\Sigma^- N})\}\varphi \\ \nu \cdot \varphi & = (\mathrm{Id} \otimes \nu \underset{N}{\cdot})\varphi, \end{array} \right. \tag{3.1}$$

for all $\varphi \in \Sigma\widetilde{M}_{|_M} = \Sigma M \otimes \Sigma N$. Here $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$ stands for the orthogonal and parallel splitting induced by the complex volume form, see e.g. [B6, Sec. 1.2.1] or [B9, Sec. 2.1]. The following Gauss-type formula holds for the spinorial Levi-Civita connections $\widetilde{\nabla}$ and $\nabla := \nabla^{\Sigma M\otimes\Sigma N}$ on $\Sigma\widetilde{M}$ and $\Sigma M\otimes\Sigma N$ respectively: for all $X\in TM$ and $\varphi\in\Gamma(\Sigma\widetilde{M}_{|_M})$,

$$\widetilde{\nabla}_X\varphi=\nabla_X\varphi+\frac{1}{2}\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\varphi, \tag{3.2}$$

where $(e_j)_{1\leq j\leq 2d}$ is any local orthonormal basis of TM and II the second fundamental form of the immersion.

Recall that, for a complex constant α, an α-Kählerian Killing spinor on a Kähler spin manifold $(\widetilde{M}^{2n},g,J)$ is a pair (ψ,ϕ) of spinors satisfying, for all $X\in T\widetilde{M}$,

$$\left|\begin{array}{ll}\widetilde{\nabla}_X\psi & =-\alpha p_-(X)\cdot\phi\\ \widetilde{\nabla}_X\phi & =-\alpha p_+(X)\cdot\psi,\end{array}\right. \tag{3.3}$$

where $p_\pm(X):=\frac{1}{2}(X\mp iJ(X))$. The existence of a non-zero α-Kählerian Killing spinor on $(\widetilde{M}^{2n},g,J)$ imposes the metric to be Einstein with scalar curvature $\widetilde{S}=4n(n+1)\alpha^2$ (in particular α must be either real or purely imaginary), the complex dimension n of $\widetilde{M}$ to be odd and the spinors ψ,ϕ to lie in particular eigenspaces of the Clifford action of $\widetilde{\Omega}$, namely

$$\left|\begin{array}{ll}\widetilde{\Omega}\cdot\psi & =-i\psi\\ \widetilde{\Omega}\cdot\phi & =i\phi.\end{array}\right. \tag{3.4}$$

Actually a Kähler spin manifold carries a non-zero α-Kählerian Killing spinor with $\alpha\in\mathbb{R}^\times$ if and only if it is the twistor-space of a quaternionic-Kähler manifold with positive scalar curvature (in particular it must be $\mathbb{C}\mathrm{P}^n$ if $n\equiv 1\,(4)$), see [B13]. For purely imaginary α only partial results are known, the prominent examples being the complex hyperbolic space [B11, Thm. 13] as well as doubly-warped products associated to some circle bundles over hyperkähler manifolds [B10].
We need the following lemma [B6, Lemme 4.4]:

Lemma 3.2.1 *Let (M^{2d},g,J) be a Kähler spin submanifold of a Kähler spin manifold $(\widetilde{M}^{2n},g,J)$ and assume the existence of an α-Kählerian Killing spinor (ψ,ϕ) on $(\widetilde{M}^{2n},g,J)$. Then*

$$(D_M^{\Sigma N})^2(\psi+\phi)=(d+1)^2\alpha^2(\psi+\phi)+\alpha^2\Omega_N\cdot\Omega_N\cdot(\psi+\phi). \tag{3.5}$$

Proof: Fix a local orthonormal basis $(e_j)_{1\leq j\leq 2n}$ of $T\widetilde{M}_{|_M}$ with $e_j\in TM$ for all $1\leq j\leq 2d$ and $e_j\in NM$ for all $2d+1\leq j\leq 2n$. Introduce the auxiliary Dirac-type operator $\widehat{D}:=\sum_{j=1}^{2d}e_j\cdot\widetilde{\nabla}_{e_j}:\Gamma(\Sigma\widetilde{M}_{|_M})\longrightarrow\Gamma(\Sigma\widetilde{M}_{|_M})$. As a consequence of the Gauss-type formula (3.2), the operators $\widehat{D}^2$ and $(D_M^{\Sigma N})^2$ are related by [B6, Lemme 4.1]

$$\widehat{D}^2\varphi=\left(D_M^{\Sigma N}\right)^2\varphi-d^2|H|^2\varphi-d\sum_{j=1}^{2d}e_j\cdot\nabla^N_{e_j}H\cdot\varphi,$$

where $H:=\frac{1}{2d}\mathrm{tr}(II)$ is the mean curvature vector field of the immersion. In particular $\widehat{D}^2$ and $(D_M^{\Sigma N})^2$ coincide as soon as the mean curvature vector field of the immersion vanishes, condition which is fulfilled here. Using $\sum_{j=1}^{2n}p_+(e_j)\cdot p_-(e_j)=i\widetilde{\Omega}-n$ and

$\sum_{j=1}^{2n} p_-(e_j)\cdot p_+(e_j) = -i\widetilde{\Omega} - n$, we compute:

$$\begin{aligned}
\widehat{D}\psi &= \sum_{j=1}^{2d} e_j \cdot \widetilde{\nabla}_{e_j}\psi \\
&\overset{(3.3)}{=} -\alpha \sum_{j=1}^{2d} e_j \cdot p_-(e_j)\cdot\phi \\
&= -\alpha \sum_{j=1}^{2d} p_+(e_j)\cdot p_-(e_j)\cdot\phi \\
&= -\alpha(i\Omega\cdot - d)\phi \\
&= -\alpha(i\widetilde{\Omega}\cdot - d)\phi + i\alpha\Omega_N\cdot\phi \\
&\overset{(3.4)}{=} (d+1)\alpha\phi + i\alpha\Omega_N\cdot\phi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{D}\phi &= \sum_{j=1}^{2d} e_j \cdot \widetilde{\nabla}_{e_j}\phi \\
&\overset{(3.3)}{=} -\alpha \sum_{j=1}^{2d} e_j \cdot p_+(e_j)\cdot\psi \\
&= -\alpha \sum_{j=1}^{2d} p_-(e_j)\cdot p_+(e_j)\cdot\psi \\
&= -\alpha(-i\Omega\cdot - d)\psi \\
&= \alpha(i\widetilde{\Omega}\cdot + d)\psi - i\alpha\Omega_N\cdot\psi \\
&\overset{(3.4)}{=} (d+1)\alpha\psi - i\alpha\Omega_N\cdot\psi,
\end{aligned}$$

so that

$$\widehat{D}(\psi+\phi) = (d+1)\alpha(\psi+\phi) + i\alpha\Omega_N\cdot(\phi-\psi).$$

To compute $\widehat{D}^2(\psi+\phi)$ we need the commutator of $\Omega_N\cdot$ with $\widehat{D}$. For any $\varphi\in\Gamma(\Sigma\widetilde{M}_{|_M})$, one has

$$\begin{aligned}
\widehat{D}(\Omega_N\cdot\varphi) &= \sum_{j=1}^{2d} e_j\cdot\widetilde{\nabla}_{e_j}(\Omega_N\cdot\varphi) \\
&= \sum_{j=1}^{2d} e_j\cdot\widetilde{\nabla}_{e_j}\Omega_N\cdot\varphi + e_j\cdot\Omega_N\cdot\widetilde{\nabla}_{e_j}\varphi \\
&= \sum_{j=1}^{2d} \Omega_N\cdot e_j\cdot\widetilde{\nabla}_{e_j}\varphi + e_j\cdot\widetilde{\nabla}_{e_j}\Omega_N\cdot\varphi \\
&= \Omega_N\cdot\widehat{D}\varphi + \sum_{j=1}^{2d} e_j\cdot\widetilde{\nabla}_{e_j}\Omega_N\cdot\varphi,
\end{aligned}$$

with, for all $X,Y\in TM$ and $\nu\in NM$,

$$\begin{aligned}
(\widetilde{\nabla}_X\Omega_N)(Y,\nu) &= -\Omega_N(\widetilde{\nabla}_X Y,\nu) \\
&= -g(J(\widetilde{\nabla}_X Y),\nu) \\
&= -g(J(II(X,Y)),\nu),
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{j=1}^{2d} e_j\cdot\widetilde{\nabla}_{e_j}\Omega_N\cdot\varphi &= -\sum_{j,k=1}^{2d}\sum_{l=2d+1}^{2n} g(J(II(e_j,e_k)),e_l)e_j\cdot e_k\cdot e_l\cdot\varphi\\
&= -\sum_{j,k=1}^{2d} e_j\cdot e_k\cdot J(II(e_j,e_k))\cdot\varphi\\
&= \sum_{j=1}^{2d} J(II(e_j,e_j))\cdot\varphi\\
&= 0,
\end{aligned}$$

since the immersion is minimal. Hence $\widehat{D}(\Omega_N\cdot\varphi)=\Omega_N\cdot\widehat{D}\varphi$ and we deduce that

$$\begin{aligned}
\widehat{D}^2(\psi+\phi) &= (d+1)\alpha\widehat{D}(\psi+\phi)+i\alpha\widehat{D}(\Omega_N\cdot(\phi-\psi))\\
&= (d+1)^2\alpha^2(\psi+\phi)+i(d+1)\alpha^2\Omega_N\cdot(\phi-\psi)+i\alpha\Omega_N\cdot\widehat{D}(\phi-\psi)\\
&= (d+1)^2\alpha^2(\psi+\phi)+i(d+1)\alpha^2\Omega_N\cdot(\phi-\psi)\\
&\quad +i\alpha\Omega_N\cdot((d+1)\alpha(\psi-\phi)-i\alpha\Omega_N\cdot(\psi+\phi))\\
&= (d+1)^2\alpha^2(\psi+\phi)+\alpha^2\Omega_N\cdot\Omega_N\cdot(\psi+\phi),
\end{aligned}$$

which concludes the proof. □

Next we formulate the main theorem of this section. Its proof requires some further notations. Given any rank-$2k$-Hermitian spin bundle $E\longrightarrow M$ with metric connection preserving the complex structure, the Clifford action of the Kähler form Ω_E of E splits the spinor bundle ΣE of E into the orthogonal and parallel sum

$$\Sigma E=\bigoplus_{r=0}^{k}\Sigma_r E, \tag{3.6}$$

where $\Sigma_r E:=\mathrm{Ker}(\Omega_E\cdot -i(2r-k)\mathrm{Id})$ is a subbundle of complex rank $\begin{pmatrix} k\\ r\end{pmatrix}$. Moreover, given any $V\in E$, one has $p_\pm(V)\cdot\Sigma_r E\subset\Sigma_{r\pm1}E$.

Theorem 3.2.2 (see [B6, Thm. 4.2]) *Let (M^{2d},g,J) be a closed Kähler spin submanifold of a Kähler spin manifold $(\widetilde{M}^{2n},g,J)$ and consider the induced spin structure on the normal bundle. Assume the existence of a complex μ-dimensional space of non-zero α-Kählerian Killing spinor on $(\widetilde{M}^{2n},g,J)$ for some $\alpha\in\mathbb{R}^\times$. Then there are μ eigenvalues λ of $(D_M^{\Sigma N})^2$ satisfying*

$$\lambda\leq\begin{cases}(d+1)^2\alpha^2 & \text{if } d \text{ is odd}\\[2ex] d(d+2)\alpha^2 & \text{if } d \text{ is even.}\end{cases} \tag{3.7}$$

If moreover (3.7) *is an equality for the smallest eigenvalue λ and some odd d, then $\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\psi=\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\phi=0$.*

Proof: Let (ψ,ϕ) be a non-zero α-Kählerian Killing spinor on $(\widetilde{M}^{2n},g,J)$. We evaluate the Rayleigh-quotient $\frac{\int_M\langle(D_M^{\Sigma N})^2(\psi+\phi),\psi+\phi\rangle v_g}{\int_M\langle\psi+\phi,\psi+\phi\rangle v_g}$ and apply the min-max principle. It can be deduced from Lemma 3.2.1 that

$$\begin{aligned}\langle (D_M^{\Sigma N})^2(\psi+\phi),\psi+\phi\rangle &= (d+1)^2\alpha^2|\psi+\phi|^2+\alpha^2\langle\Omega_N\cdot\Omega_N\cdot(\psi+\phi),\psi+\phi\rangle\\ &= (d+1)^2\alpha^2|\psi+\phi|^2-\alpha^2|\Omega_N\cdot(\psi+\phi)|^2.\end{aligned}$$

Using (3.6) for $E=NM$ we observe that $|\Omega_N\cdot(\psi+\phi)|\geq|\psi+\phi|$ if $n-d$ is odd (i.e., if d is even) and is nonnegative otherwise. The inequality follows.
If d is odd and (3.7) is an equality for the smallest eigenvalue, then $(D_M^{\Sigma N})^2(\psi+\phi)=(d+1)^2\alpha^2(\psi+\phi)$ and $\Omega_N\cdot(\psi+\phi)=0$. Since $\widetilde{\Omega}=\Omega\oplus\Omega_N$ one has $\Sigma_r\widetilde{M}_{|M}=\bigoplus_{s=0}^r\Sigma_sM\otimes\Sigma_{r-s}M$ (where each component vanishes as soon as the index exceeds its allowed bounds), so that $\psi\in\Gamma(\Sigma_{\frac{d-1}{2}}M\otimes\Sigma_{\frac{n-d}{2}}N)$ and $\phi\in\Gamma(\Sigma_{\frac{d+1}{2}}M\otimes\Sigma_{\frac{n-d}{2}}N)$. Coming back to the Gauss-type equation (3.2), one obtains

$$\left|\begin{array}{ll}\nabla_X\psi &= -\alpha p_-(X)\cdot\phi-\frac{1}{2}\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\psi\\ \nabla_X\phi &= -\alpha p_+(X)\cdot\psi-\frac{1}{2}\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\phi\end{array}\right.$$

for all $X\in TM$. Looking more precisely at the components of each side of those identities, one notices that, pointwise, $\nabla_X\psi\in\Sigma_{\frac{d-1}{2}}M\otimes\Sigma_{\frac{n-d}{2}}N$ and, using (3.1), that $p_-(X)\cdot\phi\in\Sigma_{\frac{d-1}{2}}M\otimes\Sigma_{\frac{n-d}{2}}N$. But pointwise $\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\psi\in(\Sigma_{\frac{d-3}{2}}M\otimes\Sigma_{\frac{n-d-2}{2}}N)\oplus(\Sigma_{\frac{d-3}{2}}M\otimes\Sigma_{\frac{n-d+2}{2}}N)\oplus(\Sigma_{\frac{d+1}{2}}M\otimes\Sigma_{\frac{n-d-2}{2}}N)\oplus(\Sigma_{\frac{d+1}{2}}M\otimes\Sigma_{\frac{n-d+2}{2}}N)$, in particular this term must vanish. Analogously one has $\sum_{j=1}^{2d}e_j\cdot II(X,e_j)\cdot\phi=0$. This concludes the proof. □

To test the sharpness of the estimate (3.7), we would like to first compare it to an *a priori* lower bound. This is the object of the next section.

3.3 Kirchberg-type lower bounds

In this section, we aim at giving Kirchberg type estimates for any twisted Dirac operator on closed Kähler spin manifolds. First consider a Kähler spin manifold M of complex dimension d and let E be any rank $2k$-vector bundle over M endowed with a metric connection. We define a connection on the vector bundle $\Sigma:=\Sigma M\otimes E$ by $\nabla:=\nabla^{\Sigma M\otimes E}$. The Dirac operator of M twisted with E is defined by $D_M^E:\Gamma(\Sigma)\to\Gamma(\Sigma)$, $D_M^E:=\sum_{i=1}^{2d}e_i\cdot\nabla_{e_i}$, where $\{e_i\}_{1\leq i\leq 2d}$ is any local orthonormal basis of TM and "$\cdot$" stands for the Clifford multiplication tensorized with the identity of E. The square of the Dirac-type operator D_M^E is related to the rough Laplacian via the following Schrödinger-Lichnerowicz formula [B12, Thm. II.8.17]

$$(D_M^E)^2=\nabla^*\nabla+\frac{1}{4}(\mathrm{Scal}_M+R^E),$$

where Scal_M denotes the scalar curvature of M and R^E is the endomorphism tensor field given by

$$\begin{aligned}R^E:\Sigma &\longrightarrow \Sigma\\ \psi &\longmapsto 2\sum_{i,j=1}^{2d}(e_i\cdot e_j\cdot\mathrm{Id}\otimes R^E_{e_i,e_j})\psi.\end{aligned}$$

Recall that for any eigenvalue λ of the Dirac operator, there exists an eigenspinor φ associated with λ such that $\varphi = \varphi_r + \varphi_{r+1}$, where φ_r is a section in $\Sigma_r := \Sigma_r M \otimes E$. Here $\Sigma_r M$ is the subundle $\mathrm{Ker}(\Omega \cdot -i(2r-d)\mathrm{Id})$ of ΣM. Such an eigenspinor φ is called of type $(r, r+1)$. In order to estimate the eigenvalues of the twisted Dirac operator we define, as in the classical way, on each subbundle Σ_r the *twisted twistor operator* for all $X \in \Gamma(TM)$, $\psi_r \in \Sigma_r$ by [B5]

$$P_X \psi_r := \nabla_X \psi_r + a_r p_-(X) \cdot D_+ \psi_r + b_r p_+(X) \cdot D_- \psi_r,$$

where $a_r = \frac{1}{2(r+1)}$, $b_r = \frac{1}{2(m-r+1)}$ and $D_\pm \psi_r = \sum_{i=1}^{2d} p_\pm(e_i) \cdot \nabla_{e_i} \psi_r$.

We state the following lemma:

Lemma 3.3.1 *For any eigenspinor φ of type $(r, r+1)$, we have the following inequalities*

$$\lambda^2 \geq \begin{cases} \frac{1}{4(1-a_r)} \inf\limits_{M_{\varphi_r}} (\mathrm{Scal}_M + R^E_{\varphi_r}), \\ \\ \frac{1}{4(1-b_{r+1})} \inf\limits_{M_{\varphi_{r+1}}} (\mathrm{Scal}_M + R^E_{\varphi_{r+1}}), \end{cases} \tag{3.8}$$

where $R^E_\phi := \mathfrak{Re}(R^E(\phi), \frac{\phi}{|\phi|^2})$ is defined on the set $M_\phi = \{x \in M |\ \phi(x) \neq 0\}$ for all spinor $\phi \in \Sigma$.

Proof: Using the identity $\sum_{i=1}^{2d} e_i \cdot P_{e_i} \psi_r = 0$, one can easily prove by a straightforward computation that for any spinor $\psi_r \in \Sigma_r$

$$|P\psi_r|^2 = |\nabla \psi_r|^2 - a_r |D_+ \psi_r|^2 - b_r |D_- \psi_r|^2. \tag{3.9}$$

Applying Equation (3.9) to φ_r and φ_{r+1} respectively and integrating over M, we get with the use of the Schrödinger-Lichnerowicz formula that

$$0 \leq \int_M [\lambda^2 (1-a_r) - \frac{1}{4}(\mathrm{Scal}_M + R^E_{\varphi_r})] |\varphi_r|^2.$$

Also that,

$$0 \leq \int_M [\lambda^2 (1-b_{r+1}) - \frac{1}{4}(\mathrm{Scal}_M + R^E_{\varphi_{r+1}})] |\varphi_{r+1}|^2,$$

from which the proof of the lemma follows. □

One can get rid of the dependence of the eigenspinors φ_r and φ_{r+1} in the r.h.s. of (3.8):

Corollary 3.3.2 *Let κ_1 be the smallest eigenvalue of the (pointwise) self-adjoint operator R^E. Then*

$$\lambda^2 \geq \begin{cases} \frac{d+1}{4d}(\mathrm{Scal}_0 + \kappa_1) & \text{if } d \text{ is odd} \\ \\ \frac{d}{4(d-1)}(\mathrm{Scal}_0 + \kappa_1) & \text{if } d \text{ is even}, \end{cases}$$

where Scal_0 *denotes the infimum of the scalar curvature.*

Proof: Let us choose the lowest integer $r \in \{0, 1, \cdots, d\}$ such that φ is of type $(r, r+1)$. The existence of anti-linear parallel maps on ΣM commuting with the Clifford multiplication (see e.g. [B7, Lemma 1]) allows to impose that $r \leq \frac{d-1}{2}$ if d is odd and $r \leq \frac{d-2}{2}$ if d is even. This concludes the proof. $\square$

In the following, we formulate the estimates (3.8) for the situation where M is a complex submanifold of the projective space $\mathbb{C}\mathrm{P}^n$ and E is the spinor bundle of the normal bundle NM of the immersion. To do this, we will estimate R^E_ϕ for all spinor field $\phi \in \Sigma$ in terms of the second fundamental form of the immersion.

Proposition 3.3.3 *Let (M^{2d}, g, J) be a Kähler spin submanifold of the projective space $\mathbb{C}\mathrm{P}^n$. For all spinor field $\phi \in \Sigma$, the curvature is equal to*

$$R^E_\phi = -4\mathfrak{Re}(\Omega \cdot \Omega_N \cdot \phi, \frac{\phi}{|\phi|^2}) - \sum_{i,j,p=1}^{2d} \mathfrak{Re}(e_i \cdot e_j \cdot II(e_i, e_p) \cdot II(e_j, e_p) \cdot \phi, \frac{\phi}{|\phi|^2}) + |II|^2. \tag{3.10}$$

where Ω is the Kähler form of M.

Proof: First, recall that for all $X, Y \in \Gamma(TM)$ and U, V sections in NM, the normal curvature is related to the one of $\mathbb{C}\mathrm{P}^n$ via the formula [B4, Thm. 1.1.72]

$$\begin{aligned}(R^{NM}_{X,Y}U, V) &= (R^{\mathbb{CP}^n}_{X,Y}U, V) - (B_X U, B_Y V) + (B_Y U, B_X V) \\ &= 2g(X, J(Y))g(J(U), V) - \sum_{p=1}^{2d} g(II(X, e_p), U)g(II(Y, e_p), V) \\ &\quad + \sum_{p=1}^{2d} g(II(Y, e_p), U)g(II(X, e_p), V),\end{aligned} \tag{3.11}$$

where $B_X : NM \to TM$ is the tensor field defined by $g(B_X U, Y) = -g(II(X, Y), U)$ and $\{e_p\}_{1 \leq p \leq 2d}$ is a local orthonormal basis of TM. Here we used the fact that the curvature of $\mathbb{C}\mathrm{P}^n$ is given for all $X, Y, Z \in T\mathbb{C}\mathrm{P}^n$ by

$$R^{\mathbb{CP}^n}_{X,Y}Z = (X \wedge Y + JX \wedge JY + 2g(X, JY)J)Z$$

with $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. Hence by (3.11), the normal spinorial curvature associated with any spinor field ϕ is then equal to

$$\begin{aligned}R^E_{e_i,e_j}\phi &= \frac{1}{4}\sum_{k,l=1}^{2(n-d)} g(R^{NM}_{e_i,e_j}e_k, e_l)e_k \cdot e_l \cdot \phi \\ &= \frac{1}{2}\sum_{k=1}^{2(n-d)} g(e_i, J(e_j))e_k \cdot Je_k \cdot \phi \\ &\quad -\frac{1}{2}\sum_{p=1}^{2d}[II(e_i, e_p) \cdot II(e_j, e_p) \cdot + g(II(e_i, e_p), II(e_j, e_p))]\phi.\end{aligned}$$

Thus, we deduce

$$\begin{aligned} R^E(\phi) &= 2\sum_{i,j=1}^{2d} J(e_j)\cdot e_j\cdot\Omega_N\cdot\phi - \sum_{i,j,p=1}^{2d} e_i\cdot e_j\cdot II(e_i,e_p)\cdot II(e_j,e_p)\cdot\phi \\ &\quad -e_i\cdot e_j\cdot g(II(e_i,e_p),II(e_j,e_p))\phi \\ &= -4\Omega\cdot\Omega_N\cdot\phi - \sum_{i,j,p=1}^{2d} e_i\cdot e_j\cdot II(e_i,e_p)\cdot II(e_j,e_p)\cdot\phi + |II|^2\phi. \end{aligned}$$

Finally, the scalar product of the last equality with $\frac{\phi}{|\phi|^2}$ finishes the proof. □

As we said in the proof of Corollary 3.3.2, the integer r can be chosen such that $r \leq \frac{d-1}{2}$ if d is odd and $r \leq \frac{d-2}{2}$ if d is even. However, we note that *a priori* no such choice can be made for s once r has been fixed. In particular, one cannot conclude that the smallest twisted Dirac eigenvalue of a totally geodesic M in $\widetilde{M}$ is $(d+1)^2$, even in the "simplest" case where $M = \mathbb{C}\mathrm{P}^d$ (the d-dimensional complex projective space). To test the sharpness of the estimate (3.7), we compute in the following section the spectrum of $D_M^{\Sigma N}$ for $M = \mathbb{C}\mathrm{P}^d$ canonically embedded in $\mathbb{C}\mathrm{P}^n$.

3.4 The spectrum of the twisted Dirac operator $D_M^{\Sigma N}$ on the complex projective space

In this section, we compute the spectrum of the Dirac operator of $\mathbb{C}\mathrm{P}^d$ twisted with the spinor bundle of its normal bundle when considered as canonically embedded in $\mathbb{C}\mathrm{P}^n$. The eigenvalues will be deduced from M. Ben Halima's computations [B3, Thm. 1]. We also need to compute the multiplicities in order to compare the upper bound in (3.7) with an eigenvalue which may be greater than the smallest one. The results are gathered in Theorems 3.4.7 and 3.4.8 below.

3.4.1 The complex projective space as a symmetric space

Consider the d-dimensional complex projective space $\mathbb{C}\mathrm{P}^d$ as the right quotient $\mathrm{SU}_{d+1}/\mathrm{S}(\mathrm{U}_d\times\mathrm{U}_1)$, where $\mathrm{S}(\mathrm{U}_d\times\mathrm{U}_1) := \{\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1}\end{pmatrix} \,|\, B\in\mathrm{U}_d\}$. In this section we want to describe its tangent bundle and its normal bundle when canonically embedded into $\mathbb{C}\mathrm{P}^n$ as homogeneous bundles, that is, as bundles associated to the $\mathrm{S}(\mathrm{U}_d\times\mathrm{U}_1)$-principal bundle $\mathrm{SU}_{d+1}\longrightarrow\mathbb{C}\mathrm{P}^d$ via some linear representation of $\mathrm{S}(\mathrm{U}_d\times\mathrm{U}_1)$. The one corresponding to the tangent bundle is called the *isotropy representation* of the homogeneous space $\mathrm{SU}_{d+1}/\mathrm{S}(\mathrm{U}_d\times\mathrm{U}_1)$. To compute it explicitly we consider the following $\mathrm{Ad}(\mathrm{S}(\mathrm{U}_d\times\mathrm{U}_1))$-invariant complementary subspace

$$\mathfrak{m} := \left\{\begin{pmatrix} 0 & \dots & 0 & z_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & z_d \\ -\overline{z_1} & \dots & -\overline{z_d} & 0 \end{pmatrix} \,|\, (z_1,\dots,z_d)\in\mathbb{C}^d\right\} \tag{3.12}$$

to the Lie-Algebra $\mathfrak{h}$ of $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$ in the Lie-algebra $\mathfrak{su}_{d+1} = \{X \in \mathbb{C}(d+1)\,|\,X^* = -X \text{ and } \mathrm{tr}(X) = 0\}$ and fix the (real) basis $(A_1, J(A_1), \ldots, A_d, J(A_d))$ of $\mathfrak{m}$, where:

- $(A_l)_{jk} = 1$ if $(j,k) = (l, d+1)$, -1 if $(j,k) = (d+1, l)$ and 0 otherwise;
- $(J(A_l))_{jk} = i$ if $(j,k) = (l, d+1)$ or $(j,k) = (d+1, l)$ and 0 otherwise.

It is easy to check that J defines a complex structure on $\mathfrak{m}$, which then makes $\mathfrak{m}$ into a d-dimensional complex vector space, and that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. In particular $\mathbb{C}\mathrm{P}^d$ is a symmetric space.

Lemma 3.4.1 *The isotropy representation of the symmetric space* $\mathrm{SU}_{d+1}/\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$ *is given in the complex basis* $(A_1, \ldots, A_d)$ *of* $\mathfrak{m}$ *by:*

$$\begin{array}{rcl} \alpha : \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{U}_d \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & \det(B) \cdot B. \end{array}$$

Proof: For $k \in \{1, \ldots, d\}$ and $B \in \mathrm{U}_d$ we compute

$$\begin{aligned} \mathrm{Ad}\left(\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix}\right)(A_k) &= \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} \cdot A_k \cdot \begin{pmatrix} B^* & 0 \\ 0 & \det(B) \end{pmatrix} \\ &= \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & 0 \\ 0 & \ldots & 0 & \det(B) \\ \vdots & & \vdots & 0 \\ -B^*_{k1} & \ldots & -B^*_{kd} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \ldots & 0 & \det(B)B_{1k} \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \det(B)B_{dk} \\ -\det(B)^{-1}B^*_{k1} & \ldots & -\det(B)^{-1}B^*_{kd} & 0 \end{pmatrix} \\ &= \sum_{j=1}^{d} \mathfrak{Re}(\det(B)B_{jk})A_j + \mathfrak{Im}(\det(B)B_{jk})J(A_j) \\ &= \sum_{j=1}^{d} \det(B)B_{jk}A_j, \end{aligned}$$

which gives the result. □

Recall that the tautological bundle of $\mathbb{C}\mathrm{P}^d$ is the complex line bundle $\gamma_d \longrightarrow \mathbb{C}\mathrm{P}^d$ defined by

$$\gamma_d := \{([z], v)\,|\,[z] \in \mathbb{C}\mathrm{P}^d \text{ and } v \in [z]\}.$$

It carries a canonical Hermitian metric defined by $\langle([z], v), ([z], v')\rangle := \langle v, v'\rangle$.

Lemma 3.4.2 *The normal bundle* $T^\perp\mathbb{C}\mathrm{P}^d$ *of the canonical embedding* $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^n$*,* $[z] \mapsto [z, 0_{n-d}]$*, is unitarily isomorphic to* $\gamma_d^* \otimes \mathbb{C}^{n-d}$*, where* $\gamma_d \longrightarrow \mathbb{C}\mathrm{P}^d$ *is the tautological bundle of* $\mathbb{C}\mathrm{P}^d$ *and* $\mathbb{C}^{n-d}$ *carries its canonical Hermitian inner product. In particular, the homogeneous bundle* $T^\perp\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^d$ *is associated to the* $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$*-principal* *bundle* $\mathrm{SU}_{d+1} \longrightarrow \mathbb{C}\mathrm{P}^d$ *via the representation*

$$\begin{array}{rcl} \rho : \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{U}_{n-d} \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & \det(B)\mathrm{I}_{n-d}. \end{array}$$

Proof: Consider the map

$$\begin{array}{rcl} \mathbb{C}\mathrm{P}^d \times \mathbb{C}^{n-d} & \xrightarrow{\phi} & \gamma_d \otimes T^{\perp}\mathbb{C}\mathrm{P}^d \\ ([z], v) & \longmapsto & ([z], z) \otimes d_z\pi(0_{d+1}, v), \end{array}$$

where $\pi : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}\mathrm{P}^n$ is the canonical projection. It can be easily checked that ϕ is well-defined (the identity $\pi(\lambda z) = \pi(z)$ implies $d_z\pi = \lambda d_{\lambda z}\pi$) and is a unitary vector-bundle-isomorphism. This shows the first statement. Let $(e_1, \ldots, e_{d+1})$ denote the canonical basis of $\mathbb{C}^{d+1}$. The map

$$\begin{array}{rcl} \mathrm{SU}_{d+1} \times \mathbb{C} & \longrightarrow & \gamma_d \\ (A, \lambda) & \longmapsto & ([Ae_{d+1}], \lambda Ae_{d+1}) \end{array}$$

induces a complex vector-bundle-isomorphism $\mathrm{SU}_{d+1} \times \mathbb{C}/_{\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)} \longrightarrow \gamma_d$, where the right action of $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$ onto $\mathrm{SU}_{d+1} \times \mathbb{C}$ is given by $(A, \lambda) \cdot \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} := (A \cdot \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix}, \det(B)\lambda)$. Thus γ_d is isomorphic to the homogeneous bundle over $\mathbb{C}\mathrm{P}^d$ which is associated to the $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$-principal bundle $\mathrm{SU}_{d+1} \longrightarrow \mathbb{C}\mathrm{P}^d$ via the representation $\mathrm{S}(\mathrm{U}_\mathrm{d} \times \mathrm{U}_1) \to \mathrm{U}_1$, $\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} \mapsto \det(B)^{-1}$. This concludes the proof. $\square$

Note in particular that $T^{\perp}\mathbb{C}\mathrm{P}^d$ is not trivial (and hence not flat because of $\pi_1(\mathbb{C}\mathrm{P}^d) = 0$).

3.4.2 Spin structures on $T\mathbb{C}\mathrm{P}^d$ and $T^{\perp}\mathbb{C}\mathrm{P}^d$

From now on we assume that both d and n are odd integers. Then both $T\mathbb{C}\mathrm{P}^d$ and $T\mathbb{C}\mathrm{P}^n$ are spin, in particular $T^{\perp}\mathbb{C}\mathrm{P}^d$ is spin. Since $\mathbb{C}\mathrm{P}^d$ is simply-connected, there is a unique spin structure on $T\mathbb{C}\mathrm{P}^d$ and on $T^{\perp}\mathbb{C}\mathrm{P}^d$. In this section we describe those spin structures as homogeneous spin structures. For that purpose one looks for Lie-group-homomorphisms $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) \xrightarrow{\tilde{\alpha}} \mathrm{Spin}_{2d}$ and $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) \xrightarrow{\tilde{\rho}} \mathrm{Spin}_{2(n-d)}$ lifting α and ρ through the non-trivial two-fold-covering map $\mathrm{Spin}_{2k} \xrightarrow{\xi} \mathrm{SO}_{2k}$.
First we recall the existence for any positive integer k of a Lie-group homomorphism $\mathrm{U}_k \xrightarrow{j} \mathrm{Spin}^c_{2k}$ with $\xi^c \circ j = \iota$, where $\mathrm{Spin}^c_{2k} := \mathrm{Spin}_{2k} \times \mathrm{U}_1/_{\mathbb{Z}_2}$ is the spin^c group, $\xi^c : \mathrm{Spin}^c_{2k} \longrightarrow \mathrm{SO}_{2k} \times \mathrm{U}_1$, $[u, z] \mapsto (\xi(u), z^2)$ is the canonical two-fold-covering map and $\iota : \mathrm{U}_k \longrightarrow \mathrm{SO}_{2k} \times \mathrm{U}_1$, $A \mapsto (A_{\mathbb{R}}, \det(A))$. The Lie-group homomorphism j can be explicitly described on elements of U_k of diagonal form as:

$$j(\mathrm{diag}(e^{i\lambda_1}, \ldots, e^{i\lambda_k})) = e^{\frac{i}{2}\left(\sum_{j=1}^k \lambda_j\right)} \cdot \widetilde{R}_{e_1, J(e_1)}(\frac{\lambda_1}{2}) \cdot \ldots \cdot \widetilde{R}_{e_k, J(e_k)}(\frac{\lambda_k}{2}),$$

where J is the canonical complex structure on $\mathbb{C}^k$ and, for any orthonormal system $\{v, w\}$ in $\mathbb{R}^{2k}$ and $\lambda \in \mathbb{R}$, the element $\widetilde{R}_{v,w}(\lambda) \in \mathrm{Spin}_{2k}$ is defined by

$$\widetilde{R}_{v,w}(\lambda) := \cos(\lambda) + \sin(\lambda) v \cdot w.$$

To keep the notations simple we denote by j both such Lie-group-homomorphisms $\mathrm{U}_d \longrightarrow \mathrm{Spin}^c_{2d}$ and $\mathrm{U}_{n-d} \longrightarrow \mathrm{Spin}^c_{2(n-d)}$.

Lemma 3.4.3 *Let $d < n$ be odd integers.*

1. *The spin structure on $T\mathbb{C}\mathrm{P}^d$ is associated to the $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$-principal bundle $\mathrm{SU}_{d+1} \longrightarrow \mathbb{C}\mathrm{P}^d$ via the Lie-group-homomorphism*

$$\begin{array}{rcl} \tilde{\alpha} : \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{Spin}_{2d} \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & \det(B)^{-\frac{d+1}{2}} \cdot j \circ \alpha(\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix}). \end{array}$$

2. *The spin structure on $T^{\perp}\mathbb{C}\mathrm{P}^d$ is associated to the $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$-principal bundle $\mathrm{SU}_{d+1} \longrightarrow \mathbb{C}\mathrm{P}^d$ via the Lie-group-homomorphism*

$$\begin{array}{rcl} \tilde{\rho} : \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{Spin}_{2(n-d)} \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & \det(B)^{-\frac{n-d}{2}} \cdot j \circ \rho(\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix}). \end{array}$$

Proof: It suffices to prove the results for elements of $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$ of diagonal form. Indeed any element of $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$ is conjugated in SU_{d+1} to such a diagonal matrix. Since SU_{d+1} is simply-connected the map $\mathrm{SU}_{d+1} \to \mathrm{SO}_{2k} \times \mathrm{U}_1$, $P \mapsto (PAP^{-1}, \det(A))$ (where $A \in U_k$ is arbitrary), admits a lift through $\mathrm{Spin}^c_{2k} \xrightarrow{\xi^c} \mathrm{SO}_{2k} \times \mathrm{U}_1$ which is uniquely determined by the image of one single point. Therefore the lifts under consideration are uniquely determined on diagonal elements.
For $\theta_1, \ldots, \theta_d \in \mathbb{R}$ let $M_{\theta_1,\ldots,\theta_d} := \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d}, e^{-i(\sum_{j=1}^d \theta_j)}) \in \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$. Then

$$u_{\theta_1,\ldots,\theta_d} := \widetilde{R}_{e_1,J(e_1)}(\frac{\theta_1 + \sum_{j=1}^d \theta_j}{2}) \cdot \ldots \cdot \widetilde{R}_{e_d,J(e_d)}(\frac{\theta_d + \sum_{j=1}^d \theta_j}{2})$$

lies in Spin_{2d}, only depends on $[\theta_1, \ldots, \theta_d] \in \mathbb{R}^d/2\pi\mathbb{Z}^d$ (if some θ_k is replaced by $\theta_k + 2m\pi$, then $u_{\theta_1,\ldots,\theta_d}$ is replaced by $(-1)^{m(d-1)} u_{\theta_1,\ldots,\theta_d}$, and $d-1$ is even) with $\xi(u_{\theta_1,\ldots,\theta_d}) = \alpha(M_{\theta_1,\ldots,\theta_d})$. Therefore $\tilde{\alpha}(M_{\theta_1,\ldots,\theta_d}) = u_{\theta_1,\ldots,\theta_d}$. Moreover,

$$\begin{aligned} j \circ \alpha(M_{\theta_1,\ldots,\theta_d}) &= e^{\frac{i}{2}\left(\sum_{j=1}^d \theta_j + \sum_{k=1}^d \theta_k\right)} \cdot \widetilde{R}_{e_1,J(e_1)}(\frac{\theta_1 + \sum_{j=1}^d \theta_j}{2}) \cdot \ldots \cdot \widetilde{R}_{e_d,J(e_d)}(\frac{\theta_d + \sum_{j=1}^d \theta_j}{2}) \\ &= e^{\frac{i(d+1)}{2}\sum_{j=1}^d \theta_j} \cdot \tilde{\alpha}(M_{\theta_1,\ldots,\theta_d}) \\ &= \det(\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d}))^{\frac{d+1}{2}} \cdot \tilde{\alpha}(M_{\theta_1,\ldots,\theta_d}), \end{aligned}$$

which proves 1.
The other case is much the same: setting

$$\tilde{\rho}(M_{\theta_1,\ldots,\theta_d}) := \widetilde{R}_{e_1,J(e_1)}(\frac{\sum_{j=1}^d \theta_j}{2}) \cdot \ldots \cdot \widetilde{R}_{e_{n-d},J(e_{n-d})}(\frac{\sum_{j=1}^d \theta_j}{2}),$$

one obtains a well-defined Lie-group-homomorphism $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) \xrightarrow{\tilde{\rho}} \mathrm{Spin}_{2(n-d)}$ with $\xi \circ \tilde{\rho} = \rho$ (the integer $n-d$ is even) and

$$\begin{aligned} j \circ \rho(M_{\theta_1,\ldots,\theta_d}) &= e^{\frac{i}{2}\sum_{j=1}^{n-d}\sum_{k=1}^d \theta_k} \cdot \widetilde{R}_{e_1,J(e_1)}(\frac{\sum_{j=1}^d \theta_j}{2}) \cdot \ldots \cdot \widetilde{R}_{e_{n-d},J(e_{n-d})}(\frac{\sum_{j=1}^d \theta_j}{2}) \\ &= \det(\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d}))^{\frac{n-d}{2}} \tilde{\rho}(M_{\theta_1,\ldots,\theta_d}), \end{aligned}$$

which shows 2 and concludes the proof. □

In particular, we obtain the following

Corollary 3.4.4 *Let $d < n$ be odd integers and consider the canonical embedding $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^n$ as above. Then there exists a unitary and parallel isomorphism*

$$\Sigma(T^\perp \mathbb{C}\mathrm{P}^d) \cong \bigoplus_{s=0}^{n-d} \begin{pmatrix} n-d \\ s \end{pmatrix} \cdot \gamma_d^{\frac{n-d}{2}-s},$$

where $\Sigma(T^\perp \mathbb{C}\mathrm{P}^d)$ denotes the (complex) spinor bundle of $T^\perp \mathbb{C}\mathrm{P}^d$ and, for each $s \in \{0,\ldots,n-d\}$, the factor $\begin{pmatrix} n-d \\ s \end{pmatrix}$ stands for the multiplicity with which the line bundle $\gamma_d^{\frac{n-d}{2}-s}$ appears in the splitting.

Proof: By Lemma 3.4.3 and Lemma 3.4.2, one has, for any $B \in \mathrm{U}_d$:

$$\begin{aligned} \tilde{\rho}(\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix}) &= \det(B)^{-\frac{n-d}{2}} \cdot j \circ \rho(\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix}) \\ &= \det(B)^{-\frac{n-d}{2}} \cdot j(\det(B)\mathrm{I}_{n-d}). \end{aligned}$$

Now it is elementary to prove that, for any positive integer k, any $z \in \mathrm{U}_1$ and any $s \in \{0,\ldots,k\}$,

$$\delta_{2k} \circ j(z \cdot \mathrm{I}_k)_{|_{\Sigma_{2k}^{(s)}}} = z^s \cdot \mathrm{Id}_{\Sigma_{2k}^{(s)}},$$

where $\Sigma_{2k}^{(s)}$ is the eigenspace of the Clifford action of the Kähler form to the eigenvalue $i(2s-k)$ in the spinor space Σ_{2k}. In particular $\Sigma_{2k}^{(s)}$ splits into the direct sum of $\dim_{\mathbb{C}}(\Sigma_{2k}^{(s)})$ copies of some one-dimensional representation, with $\dim_{\mathbb{C}}(\Sigma_{2k}^{(s)}) = \begin{pmatrix} k \\ s \end{pmatrix}$. Since $\Sigma_{2k} = \oplus_{s=0}^{k} \Sigma_{2k}^{(s)}$, we obtain the following splitting:

$$\begin{aligned} \delta_{2(n-d)} \circ \tilde{\rho} &= \bigoplus_{s=0}^{n-d} \det(\cdot)^{-(\frac{n-d}{2}-s)} \otimes \mathrm{Id}_{\Sigma_{2(n-d)}^{(s)}} \\ &= \bigoplus_{s=0}^{n-d} \det(\cdot)^{-(\frac{n-d}{2}-s)} \otimes \mathbf{1}_{\mathbb{C}}^{\begin{pmatrix} n-d \\ s \end{pmatrix}}, \end{aligned}$$

where $\det(\cdot) : \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) \to \mathrm{U}_1$, $\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} \mapsto \det(B)$, the trivial representation on $\mathbb{C}$ is denoted by $\mathbf{1}_{\mathbb{C}}$ and "$\mathbf{1}_{\mathbb{C}}^{l}$" means that this representation appears with multiplicity l. □

3.4.3 The twisted Dirac operator on $\mathbb{C}\mathrm{P}^d$

As a consequence of Corollary 3.4.4, the tensor product $\Sigma(T\mathbb{C}\mathrm{P}^d) \otimes \Sigma(T^\perp \mathbb{C}\mathrm{P}^d)$ splits into subbundles of the form $\Sigma(T\mathbb{C}\mathrm{P}^d) \otimes \gamma_d^m$ for some integer m. Since this splitting is orthogonal and parallel, it is also preserved by the corresponding twisted Dirac operator. Hence it suffices to describe the Dirac operator of the twisted spinor bundle $\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\gamma_d^m$ over $\mathbb{C}\mathrm{P}^d$ as an infinite sum of matrices, where $m\in\mathbb{Z}$ is an arbitrary (non-necessarily positive) integer. The Dirac eigenvalues of $\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\gamma_d^m$ have been computed by M. Ben Halima in [B3, Thm. 1]. Indeed, we have

Theorem 3.4.5 *For an odd integer d let $\mathbb{C}\mathrm{P}^d$ be endowed with its Fubini-Study metric of constant holomorphic sectional curvature 4. For an arbitrary $m\in\mathbb{Z}$ let the m^{th} power γ_d^m of the tautological bundle of $\mathbb{C}\mathrm{P}^d$ be endowed with its canonical metric and connection. Then the eigenvalues (without multiplicities) of the square of the Dirac operator of $\mathbb{C}\mathrm{P}^d$ twisted by γ_d^m are given by the following families:*

1. $2(r+l)\cdot(d+1+2(l-m-\varepsilon))$, *where* $r\in\{1,\dots,d-1\}$, $\varepsilon\in\{0,1\}$ *and* $l\in\mathbb{N}$ *with* $l\geq\max(\varepsilon,\frac{d+1}{2}-r+m)$.
2. $2l(2l+d-1-2m)$, *where* $l\in\mathbb{N}$, $l\geq\max(0,m+\frac{d+1}{2})$.
3. $2(d+l)(d+1+2(l-m))$, *where* $l\in\mathbb{N}$, $l\geq\max(0,m-\frac{d+1}{2})$.

The first family of eigenvalues corresponds to an irreducible representation of SU_{d+1} with highest weight given by [B3, Prop. 2]

$$(r+2l-\frac{d-1}{2}-m-\varepsilon,\underbrace{r+l-\frac{d-1}{2}-m,\dots,r+l-\frac{d-1}{2}-m}_{r-1},r+l-\frac{d+1}{2}-m+\varepsilon,\underbrace{r+l-\frac{d+1}{2}-m,\dots,r+l-\frac{d+1}{2}-m}_{d-r-1}).$$

Similarly, the second family of eigenvalues corresponds to the highest weight

$$(2l-\frac{d+1}{2}-m,\underbrace{l-\frac{d+1}{2}-m,\dots,l-\frac{d+1}{2}-m}_{d-1}).$$

The last family of eigenvalues corresponds to

$$(2l+\frac{d+1}{2}-m,\underbrace{l+\frac{d+1}{2}-m,\dots,l+\frac{d+1}{2}-m}_{d-1}).$$

In the following, we will determine the multiplicities of the eigenvalues in Theorem 3.4.5. Indeed, we have

Lemma 3.4.6 *Let $d\geq 1$ be an odd integer and $m\in\mathbb{Z}$.*

1. *The multiplicities of the first family of the eigenvalues are equal to*

$$\frac{d(\frac{d+1}{2}+r-m+2l-\varepsilon)}{(r+l)(\frac{d+1}{2}-m+l-\varepsilon)}\cdot\begin{pmatrix} d+l-\varepsilon\\ d\end{pmatrix}\cdot\begin{pmatrix} d-1\\ d-r-\varepsilon\end{pmatrix}\cdot\begin{pmatrix} \frac{d-1}{2}+r-m+l\\ d\end{pmatrix}.$$

2. *For the second family, we have*

$$\prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{2l-\frac{d+1}{2}-m}{d})\cdot\prod_{j=2}^{d}(1+\frac{l-\frac{d+1}{2}-m}{d-j+1}).$$

3. *For the last family of eigenvalues, the multiplicities are equal to*

$$\prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{2l+\frac{d+1}{2}-m}{d})\cdot\prod_{j=2}^{d}(1+\frac{l+\frac{d+1}{2}-m}{d-j+1}).$$

In our convention, a product taken on an empty index-set is equal to 1.

Proof: The required multiplicity can be computed with the help of the Weyl's character formula [B2]

$$\prod_{\alpha\in\Delta_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\Big),$$

where λ is a highest weight of an irreducible SU_{d+1}-representation and Δ_+ is the set of positive roots, i.e.

$$\Delta_+=\{\theta_j-\theta_k,\ 1\le j<k\le d,\ \theta_j+\sum_{k=1}^{d}\theta_k,\ 1\le j\le d\}$$

and $\delta_+=\sum_{k=1}^{d}(d-k+1)\theta_k$ is the half-sum of the positive roots of SU_{d+1}, see [B3, p. 442]. Here the scalar product $\langle\cdot,\cdot\rangle$ is the Riemannian metric on the dual of a maximal torus of SU_{d+1}, which is defined by the following product of matrices $\langle\lambda,\lambda'\rangle=\lambda.\beta.{}^t\lambda'$ where β is the matrix given by $\frac{2}{d+1}\big(-1+(d+1)\delta_{jk}\big)_{1\le j,k\le d}$. To compute the quotient in the Weyl's character formula, we treat the three cases separately:
1. Consider α of the form $\alpha=\theta_j-\theta_k$ for some $1\le j<k\le d$. Note that this form for α can only exist if $d>1$. We compute $\beta\cdot\alpha=2(\theta_j-\theta_k)$. Therefore, $\langle\delta_+,\alpha\rangle=2(k-j)$. For the highest weight λ corresponding to the first family of eigenvalues, we find that

$$1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\quad=\quad\left|\begin{array}{lll}\frac{l-\varepsilon+k-j}{k-j} & \text{case} & j=1,k\in\{2,\dots,r\}\\ \frac{l+1-2\varepsilon+k-j}{k-j} & \text{case} & j=1,k=r+1\\ \frac{l+1-\varepsilon+k-j}{k-j} & \text{case} & j=1,k\in\{r+2,\dots,d\}\\ 1 & \text{case} & j,k\in\{2,\dots,r\}\\ \frac{1-\varepsilon+k-j}{k-j} & \text{case} & j\in\{2,\dots,r\},k=r+1\\ \frac{1+k-j}{k-j} & \text{case} & j\in\{2,\dots,r\},k\in\{r+2,\dots,d\}\\ \frac{\varepsilon+k-j}{k-j} & \text{case} & j=r+1,k\in\{r+2,\dots,d\}\\ 1 & \text{case} & j,k\in\{r+2,\dots,d\}.\end{array}\right.$$

Similarly for $\alpha=\theta_j+\sum_{k=1}^{d}\theta_k$ with $j\in\{1,\dots,d\}$, we get that

$$1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\quad=\quad\left|\begin{array}{lll}\frac{u_-+1+l-\varepsilon+d-j+1}{d-j+1} & \text{case} & j=1\\ \frac{u_-+1+d-j+1}{d-j+1} & \text{case} & j\in\{2,\dots,r\}\\ \frac{u_-+\varepsilon+d-j+1}{d-j+1} & \text{case} & j=r+1\\ \frac{u_-+d-j+1}{d-j+1} & \text{case} & j\in\{r+2,\dots,d\},\end{array}\right.$$

where $u_-=r-\frac{d+1}{2}-m+l$. In order to compute the product we separate both cases $\varepsilon=0$ and $\varepsilon=1$.

• *Case* $\varepsilon = 0$: Then

$$
\begin{aligned}
\prod_{\alpha\in\Delta_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\Big) &= (\prod_{k=2}^{r}\frac{l+k-1}{k-1})\cdot(\prod_{k=r+1}^{d}\frac{l+k}{k-1})\cdot(\prod_{j=2}^{r}\prod_{k=r+1}^{d}\frac{k+1-j}{k-j})\cdot \\
&\qquad \frac{u_-+l+d+1}{d}\cdot(\prod_{j=2}^{r}\frac{u_-+d-j+2}{d-j+1})\cdot(\prod_{j=r+1}^{d}\frac{u_-+d-j+1}{d-j+1}) \\
&= \frac{(l+1)\cdot\ldots\cdot(l+r-1)\cdot(l+r+1)\cdot\ldots\cdot(l+d)}{1\cdot 2\cdot\ldots\cdot(d-1)}\cdot \\
&\qquad (\prod_{j=2}^{r}\frac{(r+2-j)\cdot\ldots\cdot(d+1-j)}{(r+1-j)\cdot\ldots\cdot(d-j)})\cdot\frac{u_-+l+d+1}{d}\cdot \\
&\qquad \frac{(u_-+d)\cdot\ldots\cdot(u_-+d-r+2)}{(d-1)\cdot\ldots\cdot(d-r+1)}\cdot\frac{(u_-+d-r)\cdot\ldots\cdot(u_-+1)}{(d-r)\cdot\ldots\cdot 2\cdot 1} \\
&= \frac{d}{l+r}\cdot\frac{(l+d)!}{d!\cdot l!}\cdot(\prod_{j=2}^{r}\frac{d+1-j}{r+1-j})\cdot\frac{u_-+l+d+1}{u_-+d-r+1}\cdot\frac{(u_-+d)!}{d!\cdot u_-!} \\
&= \frac{d}{l+r}\cdot\begin{pmatrix} l+d \\ d\end{pmatrix}\cdot\frac{(d-1)\cdot\ldots\cdot(d+1-r)}{(r-1)\cdot\ldots\cdot 2\cdot 1}\cdot\frac{u_-+l+d+1}{u_-+d-r+1}\cdot\begin{pmatrix} u_-+d \\ d\end{pmatrix} \\
&= \frac{d(u_-+l+d+1)}{(l+r)(u_-+d-r+1)}\cdot\begin{pmatrix} l+d \\ d\end{pmatrix}\cdot\begin{pmatrix} d-1 \\ r-1\end{pmatrix}\cdot\begin{pmatrix} u_-+d \\ d\end{pmatrix},
\end{aligned}
$$

which gives for the multiplicity in this case (replace u_- by $r-\frac{d+1}{2}-m+l$):

$$
\prod_{\alpha\in\Delta_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\Big) = \frac{d(\frac{d+1}{2}+r-m+2l)}{(r+l)(\frac{d+1}{2}-m+l)}\cdot\begin{pmatrix} d+l \\ d\end{pmatrix}\cdot\begin{pmatrix} d-1 \\ d-r\end{pmatrix}\cdot\begin{pmatrix} \frac{d-1}{2}+r-m+l \\ d\end{pmatrix}.
$$

• *Case* $\varepsilon = 1$: Then

$$
\begin{aligned}
\prod_{\alpha\in\Delta_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\Big) &= (\prod_{k=2}^{r+1}\frac{l+k-2}{k-1})\cdot(\prod_{k=r+2}^{d}\frac{l+k-1}{k-1})\cdot(\prod_{j=2}^{r+1}\prod_{k=r+2}^{d}\frac{k+1-j}{k-j})\cdot \\
&\qquad \frac{u_-+l+d}{d}\cdot(\prod_{j=2}^{r+1}\frac{u_-+d-j+2}{d-j+1})\cdot(\prod_{j=r+2}^{d}\frac{u_-+d-j+1}{d-j+1}) \\
&= \frac{l\cdot\ldots\cdot(l+r-1)\cdot(l+r+1)\cdot\ldots\cdot(l+d-1)}{1\cdot 2\cdot\ldots\cdot(d-1)}\cdot \\
&\qquad (\prod_{j=2}^{r+1}\frac{(r+3-j)\cdot\ldots\cdot(d+1-j)}{(r+2-j)\cdot\ldots\cdot(d-j)})\cdot\frac{u_-+l+d}{d}\cdot \\
&\qquad \frac{(u_-+d)\cdot\ldots\cdot(u_-+d-r+1)}{(d-1)\cdot\ldots\cdot(d-r)}\cdot\frac{(u_-+d-r-1)\cdot\ldots\cdot(u_-+1)}{(d-r-1)\cdot\ldots\cdot 2\cdot 1} \\
&= \frac{d}{l+r}\cdot\frac{(l+d-1)!}{d!\cdot(l-1)!}\cdot(\prod_{j=2}^{r+1}\frac{d+1-j}{r+2-j})\cdot\frac{u_-+l+d}{u_-+d-r}\cdot\frac{(u_-+d)!}{u_-!\cdot d!} \\
&= \frac{d(u_-+l+d)}{(l+r)(u_-+d-r)}\cdot\frac{(l+d-1)!}{d!\cdot(l-1)!}\cdot\frac{(d-1)!}{r!\cdot(d-r-1)!}\cdot\frac{(u_-+d)!}{u_-!\cdot d!} \\
&= \frac{d(u_-+l+d)}{(l+r)(u_-+d-r)}\cdot\begin{pmatrix} l+d-1 \\ d\end{pmatrix}\cdot\begin{pmatrix} d-1 \\ r\end{pmatrix}\cdot\begin{pmatrix} u_-+d \\ d\end{pmatrix},
\end{aligned}
$$

which, replacing u_- by $r - \frac{d+1}{2} - m + l$, gives

$$\prod_{\alpha\in\Delta_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\Big) = \frac{d(\frac{d-1}{2}+r-m+2l)}{(r+l)(\frac{d-1}{2}-m+l)} \cdot \begin{pmatrix} d+l-1 \\ d \end{pmatrix} \cdot \begin{pmatrix} d-1 \\ d-r-1 \end{pmatrix} \cdot \begin{pmatrix} \frac{d-1}{2}+r-m+l \\ d \end{pmatrix}.$$

This shows 1.

2. Consider α of the form $\alpha = \theta_j - \theta_k$ for some $1 \le j < k \le d$. We have already shown in the first part that $\langle\delta_+,\alpha\rangle = 2(k-j)$. For the highest weight λ corresponding to the second family of eigenvalues, we have

$$\langle\lambda,\alpha\rangle = \left| \begin{array}{lll} 2l & \text{case} & j=1 \\ 0 & \text{case} & j>1. \end{array} \right.$$

Similarly for $\alpha = \theta_j + \sum_{k=1}^d \theta_k$ with $j \in \{1,\dots,d\}$, we already know that $\langle\delta_+,\alpha\rangle = 2(d-j+1)$ and

$$\langle\lambda,\alpha\rangle = \left| \begin{array}{lll} 2(v+l) & \text{case} & j=1 \\ 2v & \text{case} & j>1 \end{array} \right.$$

where v denotes one of the $d-1$ last components of the weight λ. Hence the product is given by

$$\prod_{\alpha\in\Delta_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta_+,\alpha\rangle}\Big) = \prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{v+l}{d})\cdot\prod_{j=2}^{d}(1+\frac{v}{d-j+1}).$$

Of course only the central factor appears in case $d=1$. Replacing v by its respective value gives 2. and 3. and concludes the proof. □

As a consequence of Lemma 3.4.5 and Lemma 3.4.6, we obtain the

Theorem 3.4.7 *Let d be a positive odd integer and $m \in \mathbb{Z}$ be arbitrary. Denote by γ_d the tautological bundle of $\mathbb{C}\mathrm{P}^d$. Then the spectrum of the square of the Dirac operator of $\mathbb{C}\mathrm{P}^d$ twisted with γ_d^m is given by the following family of eigenvalues:*

1. $2(r+l)\cdot(d+1+2(l-m-\varepsilon))$, *where $r \in \{1,\dots,d-1\}$, $\varepsilon \in \{0,1\}$ and $l \in \mathbb{N}$ with $l \ge \max(\varepsilon, \frac{d+1}{2} - r + m)$. The multiplicity of the eigenvalue corresponding to the choice of a triple (r,ε,l) as above is given by*

$$\frac{d(\frac{d+1}{2}+r-m+2l-\varepsilon)}{(r+l)(\frac{d+1}{2}-m+l-\varepsilon)} \cdot \begin{pmatrix} d+l-\varepsilon \\ d \end{pmatrix} \cdot \begin{pmatrix} d-1 \\ d-r-\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \frac{d-1}{2}+r-m+l \\ d \end{pmatrix}.$$

2. $2l(2l+d-1-2m)$, *where $l \in \mathbb{N}$, $l \ge \max(0, m+\frac{d+1}{2})$, with multiplicity*

$$\prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{2l-\frac{d+1}{2}-m}{d})\cdot\prod_{j=2}^{d}(1+\frac{l-\frac{d+1}{2}-m}{d-j+1}).$$

3. $2(d+l)(d+1+2(l-m))$, *where $l \in \mathbb{N}$, $l \ge \max(0, m-\frac{d+1}{2})$, with multiplicity*

$$\prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{2l+\frac{d+1}{2}-m}{d})\cdot\prod_{j=2}^{d}(1+\frac{l+\frac{d+1}{2}-m}{d-j+1}).$$

Note that, since $\mathbb{C}\mathrm{P}^d$ is a symmetric space, the spectrum of every Dirac operator twisted with a homogeneous bundle over $\mathbb{C}\mathrm{P}^d$ is symmetric about the origin. Hence the spectrum of the Dirac operator of $\mathbb{C}\mathrm{P}^d$ twisted with γ_d^m can be easily deduced from that of its square.

We point out that the computations done by M. Ben Halima in [B3, Thm. 1] contain a minor mistake (his m should be replaced by $-m$). It can be also checked that, up to a factor $4(d+1)$ (his convention for the Fubini-Study metric is different from ours), our values coincide with his (his k is our l and his l is our $d-r$).

We can now formulate the

Theorem 3.4.8 *Let $d<n$ be positive odd integers. Then the spectrum of the square of the Dirac operator of $\mathbb{C}\mathrm{P}^d$ twisted with the spinor bundle of the normal bundle of the canonical embedding $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^n$ is given by the following family of eigenvalues:*

1. $2(r+l)\cdot(2d+1-n+2(s+l-\varepsilon))$, *where* $r\in\{1,\ldots,d-1\}$, $s\in\{0,\ldots,n-d\}$, $\varepsilon\in\{0,1\}$ *and* $l\in\mathbb{N}$ *with* $l\geq\max(\varepsilon,\frac{n+1}{2}-r-s)$. *The multiplicity of the eigenvalue corresponding to the choice of a* 4*-tuple* (r,s,ε,l) *as above is given by*

$$\frac{d(d-\frac{n-1}{2}+r+s+2l-\varepsilon)}{(r+l)(d-\frac{n-1}{2}+s+l-\varepsilon)}\cdot\begin{pmatrix} n-d \\ s \end{pmatrix}\cdot\begin{pmatrix} d+l-\varepsilon \\ d \end{pmatrix}\cdot\begin{pmatrix} d-1 \\ d-r-\varepsilon \end{pmatrix}\cdot\begin{pmatrix} d-\frac{n+1}{2}+r+s+l \\ d \end{pmatrix}.$$

2. $4l(l+s+d-\frac{n+1}{2})$, *where* $s\in\{0,\ldots,n-d\}$, $l\in\mathbb{N}$, $l\geq\max(0,\frac{n+1}{2}-s)$, *with multiplicity*

$$\begin{pmatrix} n-d \\ s \end{pmatrix}\cdot\prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{2l-\frac{n+1}{2}+s}{d})\cdot\prod_{j=2}^{d}(1+\frac{l-\frac{n+1}{2}+s}{d-j+1}).$$

3. $2(d+l)(2d-n+1+2(l+s))$, *where* $s\in\{0,\ldots,n-d\}$, $l\in\mathbb{N}$, $l\geq\max(0,\frac{n-1}{2}-d-s)$, *with multiplicity*

$$\begin{pmatrix} n-d \\ s \end{pmatrix}\cdot\prod_{k=2}^{d}(1+\frac{l}{k-1})\cdot(1+\frac{2l+d-\frac{n-1}{2}+s}{d})\cdot\prod_{j=2}^{d}(1+\frac{l+d-\frac{n-1}{2}+s}{d-j+1}).$$

Proof: Recall that, by Corollary 3.4.4, there exists a unitary and parallel isomorphism

$$\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\Sigma(T^\perp\mathbb{C}\mathrm{P}^d)\cong\bigoplus_{s=0}^{n-d}\begin{pmatrix} n-d \\ s \end{pmatrix}\cdot\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\gamma_d^{\frac{n-d}{2}-s},$$

where γ_d is the tautological bundle of $\mathbb{C}\mathrm{P}^d$ and $\begin{pmatrix} n-d \\ s \end{pmatrix}$ stands for the multiplicity with which the subbundle $\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\gamma_d^{\frac{n-d}{2}-s}$ appears in the splitting. Therefore, the eigenvalues of the twisted Dirac operator acting on $\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\Sigma(T^\perp\mathbb{C}\mathrm{P}^d)$ are those of $\Sigma(T\mathbb{C}\mathrm{P}^d)\otimes\gamma_d^{\frac{n-d}{2}-s}$, where s runs from 0 to $n-d$. Moreover, the multiplicity of the eigenvalue corresponding to some s is $\begin{pmatrix} n-d \\ s \end{pmatrix}$ times the multiplicity computed in

Lemma 3.4.6. Replacing m by $\frac{n-d}{2}-s$, Theorem 3.4.7 gives the result. $\square$

Note that $(d+1)^2$ is always an eigenvalue for the squared operator $(D_M^{\Sigma N})^2$: if $d=1$, take $s=\frac{n-1}{2}$ and $l=1$ in the second family of eigenvalues; if $d>1$, take $r=\frac{d+1}{2}$, $s=\frac{n-d}{2}$ and $\varepsilon=0=l$ in the first family.

Using Theorem 3.4.8, we are now able to compute the smallest eigenvalue of the twisted Dirac operator (see [B8, Proposition 4.9] for a proof):

Proposition 3.4.9 *The lowest eigenvalue for the square of the Dirac operator of* $\mathbb{C}\mathrm{P}^d$ *twisted with the spinor bundle of the normal bundle of the canonical embedding* $\mathbb{C}\mathrm{P}^d \to \mathbb{C}\mathrm{P}^n$ *is equal to* 0 *for* $d<\frac{n+1}{2}$ *and to* $(n+1)(2d+1-n)$ *for* $d\geq\frac{n+1}{2}$.

Proof: Let us consider the first family of eigenvalues with $\varepsilon=0$ (the same computation remains true for $\varepsilon=1$). For $r+s\geq\frac{n+1}{2}$, which implies $d-\frac{n-1}{2}\leq r$, the minimum is attained for $l=0$ and we find the eigenvalues $2r(2d+1-n+2s)$, which are increasing functions with respect to s with $s\geq\frac{n+1}{2}-r$. Here two cases occur:

1. Case where $\frac{n+1}{2}-r\geq 0$, the eigenvalues become $4r(d+1-r)$ and we distinguish the two subcases:
   (a) For $d\leq\frac{n+1}{2}$, then the lowest eigenvalue is equal to $4d$.
   (b) For $\frac{n+1}{2}<d$, the lowest eigenvalue is $(n+1)(2d+1-n)$.
2. Case where $\frac{n+1}{2}-r<0$ which implies $\frac{n+1}{2}<d$. Hence, the lowest eigenvalue is equal to $(n+1)(2d+1-n)$.

Now for $r+s<\frac{n+1}{2}$, we take $l=\frac{n+1}{2}-r-s$. Thus the eigenvalues are equal $2(n+1-2s)(d+1-r)$ which are decreasing functions in s with $0\leq s\leq\frac{n-1}{2}-r$. We have:

1. Case where $\frac{n-1}{2}-r\leq n-d$. We then get the eigenvalues $4(1+r)(d+1-r)$ with $d-\frac{n+1}{2}\leq r\leq\frac{n-1}{2}$. Here two cases occur:
   (a) For $d\leq\frac{n+1}{2}$, the lowest eigenvalue is equal to $8d$.
   (b) For $d>\frac{n+1}{2}$, the lowest eigenvalue is equal to $(n+3)(2d+1-n)$.
2. Case where $\frac{n-1}{2}-r>n-d$, we get the eigenvalues $2(2d-n+1)(d+1-r)$ with $1\leq r\leq d-\frac{n+3}{2}$. In this case, we have that $d>\frac{n+1}{2}$ and the lowest eigenvalue is equal to $(n+5)(2d-n+1)$.

For the second family of eigenvalues, we distinguish the cases:

1. Case where $\frac{n+1}{2}-s\leq 0$ which implies that $d\leq\frac{n-1}{2}$, we take $l=0$. The lowest eigenvalue is then equal to 0.
2. Case where $\frac{n+1}{2}-s>0$. The eigenvalues become $2d(n+1-2s)$ with $0\leq s\leq\frac{n-1}{2}$. Two cases occur
   (a) For $d\leq\frac{n+1}{2}$, the lowest eigenvalue is $4d$.
   (b) For $d>\frac{n+1}{2}$, the lowest eigenvalue is $2d(2d+1-n)$

For the last family of eigenvalues, we consider the two cases:

1. Case where $\frac{n-1}{2}-d-s>0$, which implies that $d<\frac{n-1}{2}$, we take $l=\frac{n-1}{2}-d-s$. We find the lowest eigenvalue 0 after substituting.
2. Case where $\frac{n-1}{2}-d-s\leq 0$. In this case $l=0$ and we get $2d(2d-n+1+2s)$. Here two cases occur:
   (a) For $d>\frac{n-1}{2}$, the lowest eigenvalue is $2d(2d-n+1)$.
   (b) For $d\leq\frac{n-1}{2}$, the lowest eigenvalue is 0.

□

Note that the absence of 0 in the Dirac spectrum for $d\geq\frac{n+1}{2}$ agrees with Kodaira's vanishing theorem, which implies that, if m is an integer with $|m|<\frac{d+1}{2}$, then the cohomology groups $H^q\left(\mathbb{C}\mathrm{P}^d,\gamma_d^{\frac{d+1}{2}+m}\right)$ vanish for all q, in particular the kernel of the twisted Dirac operator is trivial in that case.

Next we show that the estimate (3.7) is not always sharp. We consider the simplest case where $d=1$ and compare the multiplicities of the eigenvalues 0 and 4 with $2\begin{pmatrix} n \\ \frac{n+1}{2}\end{pmatrix}$, which is the *a priori* number of eigenvalues bounded by 4 in (3.7). The multiplicity of the eigenvalue 0 is equal to

$$\sum_{s=0}^{\frac{n-3}{2}}\begin{pmatrix} n-1 \\ s\end{pmatrix}(\frac{n-1}{2}-s)+\sum_{s=\frac{n+1}{2}}^{n-1}\begin{pmatrix} n-1 \\ s\end{pmatrix}(s-\frac{n-1}{2}),$$

which is equal to $\sum_{s=0}^{\frac{n-3}{2}}\begin{pmatrix} n-1 \\ s\end{pmatrix}(n-1-2s)$ since by replacing s by $(n-1)-s$ the second sum is equal to the first one. A short computation gives $\sum_{s=0}^{\frac{n-3}{2}}\begin{pmatrix} n-1 \\ s\end{pmatrix}(n-1-2s)=\frac{n-1}{2}\cdot\begin{pmatrix} n-1 \\ \frac{n-1}{2}\end{pmatrix}$. On the other hand, the multiplicity of the eigenvalue 4 is equal to $4\begin{pmatrix} n \\ \frac{n-1}{2}\end{pmatrix}$. Hence the sum of these two multiplicities is $(\frac{n-1}{2}+4)\cdot\begin{pmatrix} n-1 \\ \frac{n-1}{2}\end{pmatrix}$. That number is always greater than $2\begin{pmatrix} n \\ \frac{n+1}{2}\end{pmatrix}$. However, if the multiplicity of the eigenvalue 0 is smaller than $2\begin{pmatrix} n \\ \frac{n+1}{2}\end{pmatrix}$ for $n=3,5,7$, it is greater for $n\geq 9$. Thus, the equality in (3.7) is optimal for $n=3,5,7$ but is *never* optimal as soon as $n\geq 9$. In particular, the twisted Dirac operator on Kähler submanifolds behaves very differently from that on submanifolds immersed in real spaceforms, where analogous upper bounds are sharp in any dimension.

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Chapter 4

Imaginary Kählerian Killing spinors I

This chapter coincides (up to minor changes such as enumeration of pages, sections, theorems, references etc.) with the published article [32].

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Abstract. We describe and to some extent characterize a new family of Kähler spin manifolds admitting non-trivial imaginary Kählerian Killing spinors.

Keywords: Kähler manifolds, Sasakian manifolds, spin geometry

MSC classification: 53C25, 53C27, 53C55

4.1 Introduction

Let $(\widetilde{M}^{2n}, g, J)$ a Kähler manifold of real dimension $2n$ and with Kähler-form $\widetilde{\Omega}$ defined by $\widetilde{\Omega}(X,Y) := g(J(X),Y)$ for all vectors $X,Y \in T\widetilde{M}$. We denote by $p_+ : TM \longrightarrow T^{1,0}M$, $X \mapsto \frac{1}{2}(X - iJ(X))$ and $p_- : TM \longrightarrow T^{0,1}M$, $X \mapsto \frac{1}{2}(X + iJ(X))$ the projection maps. In case $\widetilde{M}^{2n}$ is spin, we denote its complex spinor bundle by $\Sigma\widetilde{M}$.

Definition 4.1.1 *Let $(\widetilde{M}^{2n}, g, J)$ a spin Kähler manifold and $\alpha \in \mathbb{C}$. A pair (ψ, ϕ) of sections of $\Sigma\widetilde{M}$ is called an α-*Kählerian Killing spinor *if and only if it satisfies, for every $X \in \Gamma(T\widetilde{M})$,*

$$\left| \begin{array}{ll} \widetilde{\nabla}_X \psi & = -\alpha p_-(X) \cdot \phi \\ \widetilde{\nabla}_X \phi & = -\alpha p_+(X) \cdot \psi. \end{array} \right.$$

An α-Kählerian Killing spinor is said to be real *(resp.* imaginary*) if and only if $\alpha \in \mathbb{R}$ (resp. $\alpha \in i\mathbb{R}^*$).*

If $\alpha = 0$, then an α-Kählerian Killing spinor is nothing but a pair of parallel spinors. The classification of Kähler spin manifolds (resp. spin manifolds) admitting real non-parallel Kählerian Killing (resp. parallel) spinors has been established by A. Moroianu in [C12] (resp. by McK. Wang in [C14]).

In this paper, we describe and partially classify those Kähler spin manifolds carrying non-trivial imaginary Kählerian Killing spinors. Note first that there is no restriction in assuming $\alpha = i$: obviously, changing (ψ, ϕ) into $(\psi, -\phi)$ changes α into $-\alpha$; moreover, (ψ, ϕ) is an α-Kählerian Killing spinor on $(\widetilde{M}^{2n}, g, J)$ if and only if it is an $\frac{\alpha}{\lambda}$-Kählerian Killing spinor on $(\widetilde{M}^{2n}, \lambda^2 g, J)$ for any constant $\lambda > 0$.

K.-D. Kirchberg, who introduced this equation (see [C9] for references), showed that, if a non-zero i-Kählerian Killing spinor (ψ, ϕ) exists on $(\widetilde{M}^{2n}, g, J)$, then necessarily the complex dimension n of $\widetilde{M}$ is odd, the manifold $(\widetilde{M}^{2n}, g)$ is Einstein with scalar curvature $-4n(n+1)$, the pair (ψ, ϕ) vanishes nowhere and satisfies $\widetilde{\Omega} \cdot \psi = -i\psi$ as well as $\widetilde{\Omega} \cdot \phi = i\phi$, see [C9] and Proposition 4.2.1 below for further properties. Moreover, he proved in the case $n = 3$ that the holomorphic sectional curvature must be constant [C9, Thm. 16], in particular only the complex hyperbolic space $\mathbb{C}\mathrm{H}^3$ occurs as simply-connected complete $(\widetilde{M}^6, g, J)$ with non-trivial i-Kählerian Killing spinors.

We extend Kirchberg's results in several ways. First, we study in detail the critical points of the length function $|\psi|$ of ψ. We show that, if the underlying Riemannian manifold $(\widetilde{M}^{2n}, g)$ is connected and complete, then $|\psi|$ has at most one critical value, which then has to be a (global) minimum and that the corresponding set of critical points is a Kähler totally geodesic submanifold (Proposition 4.2.3).
As a next step, we describe a whole family of examples of Kähler manifolds admitting non-trivial i-Kählerian Killing spinors (Theorem 4.3.9), including the complex hyperbolic space and some Kähler manifolds with non-constant holomorphic sectional curvature (Corollary 4.3.13). All arise as so-called doubly-warped products over Sasakian manifolds. A more detailed study of the induced spinor equation on that Sasakian manifold allows the complex hyperbolic space to be characterized within the family (Theorem 4.3.18).

In the last section, we show that doubly-warped products are the only possible Kähler manifolds with non-trivial i-Kählerian Killing spinors as soon as both components of (ψ, ϕ) have the same length and are exchanged through the Clifford multiplication by a (real) vector field (Theorem 4.4.1). This shows an interesting analogy with H. Baum's classification [C3, C4] of complete Riemannian spin manifolds with imaginary Killing spinors.

4.2 General integrability conditions

In this section we look for further necessary conditions for the existence of imaginary Kählerian Killing spinors. Consider the vector field V on $\widetilde{M}$ defined by

$$g(V, X) := \mathfrak{Im}(\langle p_+(X) \cdot \psi, \phi \rangle) \tag{4.1}$$

for every vector X on $\widetilde{M}$. We recall the following

Proposition 4.2.1 (see [C9]) *Let (ψ, ϕ) be an i-Kählerian Killing spinor on $(\widetilde{M}^{2n}, g, J)$ which does not vanish identically. Then the following properties hold:*

i) $\mathrm{grad}(|\psi|^2) = \mathrm{grad}(|\phi|^2) = 2V$.

ii) For all vectors $X, Y \in T\widetilde{M}$,

$$g(\widetilde{\nabla}_X V, Y) = \mathfrak{Re}\left(\langle p_-(X)\cdot\phi, p_-(Y)\cdot\phi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\right).$$

In particular,

$$\mathrm{Hess}(|\psi|^2)(X,Y) = \mathrm{Hess}(|\phi|^2)(X,Y) = 2\mathfrak{Re}\left(\langle p_-(X)\cdot\phi, p_-(Y)\cdot\phi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\right).$$

iii) $\Delta(|\psi|^2) = \Delta(|\phi|^2) = -2(n+1)(|\psi|^2+|\phi|^2)$*, where* $\Delta := -\mathrm{tr}_g(\mathrm{Hess})$.

iv) The vector field V *is holomorphic, i.e., it satisfies:* $\widetilde{\nabla}_{J(X)}V = J(\widetilde{\nabla}_X V)$ *for every* $X \in T\widetilde{M}$*. In particular, the vector field* $J(V)$ *is Killing on* $\widetilde{M}$.

v) $\mathrm{grad}(|V|^2) = 2\widetilde{\nabla}_V V$.

Note that, from Proposition 4.2.1, the identity $\Delta(|\psi|^2+|\phi|^2) = -4(n+1)(|\psi|^2+|\phi|^2)$ holds on $\widetilde{M}$, therefore $\widetilde{M}$ cannot be compact.

Next we are interested in the critical points of $|\psi|^2$ (or of $|\phi|^2$, they are the same by Proposition 4.2.1.*i*)). We need a technical lemma:

Lemma 4.2.2 *Under the hypotheses of* Proposition 4.2.1*, one has*

$$\widetilde{\nabla}_X\widetilde{\nabla}_Y V = \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V + \{2g(V,X)Y + g(V,Y)X - g(V,J(Y))J(X) + g(X,Y)V + g(J(X),Y)J(V)\}$$

for all vector fields X, Y *on* $\widetilde{M}$*. Therefore,*

$$\mathrm{Hess}(|V|^2)(X,Y) = 2g(\widetilde{\nabla}_X V, \widetilde{\nabla}_Y V) + 2\left(3g(X,V)g(Y,V) + |V|^2 g(X,Y) - g(X,J(V))g(Y,J(V))\right).$$

Proof: Using Proposition 4.2.1, we compute in a local orthonormal basis $\{e_j\}_{1\le j\le 2n}$ of $T\widetilde{M}$:

$$\begin{aligned}
\widetilde{\nabla}_X\widetilde{\nabla}_Y V &= \sum_{j=1}^{2n}\mathfrak{Re}\Big(\langle p_-(\widetilde{\nabla}_X Y)\cdot\phi, p_-(e_j)\cdot\phi\rangle + \langle p_+(\widetilde{\nabla}_X Y)\cdot\psi, p_+(e_j)\cdot\psi\rangle \\
&\qquad + \langle p_-(Y)\cdot\widetilde{\nabla}_X\phi, p_-(e_j)\cdot\phi\rangle + \langle p_-(Y)\cdot\phi, p_-(e_j)\cdot\widetilde{\nabla}_X\phi\rangle \\
&\qquad + \langle p_+(Y)\cdot\widetilde{\nabla}_X\psi, p_+(e_j)\cdot\psi\rangle + \langle p_+(Y)\cdot\psi, p_+(e_j)\cdot\widetilde{\nabla}_X\psi\rangle\Big)e_j \\
&= \sum_{j=1}^{2n}\mathfrak{Re}\Big(\langle p_-(\widetilde{\nabla}_X Y)\cdot\phi, p_-(e_j)\cdot\phi\rangle + \langle p_+(\widetilde{\nabla}_X Y)\cdot\psi, p_+(e_j)\cdot\psi\rangle \\
&\qquad - \alpha\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \alpha\langle p_-(Y)\cdot\phi, p_-(e_j)\cdot p_+(X)\cdot\psi\rangle \\
&\qquad - \alpha\langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle + \alpha\langle p_+(Y)\cdot\psi, p_+(e_j)\cdot p_-(X)\cdot\phi\rangle\Big)e_j \\
&= \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V \\
&\quad + \sum_{j=1}^{2n}\mathfrak{Im}\Big(\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle\Big)e_j \\
&\quad - \sum_{j=1}^{2n}\mathfrak{Im}\Big(\langle p_-(Y)\cdot\phi, p_-(e_j)\cdot p_+(X)\cdot\psi\rangle + \langle p_+(Y)\cdot\psi, p_+(e_j)\cdot p_-(X)\cdot\phi\rangle\Big)e_j.
\end{aligned}$$

We compute the second line of the right-hand side of the preceding equation (the treatment of the third one is analogous). Using $\langle p_+(X)\cdot\psi,\phi\rangle = 2ig(V,p_+(X))$, we obtain

$$\begin{aligned}\langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle &= \overline{\langle\psi, p_-(X)\cdot p_+(Y)\cdot p_-(e_j)\cdot\phi\rangle} + 4ig(Y,p_-(e_j))g(V,p_-(X))\\ &\quad +4ig(Y,p_-(X))g(V,p_-(e_j)).\end{aligned}$$

We deduce that, for every $j\in\{1,\ldots,2n\}$,

$$\begin{aligned}\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle &= 2\mathfrak{Re}\left(\langle\psi, p_-(X)\cdot p_+(Y)\cdot p_-(e_j)\cdot\phi\rangle\right)\\ &\quad +4ig(Y,p_-(e_j))g(V,p_-(X))\\ &\quad +4ig(Y,p_-(X))g(V,p_-(e_j)).\end{aligned}$$

The imaginary part of the right-hand side of the last equality is then given for every $j\in\{1,\ldots,2n\}$ by

$$\begin{aligned}4\mathfrak{Re}\left(g(Y,p_-(e_j))g(V,p_-(X))+g(Y,p_-(X))g(V,p_-(e_j))\right) &= g(V,X)g(Y,e_j)+g(V,J(X))g(J(Y),e_j)\\ &\quad +g(X,Y)g(V,e_j)+g(J(X),Y)g(J(V),e_j).\end{aligned}$$

This shows that

$$\begin{aligned}\sum_{j=1}^{2n}\mathfrak{Im}\Big(\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle\Big)e_j &= g(V,X)Y\\ &\quad +g(V,J(X))J(Y)\\ &\quad +g(X,Y)V\\ &\quad +g(J(X),Y)J(V).\end{aligned}$$

Similarly, one shows that

$$\begin{aligned}\sum_{j=1}^{2n}\mathfrak{Im}\Big(\langle p_-(Y)\cdot\phi, p_-(e_j)\cdot p_+(X)\cdot\psi\rangle + \langle p_+(Y)\cdot\psi, p_+(e_j)\cdot p_-(X)\cdot\phi\rangle\Big)e_j &= -g(V,Y)X\\ &\quad +g(V,J(Y))J(X)\\ &\quad -g(V,X)Y\\ &\quad +g(V,J(X))J(Y).\end{aligned}$$

Combining the computations above, we obtain

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_Y V &= \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V\\ &\quad +\left(g(V,X)Y+g(V,J(X))J(Y)+g(X,Y)V+g(J(X),Y)J(V)\right)\\ &\quad -\left(-g(V,Y)X+g(V,J(Y))J(X)-g(V,X)Y+g(V,J(X))J(Y)\right)\\ &= \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V\\ &\quad +\left(2g(V,X)Y+g(V,Y)X-g(V,J(Y))J(X)+g(X,Y)V+g(J(X),Y)J(V)\right),\end{aligned}$$

which shows the first identity. We deduce for the Hessian of $|V|^2$ that, for all vector fields X,Y on $\widetilde{M}$,

$$\begin{aligned}\mathrm{Hess}(|V|^2)(X,Y) &= 2g(\widetilde{\nabla}_X\widetilde{\nabla}_V V,Y)\\ &= 2g(\widetilde{\nabla}_{\widetilde{\nabla}_X V}V,Y)+2\Big(2g(V,X)g(V,Y)+|V|^2g(X,Y)-0+g(X,V)g(V,Y)\\ &\qquad +g(J(X),V)g(J(V),Y)\Big)\\ &= 2g(\widetilde{\nabla}_X V,\widetilde{\nabla}_Y V)+2\left(3g(X,V)g(Y,V)+|V|^2g(X,Y)-g(X,J(V))g(Y,J(V))\right),\end{aligned}$$

which is the second identity. This concludes the proof of Lemma 4.2.2. □

We can now describe more precisely the set of critical values and points of $|\psi|^2$ and $|V|^2$.

Proposition 4.2.3 *Under the hypotheses of* Proposition 4.2.1*, assume furthermore* $(\widetilde{M}^{2n}, g)$ *to be connected and complete. Then the following holds:*

i) The set $\{V = 0\}$ *of zeros of* V *coincides with* $\{\widetilde{\nabla}_V V = 0\}$*. As a consequence, the zeros of* V *are the only critical points of the function* $|V|^2$ *on* $\widetilde{M}^{2n}$*.*

ii) The subset $\{V = 0\}$ *is a (possibly empty) connected totally geodesic Kähler submanifold of complex dimension* $k < n$ *in* $(\widetilde{M}^{2n}, g, J)$*. Furthermore, for all* $x, y \in \{V = 0\}$*, every geodesic segment between* x *and* y *lies in* $\{V = 0\}$*.*

iii) The function $|\psi|^2$ *has at most one critical value on* $\widetilde{M}^{2n}$*, which is then a global minimum of* $|\psi|^2$*. Furthermore, the set of critical points of* $|\psi|^2$ *is a connected totally geodesic Kähler submanifold in* $(\widetilde{M}^{2n}, g, J)$*.*

Proof: The proof relies on simple computations and arguments.
$i)$ Proposition 4.2.1.$v)$ already implies that $\{\widetilde{\nabla}_V V = 0\}$ coincides with the set of critical points of $|V|^2$. Every zero of V is obviously a zero of $\widetilde{\nabla}_V V$, i.e., a critical point of $|V|^2$. Conversely, let $x \in \{\widetilde{\nabla}_V V = 0\}$. Then $0 = g_x(\widetilde{\nabla}_V V, V) = |p_-(V_x) \cdot \phi|^2 + |p_+(V_x) \cdot \psi|^2$, so that $p_-(V_x) \cdot \phi = 0$ and $p_+(V_x) \cdot \psi = 0$, which, in turn, implies $0 = \mathfrak{Im}\left(\langle p_+(V_x) \cdot \psi, \phi \rangle\right) = g(V_x, V_x)$, that is, $V_x = 0$. This shows $i)$.
$ii)$ The subset $\{V = 0\}$ - if non-empty - is the fixed-point-set in $\widetilde{M}^{2n}$ of the flow of the holomorphic Killing field $J(V)$, therefore it is a totally geodesic Kähler submanifold of $\widetilde{M}^{2n}$ (see e.g. [C10, Sec. II.5]); moreover, it cannot contain any open subset of $\widetilde{M}^{2n}$ since otherwise V would identically vanish as a holomorphic vector field. To show the connectedness of $\{V = 0\}$, it suffices to prove the second part of the statement. Pick any two points x_0, x_1 in $\{V = 0\}$ (or, equivalently, any critical points of $|V|^2$) and any geodesic c in $(\widetilde{M}^{2n}, g)$ with $c(0) = x_0$ and $c(1) = x_1$. Consider the real-valued function $f(t) := |V|^2_{c(t)}$ defined on $\mathbb{R}$. Then, for any $t \in \mathbb{R}$ one has $f'(t) = g(\mathrm{grad}(|V|^2), c'(t)) = 2g(\widetilde{\nabla}_{c'(t)} V, V)$ and

$$f''(t) = \mathrm{Hess}(|V|^2)(c'(t), c'(t)).$$

Lemma 4.2.2 provides the Hessian of $|V|^2$: for every $X \in T\widetilde{M}$,

$$\mathrm{Hess}(|V|^2)(X, X) = 2|\widetilde{\nabla}_X V|^2 + 2\left(3g(V, X)^2 + |V|^2|X|^2 - g(X, J(V))^2\right).$$

By Cauchy-Schwarz inequality, $|V|^2|X|^2 - g(X, J(V))^2 \geq 0$, so that $\mathrm{Hess}(|V|^2)(X, X) \geq 0$ for all X, in particular f is convex. This in turn implies that, if $f'(0) = f'(1) = 0$, then necessarily f vanishes on $[0, 1]$. This proves $ii)$.
$iii)$ Set, for any $t \in \mathbb{R}$, $h(t) := |\psi|^2_{c(t)}$ where c is an arbitrary geodesic on $(\widetilde{M}^{2n}, g)$. We show again that h is convex. As before $h''(t) = \mathrm{Hess}(|\psi|^2)(c'(t), c'(t)) \geq 0$ for every $t \in \mathbb{R}$, where $\mathrm{Hess}(|\psi|^2)(X, X) = 2(|p_-(X) \cdot \phi|^2 + |p_+(X) \cdot \psi|^2) \geq 0$ for every $X \in T\widetilde{M}$ (Proposition 4.2.1). We already know that, if $V = \frac{1}{2}\mathrm{grad}(|\psi|^2)$ vanishes at two different points of c, then it vanishes on any geodesic segment joining the two points, therefore $|\psi|^2$ is constant on it. This proves that $|\psi|^2$ has at most one critical

value. Since h is convex this critical value is necessarily a minimum. The last part of the statement is a straightforward consequence of $ii)$ since $\mathrm{grad}(|\psi|^2) = 2V$ by Proposition 4.2.1. This shows $iii)$ and concludes the proof. □

4.3 Doubly warped products with imaginary Kählerian Killing spinors

In this section, we describe the so-called doubly-warped products carrying non-zero imaginary Kählerian Killing spinors. Doubly warped products were introduced in the spinorial context by Patrick Baier in his master thesis [C1] to compute the Dirac spectrum of the complex hyperbolic space, using its representation as a doubly-warped product over an odd-dimensional sphere.

First we recall general formulas on warped products.

Lemma 4.3.1 *Let $(\widetilde{M} := M \times I, \widetilde{g} := g_t \oplus \beta dt^2)$ be a warped product, where $I \subset \mathbb{R}$ is an open interval, g_t is a smooth 1-parameter family of Riemannian metrics on M and $\beta \in C^\infty(M \times I, \mathbb{R}_+^\times)$. Denote by $\widetilde{M} \xrightarrow{\pi_1} M$ the first projection. Then, for all $X, Y \in \Gamma(\pi_1^* TM)$,*

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= -\frac{1}{2}\mathrm{grad}_{g_t}(\beta(t,\cdot)) + \frac{1}{2\beta}\frac{\partial \beta}{\partial t}\frac{\partial}{\partial t} \\
\widetilde{\nabla}_{\frac{\partial}{\partial t}} X &= \frac{\partial X}{\partial t} + \frac{1}{2} g_t^{-1}\frac{\partial g_t}{\partial t}(X,\cdot) + \frac{1}{2\beta}\frac{\partial \beta}{\partial x}(X)\frac{\partial}{\partial t} \\
\widetilde{\nabla}_X \frac{\partial}{\partial t} &= \frac{1}{2} g_t^{-1}\frac{\partial g_t}{\partial t}(X,\cdot) + \frac{1}{2\beta}\frac{\partial \beta}{\partial x}(X)\frac{\partial}{\partial t} \\
\widetilde{\nabla}_X Y &= \nabla^M_X Y - \frac{1}{2\beta}\frac{\partial g_t}{\partial t}(X,Y)\frac{\partial}{\partial t},
\end{aligned}$$

where $\frac{\partial X}{\partial t} = [\frac{\partial}{\partial t}, X]$ and ∇^M (resp. $\widetilde{\nabla}$) is the Levi-Civita covariant derivative of (M, g_t) (resp. of $(\widetilde{M}, \widetilde{g})$).

Proof: straightforward consequence of the Koszul identity. □

From now on we restrict ourselves to the following particular case: the manifold M will be equipped with a *Riemannian flow*.

Definition 4.3.2

i) A Riemannian flow *is a triple $(M, \widehat{g}, \widehat{\xi})$, where M is a smooth manifold and $\widehat{\xi}$ is a smooth unit vector field whose flow is isometric on the orthogonal distribution, i.e., $\widehat{g}(\widehat{\nabla}^M_Z \widehat{\xi}, Z') = -\widehat{g}(Z, \widehat{\nabla}^M_{Z'} \widehat{\xi})$ for all $Z, Z' \in \widehat{\xi}^\perp$, where $\widehat{\nabla}^M$ denotes the Levi-Civita covariant derivative of $(M, \widehat{g})$.*

ii) A Riemannian flow $(M, \widehat{g}, \widehat{\xi})$ is called minimal *if and only if $\widehat{\nabla}^M_{\widehat{\xi}} \widehat{\xi} = 0$, that is, if $\widehat{\xi}$ is actually a Killing vector field on M.*

Let $(M,\widehat{g},\widehat{\xi})$ be a minimal Riemannian flow. Let $\widehat{h}$ denote the endomorphism-field of $\widehat{\xi}^{\perp}$ defined by $\widehat{h}(Z) := \widehat{\nabla}^M_Z\widehat{\xi}$ for every $Z\in\widehat{\xi}^{\perp}$. Let $\widehat{\nabla}$ be the covariant derivative on $\widehat{\xi}^{\perp}$ defined for all $Z\in\Gamma(\widehat{\xi}^{\perp})$ by $\widehat{\nabla}_X Z := \begin{cases} [\widehat{\xi},Z]^{\widehat{\xi}^{\perp}} & \text{if } X=\widehat{\xi} \\ (\widehat{\nabla}^M_X Z)^{\widehat{\xi}^{\perp}} & \text{if } X\perp\widehat{\xi}\end{cases}$. Alternatively, $\widehat{\nabla}$ can be described by the following formulas: for all $Z,Z'\in\Gamma(\widehat{\xi}^{\perp})$,

$$\widehat{\nabla}^M_{\widehat{\xi}} Z = \widehat{\nabla}_{\widehat{\xi}} Z + \widehat{h}(Z) \qquad \text{and} \qquad \widehat{\nabla}^M_Z Z' = \widehat{\nabla}_Z Z' - \widehat{g}(\widehat{h}(Z),Z')\widehat{\xi}.$$

It is important to notice that, if $(M,\widehat{g},\widehat{\xi})$ is a (minimal) Riemannian flow and $g := r^2(s^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^{\perp}})$ for some constants $r,s>0$, then $(M,g,\xi := \frac{1}{rs}\widehat{\xi})$ is a (minimal) Riemannian flow with corresponding objects given by

$$h = \frac{s}{r}\widehat{h} \qquad \text{and} \qquad \nabla = \widehat{\nabla}. \tag{4.2}$$

In this language, a *Sasakian* manifold is a minimal Riemannian flow $(M,\widehat{g},\widehat{\xi})$ such that $\widehat{h}$ is a transversal Kähler structure, that is, $\widehat{h}^2 = -\mathrm{Id}_{\widehat{\xi}^{\perp}}$ and $\widehat{\nabla}\widehat{h} = 0$. Further on in the text we shall need for normalization purposes so-called *$\mathcal{D}$-homothetic deformations* of a Sasakian structure: a $\mathcal{D}$-homothetic deformation of $(M,\widehat{g},\widehat{\xi})$ is $(M,\lambda^2(\lambda^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^{\perp}}),\frac{1}{\lambda^2}\widehat{\xi})$ for some $\lambda\in\mathbb{R}_+^{\times}$. The identities (4.2) imply that $(M,\lambda^2(\lambda^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^{\perp}}),\frac{1}{\lambda^2}\widehat{\xi})$ is Sasakian as soon as $(M,\widehat{g},\widehat{\xi})$ is Sasakian.

We can now make the concept of doubly-warped product precise:

Definition 4.3.3 *A* doubly-warped product *is a warped product of the form*

$$(\widetilde{M},\widetilde{g}) := (M\times I, \rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^{\perp}})\oplus dt^2),$$

where I is an open interval, $(M,\widehat{g},\widehat{\xi})$ is a minimal Riemannian flow, $\rho,\sigma : I\longrightarrow\mathbb{R}_+^{\times}$ are smooth functions and $\widehat{g}_{\widehat{\xi}} := \widehat{g}_{|_{\mathbb{R}\widehat{\xi}\oplus\mathbb{R}\widehat{\xi}}}$, $\widehat{g}_{\widehat{\xi}^{\perp}} := \widehat{g}_{|_{\widehat{\xi}^{\perp}\oplus\widehat{\xi}^{\perp}}}$.

As for warped products, it can be easily proved that a doubly-warped product $(\widetilde{M},\widetilde{g})$ is complete as soon as $I=\mathbb{R}$ and $(M,\widehat{g})$ is complete.

It is easy to check that, setting $g_t := \rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^{\perp}})$, one has $\frac{\partial g_t}{\partial t} = 2\frac{\rho'}{\rho}g_t + \frac{2\sigma'}{\sigma}g_t(\pi_{\widehat{\xi}^{\perp}},\cdot)$ and the unit vector field providing the Riemannian flow on (M,g_t) is

$\xi = \frac{1}{\rho\sigma}\widehat{\xi}$. In particular, the formulas in Lemma 4.3.1 simplify:

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} &= 0\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\xi &= 0\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}Z &= \frac{\partial Z}{\partial t}+\frac{\rho'}{\rho}Z\\
\widetilde{\nabla}_{\xi}\frac{\partial}{\partial t} &= \frac{(\rho\sigma)'}{\rho\sigma}\xi\\
\widetilde{\nabla}_{\xi}\xi &= -\frac{(\rho\sigma)'}{\rho\sigma}\frac{\partial}{\partial t}\\
\widetilde{\nabla}_{\xi}Z &= \nabla_{\xi}Z+h(Z)\\
\widetilde{\nabla}_{Z}\frac{\partial}{\partial t} &= \frac{\rho'}{\rho}Z\\
\widetilde{\nabla}_{Z}\xi &= h(Z)\\
\widetilde{\nabla}_{Z}Z' &= \nabla_{Z}Z'-g_t(h(Z),Z')\xi-\frac{\rho'}{\rho}g_t(Z,Z')\frac{\partial}{\partial t},
\end{aligned}$$

where we have denoted the corresponding objects on (M, g_t, ξ) without the hat "$\widehat{\cdot}$".

Next we look at a possible construction of Kähler structures on doubly-warped products.

Lemma 4.3.4 *Let $(\widetilde{M},\widetilde{g}) := (M\times I, \rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp})\oplus dt^2)$ be a doubly-warped product. Assume the existence of a transversal Kähler structure J on $(M,\widehat{g},\widehat{\xi})$ and define the almost complex structure $\widetilde{J}$ on $\widetilde{M}$ by $\widetilde{J}(\xi) := \frac{\partial}{\partial t}$, $\widetilde{J}(\frac{\partial}{\partial t}) := -\xi$ and $\widetilde{J}(Z) := J(Z)$ for all $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$. Then $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is Kähler if and only if $\widehat{h} = -\frac{\rho'}{\sigma}J$ on $\{\xi,\frac{\partial}{\partial t}\}^\perp$ (in particular $\frac{\rho'}{\sigma}$ must be constant).*

Proof: Using the identities above we write down the condition $\widetilde{\nabla}\widetilde{J} = 0$. Denote by h and ∇ the objects corresponding to g_t on M. Note first that, by definition and (4.2), one has $\nabla J = 0$ on $\{\xi,\frac{\partial}{\partial t}\}^\perp$ and $\widetilde{J}_{|_{\{\xi,\frac{\partial}{\partial t}\}^\perp}} = J$, which does not depend on t. Hence we obtain, for all $Z, Z' \in \Gamma(\widehat{\xi}^{\perp})$:

$$\begin{array}{rcl}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\widetilde{J}(\frac{\partial}{\partial t})) - \widetilde{J}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}) & = & 0 \\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\widetilde{J}(\xi)) - \widetilde{J}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\xi) & = & 0 \\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\widetilde{J}(Z)) - \widetilde{J}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}Z) & = & \frac{\partial(\widetilde{J}(Z))}{\partial t} - \widetilde{J}(\frac{\partial Z}{\partial t}) = \frac{\partial(J(Z))}{\partial t} - J(\frac{\partial Z}{\partial t}) = 0 \\
\widetilde{\nabla}_{\xi}(\widetilde{J}(\frac{\partial}{\partial t})) - \widetilde{J}(\widetilde{\nabla}_{\xi}\frac{\partial}{\partial t}) & = & 0 \\
\widetilde{\nabla}_{\xi}(\widetilde{J}(\xi)) - \widetilde{J}(\widetilde{\nabla}_{\xi}\xi) & = & 0 \\
\widetilde{\nabla}_{\xi}(\widetilde{J}(Z)) - \widetilde{J}(\widetilde{\nabla}_{\xi}Z) & = & h \circ J(Z) - J \circ h(Z) \\
\widetilde{\nabla}_{Z}(\widetilde{J}(\frac{\partial}{\partial t})) - \widetilde{J}(\widetilde{\nabla}_{Z}\frac{\partial}{\partial t}) & = & -h(Z) - \frac{\rho'}{\rho}J(Z) \\
\widetilde{\nabla}_{Z}(\widetilde{J}(\xi)) - \widetilde{J}(\widetilde{\nabla}_{Z}\xi) & = & \frac{\rho'}{\rho}Z - J \circ h(Z) \\
\widetilde{\nabla}_{Z}(\widetilde{J}(Z')) - \widetilde{J}(\widetilde{\nabla}_{Z}Z') & = & -g_t(h(Z), J(Z'))\xi - \frac{\rho'}{\rho}g_t(Z, J(Z'))\frac{\partial}{\partial t} + g_t(h(Z), Z')\frac{\partial}{\partial t} - \frac{\rho'}{\rho}g_t(Z, Z')\xi.
\end{array}$$

Therefore, $\widetilde{\nabla}\widetilde{J} = 0$ implies $h = -\frac{\rho'}{\rho}J$ on $\xi^{\perp}$ which, in turn, implies $h \circ J = J \circ h$. Moreover, (4.2) implies that $h = \frac{\sigma}{\rho}\widehat{h}$, which yields $\widehat{h} = -\frac{\rho'}{\sigma}J$. The reverse implication is obvious. □

Notes 4.3.5

1. With the assumptions of Lemma 4.3.4, the function ρ' vanishes either identically or nowhere on the interval I. In the former case the vanishing of $\widehat{h}$ is equivalent to M being locally the Riemannian product of an interval with a Kähler manifold; in the latter one, we may assume, up to changing σ into $|\frac{\rho'}{\sigma}|\sigma$ (and $\widehat{g}$ into $(\frac{\sigma}{\rho'})^2\widehat{g}_{\widehat{\xi}} \oplus \widehat{g}_{\widehat{\xi}^{\perp}}$), that $\widehat{h} = -\varepsilon J$ and $\rho' = \varepsilon\sigma$ with $\varepsilon \in \{\pm 1\}$.

2. Given a Kähler doubly warped product $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ as in Lemma 4.3.4 and a real constant C, the map $(x,t) \mapsto (x, \pm t + C)$ provides a holomorphic isometry $(\widetilde{M}, \widetilde{g}, \widetilde{J}) \longrightarrow (\widetilde{M}', \widetilde{g}', \widetilde{J}')$, where $(\widetilde{M}', \widetilde{g}') := (M \times (C \pm I), g_{\pm t + C} \oplus dt^2)$ and $\widetilde{J}'$ is the corresponding complex structure (again as in Lemma 4.3.4). If furthermore M is spin, then this isometry preserves the corresponding spin structures. Thus, in the case where $\rho' \neq 0$, we may assume that $\varepsilon = 1$, i.e., that $\widehat{h} = -J$ and $\rho' = \sigma$.

Now we examine the correspondence of spinors. Let the underlying manifold M of some minimal Riemannian flow (M, g, ξ) be spin and, in case M is the total space of a Riemannian submersion with $\mathbb{S}^1$-fibres over a spin manifold N, let M carry the spin structure induced by that of N. Let ΣM denote the spinor bundle of (M, g) and "$\underset{M}{\cdot}$" its Clifford multiplication. Let the doubly warped product $\widetilde{M}$ carry the product spin structure (with Clifford multiplication denoted by "$\cdot$"). Then the transversal covariant derivative ∇ induces a covariant derivative - also denoted by ∇ - on ΣM, which is related to the spinorial Levi-Civita covariant derivative ∇^M on ΣM via (see e.g. [C7, eq. (2.4.7)] or [C8, Sec. 4])

$$\nabla^M_\xi \varphi = \nabla_\xi \varphi + \frac{1}{4}\sum_{j=1}^{2n-2} e_j \underset{M}{\cdot} h(e_j) \underset{M}{\cdot} \varphi \qquad \text{and} \qquad \nabla^M_Z \varphi = \nabla_Z \varphi + \frac{1}{2}\xi \underset{M}{\cdot} h(Z) \underset{M}{\cdot} \varphi$$

for every $\varphi \in \Gamma(\Sigma M)$, where $\{e_j\}_{1\le j\le 2n-2}$ is a local orthonormal basis of $\xi^\perp \subset TM$.

Lemma 4.3.6 *Let a minimal Riemannian flow $(M, \widehat{g}, \widehat{\xi})$ carry a transversal Kähler structure J such that the doubly-warped product $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is Kähler, where $\widetilde{J}$ is the almost-complex structure induced by J as in* Lemma 4.3.4*. Assume furthermore M to be spin. Let $\widetilde{M}$ carry the induced spin structure. Then the following identities hold for all $\varphi \in \Gamma(\Sigma\widetilde{M})$ and $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$:*

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi &= \frac{\partial \varphi}{\partial t}\\
\widetilde{\nabla}_{\xi}\varphi &= \nabla_\xi \varphi - \frac{\rho'}{2\rho}\widetilde{\Omega}\cdot\varphi - \frac{\sigma'}{2\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi\\
\widetilde{\nabla}_{Z}\varphi &= \nabla_Z \varphi - \frac{\rho'}{2\rho}(\xi\cdot J(Z) + Z\cdot\frac{\partial}{\partial t})\cdot\varphi,
\end{aligned}$$

where $\widetilde{\Omega}$ denotes the Kähler form of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$.

Proof: Let $(e_1, \ldots, e_{2n-2}, e_{2n-1} := \xi, e_{2n} := \frac{\partial}{\partial t})$ be a local positively-oriented orthonormal basis of $T\widetilde{M}$ and $(\psi_\alpha)_\alpha$ the corresponding spinorial frame. It can be assumed that $e_j = \rho^{-1}\widehat{e}_j$ with $\widehat{g}(\widehat{e}_j, \widehat{e}_k) = \delta_{jk}$ and $\frac{\partial \widehat{e}_j}{\partial t} = 0$ (extend some $\widehat{g}$-orthonormal basis independently of time). Split $\varphi = \sum_\alpha c_\alpha \psi_\alpha$, then

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi &= \frac{1}{4}\sum_\alpha c_\alpha \sum_{j,k=1}^{2n} \widetilde{g}(\widetilde{\nabla}_{\frac{\partial}{\partial t}} e_j, e_k) e_j\cdot e_k\cdot\psi_\alpha + \underbrace{\sum_\alpha \frac{\partial c_\alpha}{\partial t}\psi_\alpha}_{=:\frac{\partial\varphi}{\partial t}}\\
&= \frac{\partial\varphi}{\partial t} + \frac{1}{4}\sum_\alpha c_\alpha \sum_{j,k=1}^{2n-2} \widetilde{g}(\widetilde{\nabla}_{\frac{\partial}{\partial t}} e_j, e_k) e_j\cdot e_k\cdot\psi_\alpha\\
&= \frac{\partial\varphi}{\partial t} + \frac{1}{4}\sum_\alpha c_\alpha \sum_{j,k=1}^{2n-2} \{g_t(\frac{\partial e_j}{\partial t}, e_k) + \frac{\rho'}{\rho}\delta_{jk}\} e_j\cdot e_k\cdot\psi_\alpha\\
&= \frac{\partial\varphi}{\partial t},
\end{aligned}$$

where we have used $\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \widetilde{\nabla}_{\frac{\partial}{\partial t}}\xi = 0$ and $\frac{\partial e_j}{\partial t} = -\frac{\rho'}{\rho}e_j$ by the above choice of e_j. On the other hand, the Weingarten endomorphism field of (M, g_t) in $\widetilde{M}$ is given by $A(\xi) := -\widetilde{\nabla}_\xi \frac{\partial}{\partial t} = -\frac{(\rho\sigma)'}{\rho\sigma}\xi$ and $A(Z) := -\widetilde{\nabla}_Z \frac{\partial}{\partial t} = -\frac{\rho'}{\rho}Z$ for all $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$, so that the Gauss-Weingarten formula implies

$$\begin{aligned}
\widetilde{\nabla}_\xi\varphi &= \nabla^M_\xi\varphi + \frac{1}{2}A(\xi)\cdot\frac{\partial}{\partial t}\cdot\varphi \\
&= \nabla_\xi\varphi + \frac{1}{4}\sum_{j=1}^{2n-2} e_j \underset{M}{\cdot} h(e_j) \underset{M}{\cdot} \varphi - \frac{(\rho\sigma)'}{2\rho\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi \\
&= \nabla_\xi\varphi - \frac{\rho'}{4\rho}\sum_{j=1}^{2n-2} e_j\cdot J(e_j)\cdot\varphi - \frac{(\rho\sigma)'}{2\rho\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi \\
&= \nabla_\xi\varphi - \frac{\rho'}{2\rho}\Omega\cdot\varphi - \frac{(\rho\sigma)'}{2\rho\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi,
\end{aligned}$$

where Ω is the 2-form associated to J on $\{\xi,\frac{\partial}{\partial t}\}^\perp$, i.e., $\Omega(Z,Z') = g_t(J(Z),Z')$ for all $Z,Z'\in\{\xi,\frac{\partial}{\partial t}\}^\perp$. Since $\widetilde{\Omega} = \Omega + \xi\wedge\frac{\partial}{\partial t}$, we deduce that

$$\begin{aligned}
\widetilde{\nabla}_\xi\varphi &= \nabla_\xi\varphi - \frac{\rho'}{2\rho}\widetilde{\Omega}\cdot\varphi + \left(\frac{\rho'}{2\rho} - \frac{(\rho\sigma)'}{2\rho\sigma}\right)\xi\cdot\frac{\partial}{\partial t}\cdot\varphi \\
&= \nabla_\xi\varphi - \frac{\rho'}{2\rho}\widetilde{\Omega}\cdot\varphi - \frac{\sigma'}{2\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi.
\end{aligned}$$

For any $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$, one has

$$\begin{aligned}
\widetilde{\nabla}_Z\varphi &= \nabla^M_Z\varphi + \frac{1}{2}A(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi \\
&= \nabla_Z\varphi + \frac{1}{2}\xi \underset{M}{\cdot} h(Z) \underset{M}{\cdot} \varphi - \frac{\rho'}{2\rho}Z\cdot\frac{\partial}{\partial t}\cdot\varphi \\
&= \nabla_Z\varphi - \frac{\rho'}{2\rho}\xi\cdot J(Z)\cdot\varphi - \frac{\rho'}{2\rho}Z\cdot\frac{\partial}{\partial t}\cdot\varphi,
\end{aligned}$$

which shows the last identity and concludes the proof. □

Later on we shall need to split spinors into different components. Recall that, on any Kähler spin manifold $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$, the spinor bundle $\Sigma\widetilde{M}$ of $(\widetilde{M}^{2n},\widetilde{g})$ splits under the Clifford action of the Kähler form $\widetilde{\Omega}$ into

$$\Sigma\widetilde{M} = \bigoplus_{r=0}^{n}\Sigma_r\widetilde{M},$$

where $\Sigma_r\widetilde{M} := \mathrm{Ker}(\widetilde{\Omega}\cdot - i(2r-n)\mathrm{Id})$. Now if $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is a doubly-warped product as above, then any $\varphi\in\Sigma_r\widetilde{M}$ (with $r\in\{0,1,\ldots,n\}$) can be further split into eigenvectors for the Clifford action of $\Omega = g(J\cdot,\cdot)$. Namely, since $[\xi\wedge\frac{\partial}{\partial t},\Omega] = 0$, the automorphism $\xi\cdot\frac{\partial}{\partial t}$ of $\Sigma\widetilde{M}$ leaves $\Sigma_r\widetilde{M}$ invariant; from $(\xi\cdot\frac{\partial}{\partial t})^2 = -1$ one deduces the orthogonal decomposition $\Sigma_r\widetilde{M} = \mathrm{Ker}(\xi\cdot\frac{\partial}{\partial t} + i\mathrm{Id})\oplus\mathrm{Ker}(\xi\cdot\frac{\partial}{\partial t} - i\mathrm{Id})$. Since both Clifford actions of ξ and $\frac{\partial}{\partial t}$ are ∇-parallel, so is the latter splitting. But, for any $\varphi\in\Sigma_r\widetilde{M}$, one has

$$\begin{aligned}
\varphi\in\mathrm{Ker}(\xi\cdot\frac{\partial}{\partial t}\pm i\mathrm{Id}) \quad &\Longleftrightarrow \quad \Omega\cdot\varphi = i(2r-n)\varphi \pm i\varphi \\
&\Longleftrightarrow \quad \Omega\cdot\varphi = i(2r-n\pm 1)\varphi,
\end{aligned}$$

that is, $\Sigma_r\widetilde{M}\cap\mathrm{Ker}(\xi\cdot\frac{\partial}{\partial t}+i\mathrm{Id})=\Sigma_rM$ and $\Sigma_r\widetilde{M}\cap\mathrm{Ker}(\xi\cdot\frac{\partial}{\partial t}-i\mathrm{Id})=\Sigma_{r-1}M$, where by definition $\Sigma_rM:=\mathrm{Ker}(\Omega\cdot-i(2r-(n-1)\mathrm{Id}))$ for $r\in\{0,1,\ldots,n-1\}$ and $\{0\}$ otherwise. Out of dimensional reasons one actually has

$$\Sigma_r\widetilde{M}=\Sigma_rM\oplus\Sigma_{r-1}M \tag{4.3}$$

for every $r\in\{0,1,\ldots,n\}$. Beware here that, if r is even, then $\Sigma_r\widetilde{M}$ is a subspace of $\Sigma^+\widetilde{M}$ hence $\Sigma_r\widetilde{M}_{|_M}$ is canonically identified with a subspace of $\Sigma^+\widetilde{M}_{|_M}=\Sigma M$, whereas if r is odd then it is a subspace of $\Sigma_-\widetilde{M}$ and is also identified as a subspace of ΣM, but this time with opposite Clifford multiplication.

Lemma 4.3.7 *Under the hypotheses of* Lemma 4.3.6, *let* $\varphi\in\Gamma(\Sigma_r\widetilde{M})$ *for some* $r\in\{0,1\ldots,n\}$ *and consider its decomposition* $\varphi=\varphi_r+\varphi_{r-1}$ *w.r.t.* (4.3). *Then the identities of* Lemma 4.3.6 *read:*

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi_r &= \frac{\partial\varphi_r}{\partial t}\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi_{r-1} &= \frac{\partial\varphi_{r-1}}{\partial t}\\
\widetilde{\nabla}_{\xi}\varphi_r &= \nabla_\xi\varphi_r+\frac{i}{2}((n-2r)\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\varphi_r\\
\widetilde{\nabla}_{\xi}\varphi_{r-1} &= \nabla_\xi\varphi_{r-1}+\frac{i}{2}((n-2r)\frac{\rho'}{\rho}-\frac{\sigma'}{\sigma})\varphi_{r-1}\\
\widetilde{\nabla}_Z\varphi &= \nabla_Z\varphi_r-\frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_{r-1}+\nabla_Z\varphi_{r-1}-\frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_r
\end{aligned}$$

for all $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$*, where, as usual,* $p_\pm(Z)=\frac{1}{2}(Z\mp iJ(Z))$.

Proof: The first two identities follow from $\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\xi\wedge\frac{\partial}{\partial t})=0$ and $\frac{\partial J}{\partial t}=0$. For the third and fourth ones, note that $\widetilde{\nabla}_\xi(\xi\wedge\frac{\partial}{\partial t})=0$, so that

$$\begin{aligned}
\widetilde{\nabla}_\xi\varphi_r+\widetilde{\nabla}_\xi\varphi_{r-1} &= \nabla_\xi\varphi_r+\nabla_\xi\varphi_{r-1}-\frac{i\rho'}{2\rho}(2r-n)(\varphi_r+\varphi_{r-1})-\frac{i\sigma'}{2\sigma}(\varphi_{r-1}-\varphi_r)\\
&= \nabla_\xi\varphi_r+\frac{i}{2}((n-2r)\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\varphi_r+\nabla_\xi\varphi_{r-1}+\frac{i}{2}((n-2r)\frac{\rho'}{\rho}-\frac{\sigma'}{\sigma})\varphi_{r-1},
\end{aligned}$$

which is the result. As for the last identity, one does not have $\widetilde{\nabla}_Z(\xi\wedge\frac{\partial}{\partial t})=0$, however

$$\begin{aligned}
(\xi\cdot J(Z)+Z\cdot\frac{\partial}{\partial t})\cdot\varphi &= (-J(Z)\cdot\frac{\partial}{\partial t}\cdot\xi\cdot\frac{\partial}{\partial t}+Z\cdot\frac{\partial}{\partial t})\cdot\varphi\\
&= -iJ(Z)\cdot\frac{\partial}{\partial t}\cdot(\varphi_{r-1}-\varphi_r)+Z\cdot\frac{\partial}{\partial t}\cdot(\varphi_r+\varphi_{r-1})\\
&= 2p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_{r-1}+2p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_r
\end{aligned}$$

for all $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$. This concludes the proof. □

We now have all we need to rewrite the imaginary Kähler Killing spinor equation on doubly warped products.

Lemma 4.3.8 *Let a spin minimal Riemannian flow $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ carry a transversal Kähler structure J such that the doubly-warped product $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is Kähler, where $\widetilde{J}$ is the almost-complex structure induced by J as in* Lemma 4.3.4. *Let $\widetilde{M}$ carry the induced spin structure and assume $n \geq 3$ to be odd. Then a pair (ψ, ϕ) is an i-Kählerian Killing spinor on $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ if and only if the following identities are satisfied by the components $\phi = \phi_{\frac{n+1}{2}} + \phi_{\frac{n-1}{2}}$ and $\psi = \psi_{\frac{n-1}{2}} + \psi_{\frac{n-3}{2}}$ w.r.t.* (4.3)*:*

$$\left|\begin{array}{ll}
\frac{\partial \phi_{\frac{n+1}{2}}}{\partial t} & = 0 \\
\frac{\partial \phi_{\frac{n-1}{2}}}{\partial t} & = -i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}} \\
\frac{\partial \psi_{\frac{n-1}{2}}}{\partial t} & = -i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}} \\
\frac{\partial \psi_{\frac{n-3}{2}}}{\partial t} & = 0 \\
\nabla_\xi \phi_{\frac{n+1}{2}} & = \frac{i}{2}(\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma})\phi_{\frac{n+1}{2}} \\
\nabla_\xi \phi_{\frac{n-1}{2}} & = \frac{i}{2}(\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\phi_{\frac{n-1}{2}} - \frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}} \\
\nabla_\xi \psi_{\frac{n-1}{2}} & = -\frac{i}{2}(\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\psi_{\frac{n-1}{2}} + \frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}} \\
\nabla_\xi \psi_{\frac{n-3}{2}} & = -\frac{i}{2}(\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma})\psi_{\frac{n-3}{2}} \\
\nabla_Z \phi_{\frac{n+1}{2}} & = p_+(Z)\cdot(\frac{\rho'}{\rho}\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}} - i\psi_{\frac{n-1}{2}}) \\
\nabla_Z \phi_{\frac{n-1}{2}} & = \frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n+1}{2}} - ip_+(Z)\cdot\psi_{\frac{n-3}{2}} \\
\nabla_Z \psi_{\frac{n-1}{2}} & = \frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\psi_{\frac{n-3}{2}} - ip_-(Z)\cdot\phi_{\frac{n+1}{2}} \\
\nabla_Z \psi_{\frac{n-3}{2}} & = p_-(Z)\cdot(\frac{\rho'}{\rho}\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}} - i\phi_{\frac{n-1}{2}})
\end{array}\right. \tag{4.4}$$

for every $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$.

Proof: Since $p_+(\frac{\partial}{\partial t})\cdot\psi = \frac{1}{2}(\frac{\partial}{\partial t} + i\xi)\cdot\psi = \frac{1}{2}\frac{\partial}{\partial t}\cdot(1 + i\xi\cdot\frac{\partial}{\partial t}\cdot)\psi = \frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}$ and similarly $p_-(\frac{\partial}{\partial t})\cdot\phi = \frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}$, the i-Kählerian Killing spinor equation is satisfied by (ψ, ϕ) for $X = \frac{\partial}{\partial t}$ if and only if

$$\begin{aligned}
\frac{\partial \phi_{\frac{n+1}{2}}}{\partial t} + \frac{\partial \phi_{\frac{n-1}{2}}}{\partial t} &= -ip_+(\frac{\partial}{\partial t})\cdot\psi = -i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}} \\
\frac{\partial \psi_{\frac{n-1}{2}}}{\partial t} + \frac{\partial \psi_{\frac{n-3}{2}}}{\partial t} &= -ip_-(\frac{\partial}{\partial t})\cdot\phi = -i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}},
\end{aligned}$$

which gives the first four identities (use $[\Omega, \frac{\partial}{\partial t}] = 0$).
From $p_+(\xi)\cdot\psi = -ip_+(\frac{\partial}{\partial t})\cdot\psi = -i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}$ and $p_-(\xi)\cdot\phi = ip_-(\frac{\partial}{\partial t})\cdot\phi = i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}$ we deduce that the i-Kählerian Killing spinor equation is satisfied by (ψ, ϕ) for $X = \xi$ if and only if

$$\begin{aligned}
\nabla_\xi \phi_{\frac{n+1}{2}} + \frac{i}{2}(-\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\phi_{\frac{n+1}{2}} &= 0 \\
\nabla_\xi \phi_{\frac{n-1}{2}} - \frac{i}{2}(\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\phi_{\frac{n-1}{2}} &= -\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}} \\
\nabla_\xi \psi_{\frac{n-1}{2}} + \frac{i}{2}(\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\psi_{\frac{n-1}{2}} &= \frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}} \\
\nabla_\xi \psi_{\frac{n-3}{2}} + \frac{i}{2}(\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma})\psi_{\frac{n-3}{2}} &= 0,
\end{aligned}$$

which implies the next four equations.
Let $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$, then the i-Kählerian Killing spinor equation is satisfied by (ψ, ϕ) for $X = Z$ if and only if

$$\begin{aligned}
-ip_+(Z)\cdot\psi_{\frac{n-1}{2}} &= \nabla_Z\phi_{\frac{n+1}{2}} - \frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}\\
-ip_+(Z)\cdot\psi_{\frac{n-3}{2}} &= \nabla_Z\phi_{\frac{n-1}{2}} - \frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n+1}{2}}\\
-ip_-(Z)\cdot\phi_{\frac{n+1}{2}} &= \nabla_Z\psi_{\frac{n-1}{2}} - \frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\psi_{\frac{n-3}{2}}\\
-ip_-(Z)\cdot\phi_{\frac{n-1}{2}} &= \nabla_Z\psi_{\frac{n-3}{2}} - \frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}},
\end{aligned}$$

which concludes the proof. $\square$

Next we want to describe all doubly warped products with non-zero imaginary Kählerian Killing spinors.

Theorem 4.3.9 *For $n \geq 3$ odd let $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be a Kähler spin doubly warped product as in* Lemma 4.3.8*. If there exists a non-zero i-Kählerian Killing spinor (ψ, ϕ) on $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$, then*

- *the minimal Riemannian flow $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is Sasakian,*
- *up to changing t into $-t$, applying a $\mathscr{D}$-homothety and translating the interval I by a constant, one has either $\rho = e^t$ or $\rho = \sinh$ or $\rho = \cosh$,*
- *the components ψ_r and ϕ_r of (ψ, ϕ) w.r.t.* (4.3) *satisfy:*

i) In case $\rho = e^t$: Then $\sigma = e^t$ and, setting $\widetilde{\psi}_{\frac{n-3}{2}} := i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-3}{2}}$ and $\varphi_{\frac{n-1}{2}} := e^t(\phi_{\frac{n-1}{2}} + i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}})$, one has

$$\left|\begin{aligned}
\frac{\partial}{\partial t}\phi_{\frac{n+1}{2}} &= 0\\
\frac{\partial}{\partial t}\widetilde{\psi}_{\frac{n-3}{2}} &= 0\\
\frac{\partial}{\partial t}\varphi_{\frac{n-1}{2}} &= 0\\
\widehat{\nabla}_{\widehat{\xi}}\phi_{\frac{n+1}{2}} &= 0\\
\widehat{\nabla}_{\widehat{\xi}}\widetilde{\psi}_{\frac{n-3}{2}} &= 0\\
\widehat{\nabla}\varphi_{\frac{n-1}{2}} &= 0\\
\widehat{\nabla}_Z\phi_{\frac{n+1}{2}} &= (-1)^{\frac{n+1}{2}}p_+(Z)\,\widehat{\cdot}_M\,\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_Z\widetilde{\psi}_{\frac{n-3}{2}} &= (-1)^{\frac{n+1}{2}}p_-(Z)\,\widehat{\cdot}_M\,\varphi_{\frac{n-1}{2}}.
\end{aligned}\right.$$

If furthermore $\varphi_{\frac{n-1}{2}} = 0$, then for $\widehat{\phi}_{\frac{n-1}{2}} := e^{-t}\phi_{\frac{n-1}{2}}$ one has $\frac{\partial}{\partial t}\widehat{\phi}_{\frac{n-1}{2}} = 0$ and

$$\left|\begin{aligned}
\widehat{\nabla}\phi_{\frac{n+1}{2}} &= 0\\
\widehat{\nabla}\widetilde{\psi}_{\frac{n-3}{2}} &= 0\\
\widehat{\nabla}_{\widehat{\xi}}\widehat{\phi}_{\frac{n-1}{2}} &= 0\\
\widehat{\nabla}_Z\widehat{\phi}_{\frac{n-1}{2}} &= (-1)^{\frac{n+1}{2}}(p_-(Z)\,\widehat{\cdot}_M\,\phi_{\frac{n+1}{2}} + p_+(Z)\,\widehat{\cdot}_M\,\widetilde{\psi}_{\frac{n-3}{2}}).
\end{aligned}\right.$$

In particular, the manifold $(M^{2n-1},\widehat{g},\widehat{\xi})$ *admits a non-zero transversally parallel spinor. Conversely, every non-zero transversally parallel spinor* $\widehat{\phi}_{\frac{n-1}{2}}\in\Gamma(\Sigma_{\frac{n-1}{2}}M)$ *provides a non-zero i-Kählerian Killing spinor by setting* $\phi_{\frac{n+1}{2}}:=\psi_{\frac{n-3}{2}}:=0$ *and* $\phi_{\frac{n-1}{2}}:=e^t\widehat{\phi}_{\frac{n-1}{2}}$, $\psi_{\frac{n-1}{2}}:=-e^t i\frac{\partial}{\partial t}\cdot\widehat{\phi}_{\frac{n-1}{2}}$. *Moreover, for any i-Kählerian Killing spinor* (ψ,ϕ) *on that doubly warped product* $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$*, the component* $\phi_{\frac{n-1}{2}}$ *is transversally parallel on* $(M,\widehat{g},\widehat{\xi})$ *if and only if* $i\frac{\partial}{\partial t}\cdot\psi=-\phi$.

ii) *In case* $\rho=\sinh$*: One has* $\sigma=\cosh$ *on* $I=\mathbb{R}_+^\times$ *and there is a one-to-one correspondence between the space of i-Kählerian Killing spinors on* $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ *and that of sections* $(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ *of* $\Sigma_{\frac{n+1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-3}{2}}M\longrightarrow M$ *satisfying*

$$\left|\begin{array}{lcl}\widehat{\nabla}_{\widehat{\xi}}\overset{(\sim)}{\varphi}_r&=&\frac{(-1)^r}{2}(n-2r)\widehat{\xi}\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_r\\ \widehat{\nabla}_{\widehat{\xi}}\overset{(\sim)}{\varphi}_{r-1}&=&\frac{-(-1)^r}{2}(n-2r)\widehat{\xi}\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_{r-1}\\ \widehat{\nabla}_Z\overset{(\sim)}{\varphi}_r&=&(-1)^r p_+(Z)\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_{r-1}\\ \widehat{\nabla}_Z\overset{(\sim)}{\varphi}_{r-1}&=&(-1)^r p_-(Z)\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_r\end{array}\right. \tag{4.5}$$

on $(M^{2n-1},\widehat{g},\widehat{\xi})$*, for every* $Z\in\widehat{\xi}^\perp$ *(this means that* $(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}})$ *must satisfy* (4.5) *for* $r=\frac{n+1}{2}$ *and* $(\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ *must satisfy* (4.5) *for* $r=\frac{n-1}{2}$*).*

iii) *In case* $\rho=\cosh$*: One has* $\sigma=\sinh$ *on* $I=\mathbb{R}_+^\times$ *and there is a one-to-one correspondence between the space of i-Kählerian Killing spinors on* $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ *and that of sections* $(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ *of* $\Sigma_{\frac{n+1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-3}{2}}M\longrightarrow M$ *satisfying*

$$\left|\begin{array}{lcl}\widehat{\nabla}_{\widehat{\xi}}\overset{(\sim)}{\varphi}_r&=&-\frac{(-1)^r}{2}(n-2r)\widehat{\xi}\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_r\\ \widehat{\nabla}_{\widehat{\xi}}\overset{(\sim)}{\varphi}_{r-1}&=&\frac{(-1)^r}{2}(n-2r)\widehat{\xi}\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_{r-1}\\ \widehat{\nabla}_Z\overset{(\sim)}{\varphi}_r&=&(-1)^{\frac{n+1}{2}}p_+(Z)\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_{r-1}\\ \widehat{\nabla}_Z\overset{(\sim)}{\varphi}_{r-1}&=&(-1)^{\frac{n-1}{2}}p_-(Z)\underset{M}{\widehat{\cdot}}\overset{(\sim)}{\varphi}_r\end{array}\right. \tag{4.6}$$

on $(M^{2n-1},\widehat{g},\widehat{\xi})$*, for every* $Z\in\widehat{\xi}^\perp$ *(this means that* $(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}})$ *must satisfy* (4.6) *for* $r=\frac{n+1}{2}$ *and* $(\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ *must satisfy* (4.6) *for* $r=\frac{n-1}{2}$*).*

Proof: We first show $\rho''=\rho$ on I. In order to express all equations of (4.4) in an intrinsic way, we have to compare all objects on (M,g_t,ξ) with the corresponding ones on $(M,\widehat{g},\widehat{\xi})$. Recall that $g_t=\rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp})$ and $\xi=\frac{1}{\rho\sigma}\widehat{\xi}$. As for (4.2), it is elementary to check the following relations:

$$\widehat{\nabla}=\nabla,\quad \xi\cdot=\widehat{\xi}\widehat{\cdot},\quad \xi\underset{M}{\cdot}=\widehat{\xi}\underset{M}{\widehat{\cdot}},\quad Z\cdot=\rho Z\widehat{\cdot},\quad Z\underset{M}{\cdot}=\rho Z\underset{M}{\widehat{\cdot}},$$

for all $Z\in\xi^\perp$. Applying $\frac{\partial}{\partial t}$ onto

$$\left|\begin{array}{ll}\widehat{\nabla}_Z\phi_{\frac{n+1}{2}} & =p_+(Z)\widehat{\cdot}(\rho'\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}-i\rho\psi_{\frac{n-1}{2}})\\ \widehat{\nabla}_Z\psi_{\frac{n-3}{2}} & =p_-(Z)\widehat{\cdot}(\rho'\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}-i\rho\phi_{\frac{n-1}{2}})\end{array}\right.$$

and using $\frac{\partial\phi_{\frac{n+1}{2}}}{\partial t}=\frac{\partial\psi_{\frac{n-3}{2}}}{\partial t}=0$, one obtains

$$\begin{array}{rcl}0 & = & p_+(Z)\widehat{\cdot}(\rho''\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}+\rho'\frac{\partial}{\partial t}\cdot\frac{\partial\phi_{\frac{n-1}{2}}}{\partial t}-i\rho'\psi_{\frac{n-1}{2}}-i\rho\frac{\partial\psi_{\frac{n-1}{2}}}{\partial t})\\ & = & p_+(Z)\widehat{\cdot}(\rho''\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}+\rho'\frac{\partial}{\partial t}\cdot(-i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}})-i\rho'\psi_{\frac{n-1}{2}}-i\rho(-i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}))\\ & = & (\rho''-\rho)p_+(Z)\widehat{\cdot}\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}\end{array}$$

and analogously $(\rho''-\rho)p_-(Z)\widehat{\cdot}\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}=0$ for all $Z\in\widehat{\xi}^\perp$. Fix a local $\widehat{g}$-orthonormal basis $(e_j)_{1\leq j\leq 2n-2}$ of $\widehat{\xi}^\perp$. Putting $Z=e_j$, Clifford-multiplying by e_j and summing over j gives $(\rho''-\rho)\phi_{\frac{n-1}{2}}=(\rho''-\rho)\psi_{\frac{n-1}{2}}=0$. On the other hand, both equations involving $\frac{\partial\phi_{\frac{n-1}{2}}}{\partial t}$ and $\frac{\partial\psi_{\frac{n-1}{2}}}{\partial t}$ provide the existence of smooth sections $A^\pm_{\frac{n-1}{2}}$ of $\Sigma_{\frac{n-1}{2}}M$ (independent of t) such that $\phi_{\frac{n-1}{2}}=e^tA^+_{\frac{n-1}{2}}+e^{-t}A^-_{\frac{n-1}{2}}$ and $\psi_{\frac{n-1}{2}}=-e^ti\frac{\partial}{\partial t}\cdot A^+_{\frac{n-1}{2}}+e^{-t}i\frac{\partial}{\partial t}\cdot A^-_{\frac{n-1}{2}}$. We deduce that $(\rho''-\rho)A^+_{\frac{n-1}{2}}=(\rho''-\rho)A^-_{\frac{n-1}{2}}=0$. If both $A^+_{\frac{n-1}{2}}$ and $A^-_{\frac{n-1}{2}}$ vanished identically on M, then so would $\phi_{\frac{n-1}{2}}$ and $\psi_{\frac{n-1}{2}}$ and the identities involving $\widehat{\nabla}_Z\phi_{\frac{n-1}{2}}$ and $\widehat{\nabla}_Z\psi_{\frac{n-1}{2}}$ would provide (after contracting with the Clifford multiplication just as above) $\phi_{\frac{n+1}{2}}=\psi_{\frac{n-3}{2}}=0$, so that $(\psi,\phi)=0$, which is a contradiction. Therefore $\rho''-\rho=0$ on I.

It follows in particular that $\rho'=0$ on I cannot hold, so we may assume that $\widehat{h}=-J$ (hence $(M^{2n-1},\widehat{g},\widehat{\xi})$ is Sasakian) and $\rho'=\sigma$ (see Remarks 4.3.5). Furthermore, in the case where the constant $(\rho')^2-\rho^2$ does not vanish, up to replacing ρ by $\frac{\rho}{\sqrt{|(\rho'^2)-\rho^2|}}$ (which is equivalent to performing a $\mathscr{D}$-homothetic deformation of the Sasakian structure), we may assume that $(\rho'^2)-\rho^2=1$ or -1 on I. Next we rewrite the equations from Lemma 4.3.8 considering the new sections $\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}}$ defined by

$$\left|\begin{array}{ll}\varphi_{\frac{n+1}{2}} & :=\phi_{\frac{n+1}{2}}\\ \varphi_{\frac{n-1}{2}} & :=\rho'\phi_{\frac{n-1}{2}}+i\rho\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}\\ \widetilde{\varphi}_{\frac{n-1}{2}} & :=i\rho\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}+\rho'\psi_{\frac{n-1}{2}}\\ \widetilde{\varphi}_{\frac{n-3}{2}} & :=\psi_{\frac{n-3}{2}}.\end{array}\right.$$

Note that the linear transformation $(\phi_{\frac{n+1}{2}},\phi_{\frac{n-1}{2}},\psi_{\frac{n-1}{2}},\psi_{\frac{n-3}{2}})\mapsto(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ is invertible if and only if $(\rho')^2-\rho^2\neq 0$. From (4.4) we have, for all $Z \in \widehat{\xi}^{\perp}$:

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_{\frac{n+1}{2}} &= 0\\
\frac{\partial}{\partial t}\varphi_{\frac{n-1}{2}} &= 0\\
\frac{\partial}{\partial t}\widetilde{\varphi}_{\frac{n-1}{2}} &= 0\\
\frac{\partial}{\partial t}\widetilde{\varphi}_{\frac{n-3}{2}} &= 0\\
\widehat{\nabla}_{\widehat{\xi}}\varphi_{\frac{n+1}{2}} &= \frac{(-1)^{\frac{n+1}{2}}}{2}(n-2(\frac{n+1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\varphi_{\frac{n+1}{2}}\\
\widehat{\nabla}_{\widehat{\xi}}\varphi_{\frac{n-1}{2}} &= -\frac{(-1)^{\frac{n+1}{2}}}{2}(n-2(\frac{n+1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_{\widehat{\xi}}\widetilde{\varphi}_{\frac{n-1}{2}} &= \frac{(-1)^{\frac{n-1}{2}}}{2}(n-2(\frac{n-1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-1}{2}}\\
\widehat{\nabla}_{\widehat{\xi}}\widetilde{\varphi}_{\frac{n-3}{2}} &= -\frac{(-1)^{\frac{n-1}{2}}}{2}(n-2(\frac{n-1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-3}{2}}\\
\widehat{\nabla}_Z\varphi_{\frac{n+1}{2}} &= (-1)^{\frac{n+1}{2}}p_+(Z)\,\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_Z\varphi_{\frac{n-1}{2}} &= (-1)^{\frac{n+1}{2}}((\rho')^2-\rho^2)p_-(Z)\,\widehat{\cdot}_M\varphi_{\frac{n+1}{2}}\\
\widehat{\nabla}_Z\widetilde{\varphi}_{\frac{n-1}{2}} &= (-1)^{\frac{n-1}{2}}((\rho')^2-\rho^2)p_+(Z)\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-3}{2}}\\
\widehat{\nabla}_Z\widetilde{\varphi}_{\frac{n-3}{2}} &= (-1)^{\frac{n-1}{2}}p_-(Z)\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-1}{2}}.
\end{aligned}$$

If $(\rho')^2-\rho^2 \neq 0$ on I, then the required equations directly follow from the above ones. Moreover, since in that case the correspondence $(\phi_{\frac{n+1}{2}},\phi_{\frac{n-1}{2}},\psi_{\frac{n-1}{2}},\psi_{\frac{n-3}{2}}) \mapsto (\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ is bijective, the "If" in the assumptions is actually an "if and only if". If now $(\rho')^2-\rho^2=0$, then $\rho'=\pm\rho$ on I; since we have assumed $\rho'>0$ (up to changing t into $-t$), we only have to consider $\rho'=\rho$, hence $\rho=Ce^t$ for some positive constant C. Since translating t provides a holomorphic isometry (again see Remarks 4.3.5), one may assume that $C=1$, i.e., $\rho=e^t$. In that case, one has $\widehat{\nabla}\varphi_{\frac{n-1}{2}}=0$ on M, hence $\varphi_{\frac{n-1}{2}}$ vanishes either identically or nowhere on M (and on $\widetilde{M}$ since it is constant in t). If $\varphi_{\frac{n-1}{2}}\neq 0$, then all right members in the equations listed just above vanish except

$$\left|\begin{aligned}
\widehat{\nabla}_Z\phi_{\frac{n+1}{2}} &= (-1)^{\frac{n+1}{2}}p_+(Z)\,\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_Z\widetilde{\varphi}_{\frac{n-3}{2}} &= (-1)^{\frac{n-1}{2}}p_-(Z)\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-1}{2}},
\end{aligned}\right.$$

which together with $\widetilde{\varphi}_{\frac{n-1}{2}}=i\frac{\partial}{\partial t}\cdot\varphi_{\frac{n-1}{2}}$ gives the result. If $\varphi_{\frac{n-1}{2}}=0$ on M, then coming back to the equations from Lemma 4.3.8, one has $\widehat{\nabla}\phi_{\frac{n+1}{2}}=\widehat{\nabla}\psi_{\frac{n-3}{2}}=0$ and $\widehat{\phi}_{\frac{n-1}{2}}$ satisfies the required equations. □

Note 4.3.10 In Theorem 4.3.9.i) not every i-Kählerian Killing spinor on $\widetilde{M}$ must come from a transversally parallel spinor on M. For instance, consider the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ (for n odd) endowed with its Fubini-Study metric of constant holomorphic sectional curvature -4 and its canonical spin structure. Then $\mathbb{C}\mathrm{H}^n$ (possibly with a suitable submanifold removed) can be viewed as a doubly warped product in several ways. For example, $\mathbb{C}\mathrm{H}^n$ is a doubly-warped product over the Heisenberg group M, which admits a $\begin{pmatrix} n-1 \\ \frac{n-1}{2} \end{pmatrix}$-dimensional space of transversally parallel spinors lying pointwise in $\Sigma_{\frac{n-1}{2}}M$ (see below). However, $\mathbb{C}\mathrm{H}^n$ carries a $2\begin{pmatrix} n \\ \frac{n+1}{2} \end{pmatrix}$-dimensional space of i-Kählerian Killing spinors [C9, Sec. 3]. Therefore there exists at least one non-zero Kählerian Killing spinor on $\mathbb{C}\mathrm{H}^n$ which does not come from any transversally parallel spinor on M.

As an example for Theorem 4.3.9.i), any Heisenberg manifold of dimension $4k+1$ (with $k \geq 1$) has a spin structure for which the corresponding spinor bundle is trivialized by transversally parallel spinors. This follows from three facts: every Heisenberg manifold is an $\mathbb{S}^1$-bundle with totally geodesic fibres over a flat torus; every $\mathbb{S}^1$-bundle over a manifold carrying parallel spinors carries transversally parallel spinors for the induced spin structure, see e.g. [C6, Prop. 3.6]; the whole spinor bundle of any flat torus endowed with its so-called trivial spin structure is trivialized by parallel spinors. Note that, as a consequence of Lemma 4.3.12 below, the doubly warped product arising from a $(2n-1)$-dimensional Heisenberg manifold M choosing $\rho = \sigma = e^t$ has constant holomorphic sectional curvature -4, therefore it is holomorphically isometric to $\mathbb{C}\mathrm{H}^n$ as soon as it is simply-connected and complete.
Examples for Theorem 4.3.9.i) with non-constant holomorphic sectional curvature can be constructed out of the following lemma:

Lemma 4.3.11 *For each integer $n \equiv 1\ (4)$, let (N^{2n-2}, g_N, J) be any simply-connected closed Hodge hyperkähler manifold. Then there exists an $\mathbb{S}^1$-bundle M over N carrying an $\mathbb{S}^1$-invariant metric $\widehat{g}$ for which $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is Sasakian and for which there exists a parallel spinor lying pointwise in $\Sigma_{\frac{n-1}{2}}M$.*

Proof: Recall first that every hyperkähler manifold is spin (this follows from the structure group $\mathrm{Sp}(\frac{n-1}{2})$ being simply-connected). McK. Wang's classification [C14] of manifolds with parallel spinors provides the existence of exactly $\frac{n-1}{2}+1$ linearly independent parallel spinors on N, one of which lies pointwise in $\Sigma_{\frac{n-1}{2}}N$ if and only if $\frac{n-1}{2}$ is even [C14, (ii) p.61]. Now, for any Hodge Kähler manifold (N, g, J) ("Hodge" meaning that its Kähler class is proportional to an integral class), there exists an $\mathbb{S}^1$-bundle $M \xrightarrow{\pi} N$ carrying an $\mathbb{S}^1$-invariant metric $\widehat{g}$ for which $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is Sasakian with $\widehat{h} = -J$, see [C13, Prop. 2] (as usual $\widehat{\xi}$ denotes the fundamental vector field of the $\mathbb{S}^1$-action). By [C6, Prop. 3.6], the lift of the non-zero parallel spinor in $\Sigma_{\frac{n-1}{2}}N$ to M gives a non-zero transversal parallel spinor on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ provided the spin structure on M is induced by the one on $\pi^*(TN)$ and the trivial covering of $\mathbb{S}^1$; because of $\widehat{h} = -J$, this spinor lies pointwise in $\Sigma_{\frac{n-1}{2}}M$. □

Kodaira's embedding theorem states that a closed Kähler manifold is Hodge if and only if it is projective, i.e., if and only if it can be holomorphically embedded in some complex projective space. Therefore projective hyperkähler manifolds of complex dimension $4k$ (with $k \geq 1$) provide examples for N in Lemma 4.3.11. For instance, simply connected hyperkähler manifolds can be constructed as the Hilbert scheme of a K3-surface (cf. [C5]). Indeed, let X be a K3-surface, then the Hilbert scheme $\mathrm{Hilb}^{2k}(X)$, which is the blow-up along the diagonal of the $2k$-th symmetric product of X, is a compact, simply-connected hyperkähler manifold of complex dimension $4k$. If X is projective, e.g. a quartic, then $\mathrm{Hilb}^{2k}(X)$ is projective too and thus has an integer Kähler class.

In order to decide whether the doubly warped product we construct is the complex hyperbolic space or not, the transversal holomorphic curvature of $(M,\widehat{g},\widehat{\xi})$ and the holomorphic sectional curvature of $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ have to be compared:

Lemma 4.3.12 *Let $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ be a Kähler doubly warped product as in* Lemma 4.3.4 *with $\rho'' = \rho$, $\sigma = \rho'$ and $\widehat{h} = -J$. Then the holomorphic sectional curvature $\widetilde{K}_{\mathrm{hol}}(Z)$ of $(\widetilde{M},\widetilde{g},J)$ and the transversal holomorphic sectional curvature $\widehat{K}_{\mathrm{hol}}(Z)$ of $(M,\widehat{g},\widehat{\xi})$ are related by*

$$\widetilde{K}_{\mathrm{hol}}(Z) = \frac{1}{\rho^2}\Big(\widehat{K}_{\mathrm{hol}}(Z) - 4(\rho')^2\Big),$$

for all $Z \in \{\widehat{\xi}, \frac{\partial}{\partial t}\}^{\perp} \setminus \{0\}$. In particular, the doubly warped product $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ has constant holomorphic sectional curvature -4 if and only if the transversal holomorphic sectional curvature of $(M,\widehat{g},\widehat{\xi})$ is constant equal to $4((\rho')^2 - \rho^2)$.

Proof: Recall that $\widetilde{K}_{\mathrm{hol}}(Z)$ and $\widehat{K}_{\mathrm{hol}}(Z)$ are defined by

$$\widetilde{K}_{\mathrm{hol}}(Z) := \frac{\widetilde{g}(\widetilde{R}(Z,JZ)Z,JZ)}{\widetilde{g}(Z,Z)^2} \qquad \text{and} \qquad \widehat{K}_{\mathrm{hol}}(Z) := \frac{\widehat{g}(\widehat{R}(Z,JZ)Z,JZ)}{\widehat{g}(Z,Z)^2},$$

where $\widetilde{R}_{X,Y} := \widetilde{\nabla}_{[X,Y]} - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]$ and $\widehat{R}_{Z,Z'} := \widehat{\nabla}_{[Z,Z']} - [\widehat{\nabla}_Z, \widehat{\nabla}_{Z'}]$ are the curvature tensors associated to $\widetilde{\nabla}$ and $\widehat{\nabla}$ on $T\widetilde{M}$ and $\widehat{\xi}^{\perp}$ respectively. The following identities can be deduced from the formulas in Lemma 4.3.1, taking into account $\rho' = \sigma$ and $\rho'' = \rho$:

$$\begin{aligned}
\widetilde{g}(\widetilde{R}_{\xi,\frac{\partial}{\partial t}}\xi, \frac{\partial}{\partial t}) &= -\frac{(\rho\sigma)''}{\rho\sigma} = -4 \\
\widetilde{g}(\widetilde{R}(Z,JZ)Z,JZ) &= \widetilde{g}(\widehat{R}(Z,JZ)Z,JZ) - 4(\frac{\rho'}{\rho})^2\widetilde{g}(Z,Z)^2,
\end{aligned}$$

for every $Z \in \{\widehat{\xi}, \frac{\partial}{\partial t}\}^{\perp} \setminus \{0\}$. Using $\widetilde{g}(Z,\cdot) = \rho^2\widehat{g}(Z,\cdot)$, we obtain

$$\begin{aligned}
\widetilde{K}_{\mathrm{hol}}(Z) &= \frac{\widetilde{g}(\widehat{R}(Z,JZ)Z,JZ)}{\widetilde{g}(Z,Z)^2} - 4(\frac{\rho'}{\rho})^2 \\
&= \frac{1}{\rho^2}\frac{\widehat{g}(\widehat{R}(Z,JZ)Z,JZ)}{\widehat{g}(Z,Z)^2} - 4(\frac{\rho'}{\rho})^2,
\end{aligned}$$

which gives the first statement. Since by the computation above $\widetilde{K}_{\mathrm{hol}}(\xi) = -4$ (independently of $\widehat{g}$), the second follows from the first (note that $(\rho')^2 - \rho^2$ is constant by the assumption $\rho'' = \rho$). □

As a consequence of Theorem 4.3.9.*i*), Lemma 4.3.11 and Lemma 4.3.12, we obtain:

Corollary 4.3.13 *For an integer $n \equiv 1\ (4)$, let (N^{2n-2}, g_N, J) be any simply-connected closed Hodge hyperkähler manifold. Let $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ be constructed from N as in* Lemma 4.3.11 *and $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be the Kähler spin doubly warped product constructed from M as in* Lemma 4.3.6 *with $\rho = \sigma = e^t$. Then $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ carries a non-zero i-Kählerian Killing spinor but has non-constant holomorphic sectional curvature.*

Proof: The existence of a non-zero *i*-Kählerian Killing spinor follows from Theorem 4.3.9.*i*) and Lemma 4.3.11. In case $\rho = \sigma = e^t$, Lemma 4.3.12 implies that the holomorphic sectional curvature of the doubly warped product $(\widetilde{M}^{2n}, \widetilde{g}, J)$ is -4 if and only if the transversal holomorphic sectional curvature of $(M, \widehat{g}, \widehat{\xi})$ vanishes, that is, if and only if its transversal curvature vanishes (see e.g. [C11, Prop. 7.1 p.166]). Now for any $\mathbb{S}^1$-bundle as in Lemma 4.3.11, the transversal (holomorphic) sectional curvature of M and the (holomorphic) sectional curvature of N coincide. Since simply-connected closed hyperkähler manifolds cannot be flat, the Kähler manifold $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ cannot have constant holomorphic sectional curvature. □

Corollary 4.3.13 provides the first family of examples of Kähler spin manifolds of non-constant holomorphic sectional curvature carrying non-zero imaginary Kählerian Killing spinors.

The two other subcases $(\rho')^2 - \rho^2 = 1$ and $(\rho')^2 - \rho^2 = -1$ are geometrically more simple to describe. We do it in separate lemmas.

Lemma 4.3.14 *Let (M^{2n-1}, g, ξ) be a Sasakian spin manifold with $h = -J$ and fix $r \in \{0, 1, \ldots, n\}$. Then a section (ψ_r, ψ_{r-1}) of $\Sigma_r M \oplus \Sigma_{r-1} M$ satisfies* (4.5) *if and only if $\psi := \psi_r + \psi_{r-1}$ is a $\frac{(-1)^r}{2}$-Killing spinor on (M, g).*

Proof: Let Ω be the 2-form associated to J on $\xi^\perp$, i.e., $\Omega(Z, Z') = g(J(Z), Z')$ for all $Z, Z' \perp \xi$. Using $\Omega \underset{M}{\cdot} \psi_r = (-1)^{r+1}(2r - n + 1)\xi \underset{M}{\cdot} \psi_r$ (for all r) we have on the one hand

$$\begin{aligned}
\nabla_\xi \psi &= \nabla^M_\xi \psi + \frac{1}{2}\Omega \underset{M}{\cdot} \psi \\
&= \nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{1}{2}\Omega \underset{M}{\cdot} \psi \\
&= \nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi - \frac{(-1)^r}{2}(2r - n + 1)\xi \underset{M}{\cdot} \psi_r \\
&\quad + \frac{(-1)^r}{2}(2(r-1) - n + 1)\xi \underset{M}{\cdot} \psi_{r-1} \\
&= \nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{(-1)^r}{2}(n - 2r)\xi \underset{M}{\cdot} \psi_r + \frac{(-1)^r}{2}(2(r-1) - n + 2)\xi \underset{M}{\cdot} \psi_{r-1},
\end{aligned}$$

which implies

$$\left| \begin{aligned}
\nabla_\xi \psi_r &= (\nabla^M_\xi \psi - \tfrac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi)_r + \tfrac{(-1)^r}{2}(n - 2r)\xi \underset{M}{\cdot} \psi_r \\
\nabla_\xi \psi_{r-1} &= (\nabla^M_\xi \psi - \tfrac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi)_{r-1} - \tfrac{(-1)^r}{2}(n - 2r)\xi \underset{M}{\cdot} \psi_{r-1}.
\end{aligned} \right. \tag{4.7}$$

On the other hand, for every $Z \in \xi^\perp$ one has,

$$\begin{aligned}
\nabla_Z \psi &= \nabla_Z^M \psi - \frac{1}{2}\xi \underset{M}{\cdot} h(Z) \underset{M}{\cdot} \psi \\
&= \nabla_Z^M \psi - \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi + \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi - \frac{1}{2} J(Z) \underset{M}{\cdot} \xi \underset{M}{\cdot} \psi \\
&= \nabla_Z^M \psi - \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi + \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi - \frac{1}{2} J(Z) \underset{M}{\cdot} \{(-1)^{r+1} i \psi_r + (-1)^r i \psi_{r-1}\} \\
&= \nabla_Z^M \psi - \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi + \frac{(-1)^r}{2} (Z + iJ(Z)) \cdot \psi_r + \frac{(-1)^r}{2} (Z - iJ(Z)) \cdot \psi_{r-1},
\end{aligned}$$

which implies

$$\left| \begin{array}{ll} \nabla_Z \psi_r & = (\nabla_Z^M \psi - \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi)_r + (-1)^r p_+(Z) \cdot \psi_{r-1} \\ \nabla_Z \psi_{r-1} & = (\nabla_Z^M \psi - \frac{(-1)^r}{2} Z \underset{M}{\cdot} \psi)_{r-1} + (-1)^r p_-(Z) \cdot \psi_r. \end{array} \right. \tag{4.8}$$

Therefore the pair (ψ_r, ψ_{r-1}) satisfies (4.5) if and only if $\psi := \psi_r + \psi_{r-1}$ satisfies $\nabla_X^M \psi = \frac{(-1)^r}{2} X \underset{M}{\cdot} \psi$ for all $X \in TM$, that is, if and only if ψ is a $\frac{(-1)^r}{2}$-Killing spinor on (M, g). $\square$

The case $(\rho')^2 - \rho^2 = -1$ is analogous to the case $(\rho')^2 - \rho^2 = 1$ up to a Lorentzian detour. We call (4.9) the following system of equations:

$$\left| \begin{array}{ll} \nabla_\xi \psi_r & = -\frac{(-1)^r}{2}(n-2r)\xi \underset{M}{\cdot} \psi_r \\ \nabla_\xi \psi_{r-1} & = \frac{(-1)^r}{2}(n-2r)\xi \underset{M}{\cdot} \psi_{r-1} \\ \nabla_Z \psi_r & = (-1)^r \varepsilon p_+(Z) \underset{M}{\cdot} \psi_{r-1} \\ \nabla_Z \psi_{r-1} & = -(-1)^r \varepsilon p_-(Z) \underset{M}{\cdot} \psi_r \end{array} \right. \tag{4.9}$$

for all $Z, Z' \in \xi^\perp$, where $\varepsilon \in \{\pm 1\}$.

Lemma 4.3.15 *Let (M^{2n-1}, g, ξ) be a Sasakian spin manifold with $h = -J$ and fix $r \in \{0, 1, \ldots, n\}$ as well as $\varepsilon \in \{\pm 1\}$. Then a section (ψ_r, ψ_{r-1}) of $\Sigma_r M \oplus \Sigma_{r-1} M$ satisfies* (4.9) *if and only if $\psi := \psi_r + i\varepsilon\psi_{r-1}$ is a $\frac{(-1)^{r+1} i}{2}$-Killing spinor on the Lorentzian manifold $(M, -g_\xi \oplus g_{\xi^\perp})$.*

Proof: First, there exists the analog of Riemannian flow in the Lorentzian context. A Lorentzian flow is given by a triple $(M, \widehat{g}, \widehat{\xi})$, where $(M, \widehat{g})$ is a Lorentzian manifold and $\widehat{\xi}$ a smooth tangent vector field on M with $\widehat{g}(\widehat{\xi}, \widehat{\xi}) = -1$ and $\widehat{g}(\widehat{\nabla}_Z^M \widehat{\xi}, Z') = -\widehat{g}(\widehat{\nabla}_{Z'}^M \widehat{\xi}, Z)$ for all $Z, Z' \in \widehat{\xi}^\perp$. Note that $(M, \widehat{g})$ is necessarily time-oriented because of the existence of $\widehat{\xi}$. Setting $\widehat{\nabla}_X Z := \left| \begin{array}{ll} [\widehat{\xi}, Z]^{\widehat{\xi}^\perp} & \text{if } X = \widehat{\xi} \\ (\widehat{\nabla}_X^M Z)^{\widehat{\xi}^\perp} & \text{if } X \perp \widehat{\xi} \end{array} \right.$ for all $Z \in \Gamma(\widehat{\xi}^\perp)$ and $\widehat{h} := \widehat{\nabla}^M \widehat{\xi}$, one obtains a metric connection $\widehat{\nabla}$ and a skew-symmetric endomorphism-field $\widehat{h}$ on $\widehat{\xi}^\perp$ such that

$$\left| \begin{array}{ll} \widehat{\nabla}_{\widehat{\xi}}^M Z & = \widehat{\nabla}_{\widehat{\xi}} Z + \widehat{h}(Z) + \widehat{g}(\widehat{\nabla}_{\widehat{\xi}}^M \widehat{\xi}, Z)\widehat{\xi} \\ \widehat{\nabla}_Z^M Z' & = \widehat{\nabla}_Z Z' + \widehat{g}(\widehat{h}(Z), Z')\widehat{\xi} \end{array} \right.$$

for all $Z, Z' \in \Gamma(\widehat{\xi}^\perp)$. Moreover, in case M is spin, the corresponding Gauss-type formula for spinors reads

$$\left|\begin{array}{ll}\widehat{\nabla}_{\widehat{\xi}}\varphi & = \widehat{\nabla}^M_{\widehat{\xi}}\varphi - \frac{1}{2}\widehat{\Omega}\,\widehat{\cdot}_M\,\varphi + \frac{1}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\widehat{\nabla}^M_{\widehat{\xi}}\widehat{\xi}\,\widehat{\cdot}_M\,\varphi\\ \widehat{\nabla}_Z\varphi & = \widehat{\nabla}^M_Z\varphi + \frac{1}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\widehat{h}(Z)\,\widehat{\cdot}_M\,\varphi\end{array}\right.$$

for all $\varphi \in \Gamma(\Sigma M)$ and $Z \in \widehat{\xi}^\perp$, where $\widehat{\Omega}(Z, Z') := \widehat{g}(\widehat{h}(Z), Z')$. In case $(M, \widehat{g}, \widehat{\xi})$ is Lorentzian Sasakian, i.e., if furthermore $\widehat{\nabla}^M_{\widehat{\xi}}\widehat{\xi} = 0$, $\widehat{h}^2 = -\mathrm{Id}$ and $\widehat{\nabla}\widehat{h} = 0$, then we still have the $\widehat{\nabla}$-parallel decomposition $\Sigma M = \oplus_{r=0}^{n-1}\Sigma_r M$ with $\Sigma_r M := \mathrm{Ker}(\widehat{\Omega}\,\widehat{\cdot}_M - i(2r-(n-1)\mathrm{Id}))$. This time one has $\widehat{\xi}\,\widehat{\cdot}_M\,\varphi_r = (-1)^{r+1}\varphi_r$ for all $\varphi_r \in \Sigma_r M$.

Assume now $(M, \widehat{g}, \widehat{\xi})$ to be Lorentzian Sasakian and pick a section $\psi = \psi_r + \psi_{r-1}$ of $\Sigma_r M \oplus \Sigma_{r-1} M$, then the formulas above imply

$$\begin{aligned}
\widehat{\nabla}_{\widehat{\xi}}\psi &= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{1}{2}\widehat{\Omega}\,\widehat{\cdot}_M\,\psi\\
&= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi + \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi - \frac{i}{2}\big((2r-(n-1))\psi_r + (2(r-1)-(n-1))\psi_{r-1}\big)\\
&= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi + \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi\\
&\quad + \frac{(-1)^r i}{2}(2r-(n-1))\widehat{\xi}\,\widehat{\cdot}_M\,\psi_r - \frac{(-1)^r i}{2}(2(r-1)-(n-1))\widehat{\xi}\,\widehat{\cdot}_M\,\psi_{r-1}\\
&= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi + \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\,\psi_r - \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\,\psi_{r-1},
\end{aligned}$$

that is,

$$\left|\begin{array}{ll}\widehat{\nabla}_{\widehat{\xi}}\psi_r & = \big(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi\big)_r + \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\,\psi_r\\ \widehat{\nabla}_{\widehat{\xi}}\psi_{r-1} & = \big(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\psi\big)_{r-1} - \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\,\psi_{r-1}.\end{array}\right.$$

This is still valid for $r = 0$ or $r = n$ (setting $\psi_{-1} := \psi_n := 0$). Similarly, for all $Z \in \widehat{\xi}^\perp$,

$$\begin{aligned}
\widehat{\nabla}_Z\psi &= \widehat{\nabla}^M_Z\psi + \frac{1}{2}\widehat{\xi}\,\widehat{\cdot}_M\,\widehat{h}(Z)\,\widehat{\cdot}_M\,\psi\\
&= \widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\,\psi + \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\,\psi - \frac{(-1)^{r+1}}{2}\widehat{h}(Z)\,\widehat{\cdot}_M\,\psi_r + \frac{(-1)^{r+1}}{2}\widehat{h}(Z)\,\widehat{\cdot}_M\,\psi_{r-1}\\
&= \widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\,\psi + (-1)^{r+1}ip_-(Z)\,\widehat{\cdot}_M\,\psi_r + (-1)^{r+1}ip_+(Z)\,\widehat{\cdot}_M\,\psi_{r-1},
\end{aligned}$$

that is,

$$\left|\begin{array}{ll}\widehat{\nabla}_Z\psi_r & = \big(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\,\psi\big)_r + (-1)^{r+1}ip_+(Z)\,\widehat{\cdot}_M\,\psi_{r-1}\\ \widehat{\nabla}_Z\psi_{r-1} & = \big(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\,\psi\big)_{r-1} + (-1)^{r+1}ip_-(Z)\,\widehat{\cdot}_M\,\psi_r.\end{array}\right.$$

If one changes the Lorentzian metric $\widehat{g}$ into $g := -\widehat{g}_{\widehat{\xi}} \oplus \widehat{g}_{\widehat{\xi}^\perp}$, then one obtains a smooth Riemannian metric g on M and the triple $(M, g, \xi := \widehat{\xi})$ is a Riemannian flow with

$$\left|\begin{array}{ll}\nabla^M_\xi\xi & = -\widehat{\nabla}^M_{\widehat{\xi}}\widehat{\xi}\\ h & = -\widehat{h}\\ \nabla & = \widehat{\nabla}.\end{array}\right.$$

Moreover, the Clifford multiplications are related by

$$\left|\begin{array}{ll} \xi \underset{M}{\cdot} & = i\widehat{\xi}\,\widehat{\underset{M}{\cdot}} \\ Z \underset{M}{\cdot} & = Z\,\widehat{\underset{M}{\cdot}}, \end{array}\right.$$

for all $Z \in \xi^\perp = \widehat{\xi}^\perp$. Therefore the equations above become on (M,g,ξ)

$$\left|\begin{array}{ll} \nabla_\xi \psi_r & = \left(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\underset{M}{\cdot}}\,\psi\right)_r - \frac{(-1)^r}{2}(n-2r)\xi \underset{M}{\cdot} \psi_r \\ \nabla_\xi \psi_{r-1} & = \left(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\underset{M}{\cdot}}\,\psi\right)_{r-1} + \frac{(-1)^r}{2}(n-2r)\xi \underset{M}{\cdot} \psi_{r-1} \\ \nabla_Z \psi_r & = \left(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\underset{M}{\cdot}}\,\psi\right)_r + (-1)^{r+1}ip_+(Z) \underset{M}{\cdot} \psi_{r-1} \\ \nabla_Z \psi_{r-1} & = \left(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\underset{M}{\cdot}}\,\psi\right)_{r-1} + (-1)^{r+1}ip_-(Z) \underset{M}{\cdot} \psi_r. \end{array}\right.$$

Therefore, $\psi_r - i\varepsilon\psi_{r-1}$ satisfies (4.9) if and only if ψ is a $\frac{(-1)^{r+1}i}{2}$-Killing spinor on $(M,\widehat{g},\widehat{\xi})$. □

Round spheres provide examples of spin Sasakian manifolds where (4.5) is fulfilled for the right r.

Lemma 4.3.16 *For any odd $n \geq 3$, the $(2n-1)$-dimensional round sphere M with its canonical Sasakian and spin structures admits a $2\begin{pmatrix} n \\ \frac{n+1}{2}\end{pmatrix}$-dimensional space of sections of $\Sigma_{\frac{n+1}{2}}M \oplus \Sigma_{\frac{n-1}{2}}M \oplus \Sigma_{\frac{n-1}{2}}M \oplus \Sigma_{\frac{n-3}{2}}M$ satisfying* (4.5).

Proof: Consider the standard embedding $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$, with unit normal $\nu_x = x$ and hence Weingarten-endomorphism field $A = -\mathrm{Id}_{TM}$. Set $\xi := -i\nu$. It is well-known that $(\mathbb{S}^{2n-1}, g, \xi)$ is a Sasakian spin manifold with $h = -J$ on $\xi^\perp \subset TM$, where J is the standard complex structure induced from $\mathbb{C}^n$. Let $\psi \in \Sigma_r\mathbb{C}^n$ with $r \in \{0,1\ldots,n\}$ (i.e., $\widetilde{\Omega}\cdot\psi = i(2r-n)\psi$ where $\widetilde{\Omega}$ is the standard Kähler form of $\mathbb{C}^n$). If r is even then $\psi \in \Sigma^+\mathbb{C}^n$. In that case the spinorial Gauss formula reads

$$\nabla^M_X\varphi = \nabla^{\mathbb{C}^n}_X\varphi - \frac{1}{2}A(X) \underset{M}{\cdot} \varphi$$

so that the restriction of ψ on $\mathbb{S}^{2n-1}$ satisfies $\nabla^M_X\psi = \frac{1}{2}X \underset{M}{\cdot} \psi$, i.e., is a $\frac{1}{2}$-Killing spinor. If r is odd, then $\psi \in \Sigma_-\mathbb{C}^n$. The spinorial Gauss formula for a section $\varphi \in \Sigma^-\mathbb{C}^n_{|_{\mathbb{S}^{2n-1}}}$, which can be identified with $\Sigma\mathbb{S}^{2n-1}$ provided we change the sign of the Clifford multiplication, reads then

$$\nabla^M_X\varphi = \nabla^{\mathbb{C}^n}_X\varphi + \frac{1}{2}A(X) \underset{M}{\cdot} \varphi$$

for every $X \in TM$. We deduce that $\nabla^M_X\psi = -\frac{1}{2}X \underset{M}{\cdot} \psi$ for every $X \in TM$, that is, the restriction of ψ to $\mathbb{S}^{2n-1}$ is a $-\frac{1}{2}$-Killing spinor. To sum up, the restriction of a constant section $\psi \in \Sigma_r\mathbb{C}^n$ to $M := \mathbb{S}^{2n-1}$ is a $\frac{(-1)^r}{2}$-Killing spinor on M. Decompose such a ψ into $\psi = \psi_r + \psi_{r-1}$, see (4.3). From Lemma 4.3.14 and $\mathrm{rk}_{\mathbb{C}}(\Sigma_r\mathbb{C}^n) = \begin{pmatrix} n \\ r \end{pmatrix}$ we conclude. $\square$

The analog of $\mathbb{S}^{2n-1}$ in the Lorentzian context is the Anti-deSitter spacetime $\mathbb{H}^{2n-1}$, that can be defined by

$$\mathbb{H}^{2n-1} := \{z \in \mathbb{C}^n \mid -|z_0|^2 + \sum_{j=1}^{n-1} |z_j|^2 = -1\}.$$

Lemma 4.3.17 *For any odd $n \geq 3$, the $(2n-1)$-dimensional Anti-deSitter spacetime $M := \mathbb{H}^{2n-1}$ with its induced Lorentzian Sasakian structure (with $\widehat{\xi}_x = ix$ and $\widehat{h} = J$) and induced spin structure admits an $\begin{pmatrix} n \\ r \end{pmatrix}$-dimensional space of $\frac{(-1)^{r+1}i}{2}$-Killing spinors lying pointwise in $\Sigma_r M \oplus \Sigma_{r-1} M$. In particular, if one considers the (Riemannian) Sasakian metric given by $-\widehat{g}_{\widehat{\xi}} \oplus \widehat{g}_{\widehat{\xi}^\perp}$, where $\widehat{g}$ is the canonical Lorentzian metric of sectional curvature -1, then $\mathbb{H}^{2n-1}$ admits a $2\begin{pmatrix} n \\ \frac{n+1}{2} \end{pmatrix}$-dimensional space of sections of $\Sigma_{\frac{n+1}{2}} M \oplus \Sigma_{\frac{n-1}{2}} M \oplus \Sigma_{\frac{n-1}{2}} M \oplus \Sigma_{\frac{n-3}{2}} M$ satisfying* (4.6).

Proof: First recall that M is a Lorentzian Sasakian manifold and simultaneously an $\mathbb{S}^1$-bundle with totally geodesic fibres over $\mathbb{C}\mathrm{H}^{n-1}$. Just as for the sphere, one can restrict spinors from $\mathbb{C}^n$ onto M so that the following Gauss-Weingarten-formula holds for all $\psi \in C^\infty(\mathbb{C}^n, \Sigma_{2n})$ and all $X \in TM$:

$$\begin{aligned} \nabla^M_X \psi &= -\frac{A(X)}{2} \cdot \nu \cdot \psi \\ &= \left| \begin{array}{ll} \frac{iA(X)}{2} \underset{M}{\cdot} \psi & \text{if } \psi(x) \in \Sigma^+_{2n}\ \forall x \\ -\frac{iA(X)}{2} \underset{M}{\cdot} \psi & \text{if } \psi(x) \in \Sigma^-_{2n}\ \forall x, \end{array} \right. \end{aligned}$$

where $A(X) := \widetilde{\nabla}_X \nu$ is the Weingarten endormorphism of M in $\mathbb{C}^n$. Moreover, there still exists a $\widetilde{\nabla}$-parallel splitting $\Sigma_{2n} = \oplus_{r=0}^n \Sigma_{2n,r}$ where $\Sigma_{2n,r} := \mathrm{Ker}(\widetilde{\Omega} \cdot -i(2r-n)\mathrm{Id})$ (with dimension $\begin{pmatrix} n \\ r \end{pmatrix}$) and $\widetilde{\Omega}$ is the Kähler form associated to the standard complex structure J on $\widetilde{M}$. Choosing $\nu_x := -x$ as unit normal on M, one has $A = -\mathrm{Id}_{TM}$, so that the restriction of any constant section of $\mathbb{C}^n \times \Sigma_{2n,r}$ onto M provides a $\frac{(-1)^{r+1}i}{2}$-Killing spinor. Since again $\Sigma_r \widetilde{M}_{|_M} = \Sigma_r M \oplus \Sigma_{r-1} M$, the first statement follows. The second statement is a consequence of the first one together with Lemma 4.3.15. $\square$

The doubly warped product of Theorem 4.3.9.$ii)$ corresponding to $M = \mathbb{S}^{2n-1}$ is the complement of a point in the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ with its canonical Fubini-Study metric of constant holomorphic sectional curvature -4 (compare with [C1, Satz 5.1]). Therefore we obtain a new description of the imaginary Kählerian Killing spinors on $\mathbb{C}\mathrm{H}^n$ after the explicit one by K.-D. Kirchberg [C9, Sec. 3]. Actually $\mathbb{C}\mathrm{H}^n$ is essentially the only example occurring in Theorem 4.3.9.$ii)$:

Theorem 4.3.18 *For $n \geq 3$ odd let $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be a Kähler doubly warped product as in* Lemma 4.3.6 *with $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ complete, Sasakian, simply-connected, spin, $I = \mathbb{R}_+^\times$, $\rho = \sinh$ and $\sigma = \cosh$. Let $\widetilde{M}$ carry the induced spin structure and assume $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ admits a non-zero i-Kählerian Killing spinor (ψ, ϕ).*
Then $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is holomorphically isometric to $\mathbb{C}\mathrm{H}^n \setminus \{x\}$ for some $x \in \mathbb{C}\mathrm{H}^n$.

Proof: It suffices to show that $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is $\mathbb{S}^{2n-1}$ with its standard Sasakian structure. By assumption and Lemma 4.3.14, the section $\varphi_{\frac{n+1}{2}} + \varphi_{\frac{n-1}{2}}$ is a $\frac{(-1)^{\frac{n+1}{2}}}{2}$-Killing spinor on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ lying pointwise in $\Sigma_{\frac{n+1}{2}} M \oplus \Sigma_{\frac{n-1}{2}} M$ and the section $\widetilde{\varphi}_{\frac{n-1}{2}} + \widetilde{\varphi}_{\frac{n-3}{2}}$ is a $-\frac{(-1)^{\frac{n+1}{2}}}{2}$-Killing spinor on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ lying pointwise in $\Sigma_{\frac{n-1}{2}} M \oplus \Sigma_{\frac{n-3}{2}} M$. At least one of them does not vanish. Now C. Bär's classification (see in particular [C2, Thm. 3]) implies that either $M = \mathbb{S}^{2n-1}$ or M is a compact Einstein-Sasakian manifold with exactly one non-zero $\frac{1}{2}$- and one non-zero $-\frac{1}{2}$-Killing spinor. Moreover, each Killing spinor induces a parallel spinor on the Riemannian cone $\overline{M}$ over M [C2]. But coming back to McK. Wang's classification of simply-connected complete Riemannian spin manifolds with parallel spinors, it turns out that, in the latter case, the reduced holonomy of $\overline{M}$ is SU_n (where n is its complex dimension) and the parallel spinors lie in $\Sigma_0 \overline{M}$ and $\Sigma_n \overline{M}$ (see [C14, (i) p.61]), in particular not in $\Sigma_{\frac{n\pm 1}{2}} \overline{M}$. Thus only $\mathbb{S}^{2n-1}$ occurs. □

In case $M = \mathbb{H}^{2n-1}$ is equipped with its associated Riemannian Sasakian structure, the corresponding doubly warped product with $\rho = \cosh$ and $\sigma = \sinh$ has again constant holomorphic sectional curvature -4 by Lemma 4.3.12. It is actually the complement in $\mathbb{C}\mathrm{H}^n$ of some submanifold. We conjecture that, up to covering, $\mathbb{H}^{2n-1}$ is the only Lorentzian Sasakian manifold having non-zero imaginary Killing spinors lying pointwise in the "middle" eigenspaces $\Sigma_r M$ (with $r \in \{\frac{n-3}{2}, \ldots, \frac{n+1}{2}\}$) of the Clifford action of the transversal Kähler form. If this happens, then only the complex hyperbolic space can occur as (simply-connected complete) example of doubly warped product in Theorem 4.3.9.*iii*).

4.4 Classification in a particular case

In this section, we show that the structure of a doubly warped product can be recovered from the length function of a non-zero imaginary Kählerian Killing spinor satisfying certain supplementary assumption on the Kähler manifold $\widetilde{M}$. The following result can be seen as analogous to H. Baum's one [C3] about imaginary Killing spinors of so-called *type I*. Recall for the next theorem that V was defined by (4.1).

Theorem 4.4.1 *Let $(\widetilde{M}^{2n}, g, J)$ be a connected complete Kähler spin manifold carrying a non-zero i-Kählerian Killing spinor (ψ, ϕ). Assume $|\psi| = |\phi|$ and the existence of a real vector field W on $\widetilde{M}$ together with a non-identically vanishing continuous function $\mu : \widetilde{M} \longrightarrow \mathbb{C}$ such that $W \cdot \psi = \mu \phi$. Then the vector field V has no zero, the Kähler manifold $(\widetilde{M}^{2n}, g, J)$ is a doubly warped product as in* Theorem 4.3.9.*i*) *and (ψ, ϕ) comes from a transversally parallel spinor on $(M, \widehat{g}, \widehat{\xi})$.*

Proof: We construct a holomorphic isometry between $(\widetilde{M}^{2n},g,J)$ and some doubly warped product. This isometry is provided by the flow of some vector field associated to the Kählerian Killing spinor (compare with the case of imaginary Killing spinors [C3]).
First note that, if $|\psi|=|\phi|$, then both ψ and ϕ have no zero on $\widetilde{M}$. Because of $|W|\cdot|\psi|=|W\cdot\psi|=|\mu|\cdot|\phi|$, this already implies $|W|=|\mu|$ on $\widetilde{M}$. Fix a neighbourhood U of a point x with $\mu(x)\neq 0$ for all $x\in U$. It follows from the definition of V that

$$\mu = 2i\frac{g(p_+(W),V)}{|\phi|^2} \tag{4.10}$$

on U, in particular $W(x)\neq 0$ and $V(x)\neq 0$ for all $x\in U$. Now Cauchy-Schwarz inequality with $X=V$ in (4.1) gives $|V|\leq|\psi|\cdot|\phi|$ on $\widetilde{M}$. With (4.10) we deduce that

$$\begin{aligned}
|\mu|^2 &= \frac{|V|^2}{|\phi|^4}\Big(g(W,\frac{V}{|V|})^2+g(W,\frac{J(V)}{|V|})^2\Big)\\
&\leq \frac{|V|^2|W|^2}{|\phi|^4}\\
&\leq |W|^2
\end{aligned}$$

on U, which together with $|\mu|=|W|$ provides $|V|=|\phi|^2$. By the equality case in Cauchy-Schwarz inequality, we obtain $V\cdot\psi=i|V|\phi$ and $V\cdot\phi=i|V|\psi$ on U. This identity holds on $\widetilde{M}$ because of the analyticity of all objects involved (by definition, ψ is anti-holomorphic and ϕ is holomorphic). This in turn implies $|V|=|\phi|^2$ on $\widetilde{M}$, in particular $\{V=0\}=\varnothing$ and $\frac{V}{|V|}\cdot\psi=i\phi$ as well as $\frac{V}{|V|}\cdot\phi=i\psi$ on $\widetilde{M}$.
Next we look at the level hypersurfaces $M_r:=\{x\in\widetilde{M},\,|\phi(x)|=r\}$ (with $r\in\mathbb{R}_+^\times$) which, if non-empty, are smooth because of $\{V=0\}=\varnothing$ and Proposition 4.2.1. A unit normal to M_r is given by $\nu:=\frac{V}{|V|}$ and the associated Weingarten endomorphism field is

$$\begin{aligned}
A(X) &:= -\widetilde{\nabla}_X\nu\\
&= -\frac{1}{|V|}\left(\widetilde{\nabla}_XV-g(\widetilde{\nabla}_XV,\frac{V}{|V|})\frac{V}{|V|}\right)
\end{aligned}$$

for every $X \in \nu^\perp$. Setting $\xi := -J(\nu)$ (note that the vector field ξ is pointwise tangent to M_r), using $\nu \cdot \psi = i\phi$ and Proposition 4.2.1.*ii*), we compute, for all $X, Y \in \nu^\perp$,

$$\begin{aligned}
g(A(X),Y) &= -\frac{1}{|V|} g(\widetilde{\nabla}_X V, Y) \\
&= -\frac{1}{|V|} \mathfrak{Re}\left(\langle p_-(X)\cdot\phi, p_-(Y)\cdot\phi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\right) \\
&= -\frac{1}{|V|} \mathfrak{Re}\left(\langle p_-(X)\cdot\nu\cdot\psi, p_-(Y)\cdot\nu\cdot\psi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\right) \\
&= -\frac{1}{|V|} \mathfrak{Re}\Big(-\langle p_-(X)\cdot\nu\cdot\psi, \nu\cdot p_-(Y)\cdot\psi\rangle - 2\overline{g(\nu, p_-(Y))}\langle p_-(X)\cdot\nu\cdot\psi, \psi\rangle \\
&\qquad + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\Big) \\
&= -\frac{1}{|V|} \mathfrak{Re}\Big(\langle \nu\cdot p_-(X)\cdot\psi, \nu\cdot p_-(Y)\cdot\psi\rangle + 2g(\nu, p_-(X))\langle \psi, \nu\cdot p_-(Y)\cdot\psi\rangle \\
&\qquad - 2\overline{g(\nu, p_-(Y))}\langle p_-(X)\cdot\nu\cdot\psi, \psi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\Big) \\
&= -\frac{1}{|V|} \mathfrak{Re}\Big(\langle X\cdot\psi, Y\cdot\psi\rangle + ig(\nu, J(X))\langle \psi, \nu\cdot p_-(Y)\cdot\psi\rangle + ig(\nu, J(Y))\langle p_-(X)\cdot\nu\cdot\psi, \psi\rangle\Big) \\
&= -\frac{1}{|V|} \Big(|\psi|^2 g(X,Y) + g(\nu, J(X))\mathfrak{Re}(\underbrace{\langle \phi, p_-(Y)\cdot\psi\rangle}_{0}) - g(\nu, J(Y))\mathfrak{Re}(\langle p_-(X)\cdot\phi, \psi\rangle)\Big) \\
&= -\frac{1}{|V|} \Big(|\psi|^2 g(X,Y) + g(\nu, J(Y)) g(J(X), V)\Big) \\
&= -(g(X,Y) + g(\xi, X) g(\xi, Y)),
\end{aligned}$$

that is, $A = -\mathrm{Id}_{TM_r} - \xi^\flat \otimes \xi$. In particular, the Gauß-Weingarten formula for the inclusion $M_r \subset \widetilde{M}$ reads $\widetilde{\nabla}_X Y = \nabla^{M_r}_X Y - (g(X,Y) + g(\xi,X)g(\xi,Y))\nu$ for all vector fields X, Y tangent to M_r.

We begin with the reconstruction of the doubly warped product structure of Theorem 4.3.9.*i*). From $A(\xi) = -2\xi$, we deduce that $A(J(V)) = -2J(V)$, hence $\widetilde{\nabla}_{J(V)}\nu = 2J(V)$. Proposition 4.2.1.*ii*) gives

$$J(V)(|V|) = \frac{g(\widetilde{\nabla}_V V, J(V))}{|V|} = \frac{1}{|V|}\mathfrak{Re}\left(\langle p_-(V)\cdot\phi, p_-(J(V))\cdot\phi\rangle + \langle p_+(V)\cdot\psi, p_+(J(V))\cdot\psi\rangle\right) = 0.$$

Therefore $\widetilde{\nabla}_{J(V)} V = 2|V|J(V)$, that is, $\widetilde{\nabla}_V V = 2|V|V$ using $\widetilde{\nabla}_{J(X)} V = J(\widetilde{\nabla}_X V)$ for all X. This implies for the commutator of ξ and ν (which we need later for the identification of the metric and of the Sasakian structure)

$$\begin{aligned}
[\xi,\nu] &= -[J(\nu),\nu] \\
&= -[\frac{J(V)}{|V|},\frac{V}{|V|}] \\
&= -\frac{1}{|V|}\underbrace{J(V)(\frac{1}{|V|})}_{0}V-\frac{1}{|V|}[\frac{J(V)}{|V|},V] \\
&= \frac{1}{|V|}V(\frac{1}{|V|})J(V)-\frac{1}{|V|^2}[J(V),V] \\
&= -\frac{g(\widetilde{\nabla}_V V,V)}{|V|^3}J(\frac{V}{|V|})-\frac{1}{|V|^2}J(\underbrace{[V,V]}_{0}) \\
&= 2\xi. \qquad (4.11)
\end{aligned}$$

We show now that each (non-empty) $(M_r,g_{|M_r},\xi_{|M_r})$ is Sasakian. For every $X\in TM_r$, one has

$$\begin{aligned}
\widetilde{\nabla}_X\xi &= -\widetilde{\nabla}_X(J(\nu)) \\
&= -J(\widetilde{\nabla}_X\nu) \\
&= J(A(X)) \\
&= -J(X)-g(\xi,X)\nu,
\end{aligned}$$

so that $\widetilde{\nabla}_\xi\xi=-2\nu$, from which $\nabla^{M_r}_\xi\xi=0$ follows and, for every $Z\in\{\xi,\nu\}^\perp$, the identity $\widetilde{\nabla}_Z\xi=-J(Z)$ implies $\nabla^{M_r}_Z\xi=-J(Z)$. In particular, $\xi_{|M_r}$ defines a minimal Riemannian flow on $(M_r,g_{|M_r})$ and $h=-J$ is an almost Hermitian structure on $\xi^\perp\subset TM_r$. It remains to show that h - or, equivalently, J - is transversally parallel on $\xi^\perp$. Recall that, from the definition of the transversal covariant derivative ∇ one has, for all sections Z,Z' of $\xi^\perp$,

$$\begin{aligned}
\nabla_\xi Z &= \nabla^{M_r}_\xi Z-h(Z) \\
&= \widetilde{\nabla}_\xi Z-g(A(\xi),Z)\nu+J(Z) \\
&= \widetilde{\nabla}_\xi Z+J(Z)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_Z Z' &= \nabla^{M_r}_Z Z'+g(h(Z),Z')\xi \\
&= \widetilde{\nabla}_Z Z'-g(A(Z),Z')\nu-g(J(Z),Z')\xi \\
&= \widetilde{\nabla}_Z Z'+g(Z,Z')\nu-g(J(Z),Z')\xi,
\end{aligned}$$

from which one deduces that

$$\begin{aligned}
(\nabla_\xi J)(Z) &= \nabla_\xi(J(Z))-J(\nabla_\xi Z) \\
&= \widetilde{\nabla}_\xi(J(Z))-Z-J(\widetilde{\nabla}_\xi Z)+Z \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}(\nabla_Z J)(Z') &= \nabla_Z(J(Z')) - J(\nabla_Z Z') \\ &= \widetilde{\nabla}_Z(J(Z')) + g(Z, J(Z'))\nu - g(J(Z), J(Z'))\xi \\ &\quad -J(\widetilde{\nabla}_Z Z') + g(Z, Z')\xi + g(J(Z), Z')\nu \\ &= 0,\end{aligned}$$

i.e., $\nabla J = 0$, which proves that $(M_r, g_{|_{M_r}}, \xi_{|_{M_r}})$ is Sasakian.

We come to the holomorphic isometry. Denote $M := M_1$, $\widehat{g} := g_{|_M}$ and $\widehat{\xi} := \xi_{|_M}$. Up to rescaling (ψ, ϕ) by a positive constant (this does not influence both conditions on (ψ, ϕ)), we may assume that $M \neq \emptyset$. Let F_t^ν be the flow of ν on $\widetilde{M}$. The vector field ν is complete since ν is bounded and $(\widetilde{M}, g)$ is complete. Consider the map

$$\begin{aligned} F : M \times \mathbb{R} &\longrightarrow \widetilde{M} \\ (x, t) &\longmapsto F_t^\nu(x). \end{aligned}$$

We first show that F is a diffeomorphism. If $F_t^\nu(x) = F_{t'}^\nu(x')$ for some $t, t' \in \mathbb{R}$ and $x, x' \in M$, then x and x' lie on the same integral curve of ν. Let now c be any integral curve of ν on $\widetilde{M}$ with $c(0) \in M$ and set $f(t) := |V|_{c(t)}$ (note that f *a priori* depends on the curve and in particular on the chosen starting point). Then f is smooth with first derivative given by $f'(t) = \frac{g(\widetilde{\nabla}_V V, V)}{|V|^2}(c(t)) = 2|V|_{c(t)} = 2f(t)$ for all t, so that $f = f(0)e^{2t} = e^{2t}$. This has several consequences. On the one hand, f is injective, so that c meets M at most once, hence $x = x'$ and $t = t'$, which proves the injectivity of F. On the other hand, f does *a posteriori* not depend on the chosen starting point on M, in particular F_t^ν preserves the foliation by the level hypersurfaces M_r of $|\phi|$ and hence the orthogonal splitting $TM_r \oplus \mathbb{R}\nu$. Together with the surjectivity of $f : \mathbb{R} \to \mathbb{R}_+^\times$, we obtain that of F and the pointwise invertibility of the differential of F. Therefore F is a diffeomorphism.

Next we determine the metric F^*g. The map F sends $\frac{\partial}{\partial t}$ onto ν, so that obviously $F^*g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$. The preceding considerations also yield $F^*g(\frac{\partial}{\partial t}, X) = 0$ for all $t \in \mathbb{R}$ and $X \in TM$. Since

$$\frac{\partial}{\partial s}(F_s^\nu)_*\xi_{|_{s=t}} = (F_t^\nu)_*\frac{\partial}{\partial s}(F_s^\nu)_*\xi_{|_{s=0}} = (F_t^\nu)_*[\xi, \nu] \overset{(4.11)}{=} 2(F_t^\nu)_*\xi,$$

we have

$$(F_t^\nu)_*\xi = e^{2t}\xi \tag{4.12}$$

for every $t \in \mathbb{R}$. Moreover, the Lie derivative of g in direction of ν is given for all $X, Y \in \nu^\perp$ by

$$\begin{aligned}(\mathscr{L}_\nu g)(X, Y) &= g(\widetilde{\nabla}_X \nu, Y) + g(\widetilde{\nabla}_Y \nu, X) \\ &= -2g(A(X), Y) \\ &= 2(g(X, Y) + g(\xi, X)g(\xi, Y)),\end{aligned}$$

that is, $(\mathscr{L}_\nu g)_{|_{\nu^\perp}} = 2(g+\xi^\flat\otimes\xi^\flat)$. The identity $\frac{\partial}{\partial s}(F_s^\nu)^*g_{|s=t} = (F_t^\nu)^*\mathscr{L}_\nu g$ provides, for any $X,Y\in TM$ and $t\in\mathbb{R}$

$$\begin{aligned}
\frac{\partial}{\partial s}((F_s^\nu)^*g(X,Y))_{|s=t} &= (\frac{\partial}{\partial s}(F_s^\nu)^*g_{|s=t})(X,Y)\\
&= \{(F_t^\nu)^*\mathscr{L}_\nu g\}(X,Y)\\
&= \mathscr{L}_\nu g((F_t^\nu)_*X,(F_t^\nu)_*Y)\circ F_t^\nu\\
&= 2\Big(g((F_t^\nu)_*X,(F_t^\nu)_*Y)+g(\xi,(F_t^\nu)_*X)g(\xi,(F_t^\nu)_*Y)\Big)\circ F_t^\nu\\
&= 2\Big((F_t^\nu)^*g(X,Y)+(F_t^\nu)^*g((F_{-t}^\nu)_*\xi,X)(F_t^\nu)^*g((F_{-t}^\nu)_*\xi,Y)\Big)\\
&\overset{(4.12)}{=} 2\Big((F_t^\nu)^*g(X,Y)+e^{-4t}(F_t^\nu)^*g(\xi,X)(F_t^\nu)^*g(\xi,Y)\Big). \quad (4.13)
\end{aligned}$$

Since $(F_t^\nu)^*g(\xi,\xi)=g((F_t^\nu)_*\xi,(F_t^\nu)_*\xi)\circ F_t^\nu \overset{(4.12)}{=} (e^{4t}g(\xi,\xi))\circ F_t^\nu = e^{4t}$, we deduce from (4.13) that, for $X=\xi$,

$$\frac{\partial}{\partial s}((F_s^\nu)^*g(\xi,Y))_{|s=t} = 4(F_t^\nu)^*g(\xi,Y),$$

from which $(F_t^\nu)^*g(\xi,Y)=e^{4t}g(\xi,Y)$ follows. In particular, $(F_t^\nu)^*g(\xi,Y)=0$ for every $Y\in\{\xi,\nu\}^\perp$. For $X,Y\in\{\xi,\nu\}^\perp$, the identity (4.13) becomes

$$\frac{\partial}{\partial s}((F_s^\nu)^*g(X,Y))_{|s=t} = 2(F_t^\nu)^*g(X,Y),$$

which implies $(F_t^\nu)^*g(X,Y)=e^{2t}g(X,Y)$. To sum up, the pull-back metric on $M\times\mathbb{R}$ is given by

$$F^*g = e^{2t}(e^{2t}\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp})\oplus dt^2,$$

where $\widehat{g}_{\widehat{\xi}} = \widehat{\xi}^\flat\otimes\widehat{\xi}^\flat = \widehat{g}(\widehat{\xi},\cdot)\otimes\widehat{g}(\widehat{\xi},\cdot)$ and, as in the beginning of this section, $\widehat{g}_{\widehat{\xi}^\perp}$ denotes the restriction of $\widehat{g}$ onto the subspace $\{\widehat{\xi},\frac{\partial}{\partial t}\}^\perp\subset TM$. Hence the map F provides an isometry with the doubly warped product of Theorem 4.3.9.*i*). This isometry pulls the spin structure of $\widetilde{M}$ back onto the product spin structure of $M\times\mathbb{R}$, where M carries the spin structure induced by its embedding in $\widetilde{M}$. It remains to show that F identifies the complex structures. This follows from the definition of the complex structure on the doubly warped product $M\times\mathbb{R}$ (see Lemma 4.3.4), from $(F_t^\nu)_*\nu=\nu$, $(F_t^\nu)_*(e^{-2t}\widehat{\xi})=\xi$ and from $[J(Z),\nu]=\widetilde{\nabla}_{J(Z)}\nu-\widetilde{\nabla}_\nu J(Z) = -A(J(Z))-J(\widetilde{\nabla}_\nu Z) = J(Z)-J(\widetilde{\nabla}_\nu Z) = J([Z,\nu])$ for every section Z of $\{\xi,\nu\}^\perp$ (use the computation of A above).
Last but not the least, the identity $\nu\cdot\psi=i\phi$ implies that ϕ (or, equivalently, ψ) is transversally parallel on $(M,\widehat{g},\widehat{\xi})$ by Theorem 4.3.9.*i*). This concludes the proof of Theorem 4.4.1. □

It is important to note that only the condition $W\cdot\psi=\mu\phi$ for some real vector field W is restrictive, since by [C9, Thm. 11] the identity $|\psi|=|\phi|$ can always be assumed.

We conjecture that the examples of Section 4.3 describe all Kähler spin manifolds admitting non-trivial imaginary Kählerian Killing spinors. This will be the object of a forthcoming paper.

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Chapter 5

The Yamabe problem on Lorentzian manifolds

5.1 Introduction and first results

We first state the problem and discuss it locally as well as in dimension 2.

Let (M^n, g) be an n-dimensional Lorentzian manifold, where the signature of the metric is $(- + \ldots +)$. Let $\Box := \delta_g \circ d = -\mathrm{tr}_g(\nabla \circ d)$ denote the scalar d'Alembert operator on (M^n, g). If S_g stands for the scalar curvature of (M^n, g), then the transformation formulas for scalar curvature under conformal changes of metric read

$$e^{2u} S_{\overline{g}} = S_g + 2\Box u \tag{5.1}$$

for $n = 2$ and $\overline{g} := e^{2u} g$ (here $u \in C^\infty(M, \mathbb{R})$) and

$$\frac{n-2}{4(n-1)} S_{\overline{g}} \varphi^{\frac{n+2}{n-2}} = \Box \varphi + \frac{n-2}{4(n-1)} S_g \varphi \tag{5.2}$$

for $n \geq 3$ and $\overline{g} := \varphi^{\frac{4}{n-2}} g$ (here $\varphi \in C^\infty(M, \mathbb{R}_+^\times)$). As in the Riemannian context (see H. Yamabe [D31]), the *Yamabe problem* can be formulated as follows:

Yamabe problem: *Given a Lorentzian metric g on M, find a metric $\overline{g}$ conformal to g with constant scalar curvature on M.*

From both identities above this is equivalent to solving (5.1) in dimension $n = 2$ and (5.2) in dimension $n \geq 3$ respectively: given a constant $S_{\overline{g}} \in \mathbb{R}$, look for $u \in C^\infty(M, \mathbb{R})$ (resp. $\varphi \in C^\infty(M, \mathbb{R}_+^\times)$) satisfying (5.1) (resp. (5.2)).

Both (5.1) and (5.2) are semilinear (and nonlinear in case $S_{\overline{g}} \neq 0$) wave equations. Since such an equation can be locally put into the form of a symmetric (or symmetrizable) hyperbolic system and such systems always have local smooth solutions (see e.g. [D30, Ch. 16]), both (5.1) and (5.2) are locally solvable on any spacetime.

To prove global existence (and possibly uniqueness) of solutions, it is convenient to restrict the geometric category of Lorentzian manifolds. First, we assume M to admit a time-orientation (such Lorentzian manifolds will be called *spacetimes*). We shall mainly focus on so-called *globally hyperbolic* spacetimes:

Definition 5.1.1 *A spacetime (M^n,g) is called* globally hyperbolic *if and only if there exists a Cauchy hypersurface in M, that is, a subset Σ of M which is met exactly once by every inextendible timelike curve*[1] *in M.*

By [D7, Thm. 3.2], a spacetime is globally hyperbolic if and only if it has no closed (future- or past-directed) causal curve and all subsets of the form $J_+^M(p)\cap J_-^M(q)$, $p,q\in M$, are compact. If Σ is a *smooth spacelike* Cauchy hypersurface of M, then actually it is met exactly once by any inextendible *causal* curve in M. We also recall the following smooth splitting theorem for globally hyperbolic spacetimes:

Theorem 5.1.2 (A. Bernal & M. Sánchez [D5, D6]) *Let (M^n,g) be a spacetime.*

i) If (M^n,g) is globally hyperbolic, then it is isometric to $(\mathbb{R}\times\Sigma,-\beta dt^2\oplus g_t)$, where each $\{t\}\times\Sigma$ corresponds to a smooth spacelike *Cauchy hypersurface of M, $\beta\in C^\infty(\mathbb{R}\times\Sigma,\mathbb{R}_+^\times)$ and $(g_t)_t$ is a smooth 1-parameter family of Riemannian metrics on Σ.*

ii) If $\Sigma\subset M$ is any given *smooth spacelike Cauchy hypersurface in the (globally hyperbolic) spacetime (M^n,g), then for any $t_0\in\mathbb{R}$ there is an isometry $(M^n,g)\cong(\mathbb{R}\times\Sigma,-\beta dt^2\oplus g_t)$ as above and where Σ* identifies with *$\{t_0\}\times\Sigma$.*

For instance, the warped product $(M,g)=(I\times\Sigma,-dt^2\oplus b(t)^2g_\Sigma)$ of an open interval $I\subset\mathbb{R}$ with a Riemannian manifold (Σ,g_Σ) (where $b\in C^\infty(I,\mathbb{R}_+^\times)$ is arbitrary) is globally hyperbolic if and only if (Σ,g_Σ) is complete, see e.g. [D4, Thm. 3.66] or [D3, Lemma A.5.14]. This class contains for instance all Robertson-Walker spacetimes, in particular the Minkowski and the de Sitter spacetimes.

It is however important to note that, in general, Theorem 5.1.2 only *implies* the existence of a smooth splitting in the form $(\mathbb{R}\times\Sigma,-\beta dt^2\oplus g_t)$, and that the induced Riemannian metric g_t on Σ need not be complete. Namely, not every product of the form $(I\times\Sigma,-\beta dt^2\oplus g_t)$ – even with complete g_t – is globally hyperbolic. For instance, every hypersurface of the form $\{t\}\times\mathbb{S}_+^{n-1}=\{t\}\times\left\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n\,|\,\sum_{j=1}^n x_j^2=1 \text{ and } x_n>0\right\}$ in the (universal cover of the) anti de Sitter spacetime $(\mathbb{R}\times\mathbb{S}_+^{n-1},\frac{1}{x_n^2}(-dt^2\oplus\langle\cdot\,,\cdot\rangle)$ is complete w.r.t. $\frac{1}{x_n^2}\langle\cdot\,,\cdot\rangle$ (it is isometric to the hyperbolic space), nevertheless the anti de Sitter spacetime is not globally hyperbolic, in particular no $\{t\}\times\mathbb{S}_+^{n-1}$ can be a Cauchy hypersurface. Moreover, there may exist incomplete spacelike Cauchy hypersurfaces in globally hyperbolic spacetimes, as noticed in e.g. [D1, Sec. 2.5]: take for example the flat 2-dimensional Minkowski space $(M^2,g)=(\mathbb{R}^2,\langle\langle\cdot\,,\cdot\rangle\rangle)$ in null coordinates, i.e., with metric $\langle\langle\cdot\,,\cdot\rangle\rangle=dx_1\otimes dx_2+dx_2\otimes dx_1$, then the graph of any monotonously increasing diffeomorphism $f:\mathbb{R}\to\mathbb{R}$ with $\int_0^\infty\sqrt{f'(s)}ds<\infty$ is an incomplete spacelike Cauchy hypersurface of (M^2,g). Let us also mention that any product of the form $(I\times\Sigma,-\beta dt^2\oplus g_t)$ with *closed* Σ is globally hyperbolic and contains every $\{t\}\times\Sigma$ as a Cauchy hypersurface [D26, Cor. 3.3].

[1] meaning that every timelike curve which is inextendible *as a curve* meets Σ exactly once.

Since the causal type for vectors does not change when rescaling pointwise the metric, it is easy to see that (M^n, g) is globally hyperbolic if and only if $(M^n, \overline{g})$ is globally hyperbolic, for any metric $\overline{g}$ conformal to g. By conformal invariance of the Yamabe problem, we can therefore – and will in most cases – assume that $\beta = 1$, that is, that $g = -dt^2 \oplus g_t$ on $I \times \Sigma$. Before studying the above equations in particular cases, we give the following useful formulas:

Lemma 5.1.3 *Let a spacetime (M^n, g) be of the form $(I \times \Sigma, -\beta dt^2 \oplus g_t)$ where $\beta \in C^\infty(I \times \Sigma, \mathbb{R}_+^\times)$ and $(g_t)_t$ is a smooth 1-parameter family of Riemannian metrics on Σ. Then the following identities hold.*

1. *For every $f \in C^\infty(M, \mathbb{R})$,*

$$\begin{aligned}\Box f &= \frac{1}{\beta}\frac{\partial^2 f}{\partial t^2} + \frac{1}{2\beta}\Big(\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t}) - \frac{1}{\beta}\frac{\partial \beta}{\partial t}\Big)\frac{\partial f}{\partial t} \\ &\quad -\frac{1}{2\beta} g_t(\mathrm{grad}_{g_t}(\beta(t,\cdot)), \mathrm{grad}_{g_t}(f(t,\cdot))) + \Delta_{g_t} f(t,\cdot), \end{aligned} \tag{5.3}$$

   where $\Delta_{g_t} := \delta^\Sigma_{g_t} \circ d = -\mathrm{tr}_{g_t}(\mathrm{Hess}^\Sigma_{g_t}()) : C^\infty(\Sigma, \mathbb{R}) \to C^\infty(\Sigma, \mathbb{R})$.

2. *In case $\beta = 1$, we have*

$$\Box = \frac{\partial^2}{\partial t^2} + \frac{1}{2}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\frac{\partial}{\partial t} + \Delta_{g_t}. \tag{5.4}$$

3. *In case $\beta = 1$ and $g_t = b(t)^2 g_\Sigma$ for some $b \in C^\infty(I, \mathbb{R}_+^\times)$ and some Riemannian metric g_Σ on Σ, one has*

$$\begin{aligned}\Box + a_n S_g &= \frac{\partial^2}{\partial t^2} + (n-1)\frac{b'}{b}\frac{\partial}{\partial t} + \frac{1}{b^2}\Delta_{g_\Sigma} \\ &\quad + \frac{a_n}{b^2}\Big(S_{g_\Sigma} + 2(n-1)bb'' + (n-1)(n-2)(b')^2\Big), \end{aligned} \tag{5.5}$$

   where $a_n := \frac{n-2}{4(n-1)}$ and where S_g and S_{g_Σ} are the scalar curvatures of (M, g) and (Σ, g_Σ) respectively.

4. *In case $\beta = 1$ and $g_t = g_\Sigma$ for some Riemannian metric g_Σ on Σ, one has*

$$\Box + a_n S_g = \frac{\partial^2}{\partial t^2} + L_{g_\Sigma}, \tag{5.6}$$

   where $L_{g_\Sigma} := \Delta_{g_\Sigma} + a_n S_{g_\Sigma}$.

We first deal with the case $n = 2$. The following theorem is the exact analogue of Theorem 5.2.9 below in dimension 2.

Theorem 5.1.4 *Let (M^2, g) be a connected 2-dimensional globally hyperbolic spacetime.*

1) *Then (M^2, g) is conformally equivalent to the product $(I' \times \Sigma, -dt^2 \oplus ds^2)$ of an open interval $I' \subset \mathbb{R}$ with either $\Sigma = \mathbb{S}^1$ (circle of arbitrary radius) or $\Sigma = \mathbb{R}$. In particular, (M^2, g) is conformally flat, i.e.,* (5.1) *with $S_{\overline{g}} = 0$ always has a global smooth solution on M.*

2) *If* $S_{\overline{g}} \in \mathbb{R}^\times$*, then there is no solution to* (5.1) *on* $(\mathbb{R}\times\mathbb{S}^1, -dt^2 \oplus ds^2)$.

Proof: Theorem 5.1.2 yields a smooth splitting $(M^2, g) = (I \times \Sigma, -\beta dt^2 \oplus g_t)$, where $I \subset \mathbb{R}$ is an open interval, $\beta \in C^\infty(M, \mathbb{R}_+^\times)$, each $\{t\}\times\Sigma$ is a smooth spacelike Cauchy hypersurface in M and $(g_t)_{t\in I}$ is a smooth one-parameter family of Riemannian metrics on Σ. By conformal invariance of the Yamabe problem, we may assume $\beta = 1$. Because Σ is 1-dimensional, one has $g_t = b(t)^2 ds^2$, where ds^2 is a fixed metric on Σ and $b \in C^\infty(I, \mathbb{R}_+^\times)$. It is easy to see that $(I \times \Sigma, -dt^2 \oplus b(t)^2 ds^2)$ is conformally equivalent to $(I' \times \Sigma, -dt^2 \oplus ds^2)$, where I' is determined by b (see Section 5.2 below). Since global hyperbolicity is a conformal invariant, $(I' \times \Sigma, -dt^2 \oplus ds^2)$ is globally hyperbolic; in turn, this forces $\Sigma = \mathbb{S}^1$ or $\Sigma = \mathbb{R}$. This shows 1). Note that, as an alternative proof of 1), we may solve directly the Cauchy problem associated to (5.1): fixing a (smooth spacelike) Cauchy hypersurface Σ of M with future unit normal ν as well as $u_0, u_1 \in C^\infty(\Sigma, \mathbb{R})$, the Cauchy problem with smooth (but not necessarily compactly-supported) data $\Box u = -\frac{S_g}{2}$, $u_{|\Sigma} = u_0$, $\partial_\nu u_{|\Sigma} = u_1$ is *linear* (inhomogeneous), hence always solvable on any globally hyperbolic spacetime, see e.g. [D12, Cor. 5].
Let $S_{\overline{g}} \in \mathbb{R}^\times$ be arbitrary. Assume the existence of $u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, \mathbb{R})$ solving (5.1), i.e., $\Box u = \frac{S_{\overline{g}}}{2} e^{2u}$ on $\mathbb{R}\times\mathbb{S}^1$. Setting $y : \mathbb{R} \to \mathbb{R}$, $t \mapsto \int_{\mathbb{S}^1} u(t,x)dx$, the function y is smooth with

$$\begin{aligned} y''(t) &= \int_{\mathbb{S}^1} \frac{\partial^2 u}{\partial t^2}(t,x)dx \\ &= \int_{\mathbb{S}^1} (\Box u)(t,x)dx \qquad \text{since } \int_{\mathbb{S}^1} \frac{\partial^2 u}{\partial x^2}(t,x)dx = 0 \\ &= \frac{S_{\overline{g}}}{2} \int_{\mathbb{S}^1} e^{2u(t,x)}dx, \end{aligned}$$

compare with the proof of Theorem 5.2.9 below. Assume $S_{\overline{g}} > 0$. Denoting by $L > 0$ the length of $\mathbb{S}^1$, Jensen's inequality yields

$$y'' \geq \frac{S_{\overline{g}} L}{2} \exp\left(\frac{1}{L}\int_{\mathbb{S}^1} 2u(t,x)dx\right) = \frac{S_{\overline{g}} L}{2} e^{\frac{2y}{L}}$$

on $\mathbb{R}$. But no function satisfying that differential inequality can exist on $\mathbb{R}$, see also the proof of Theorem 5.2.9 below. Namely, up to replacing y by $t \mapsto y(\alpha t)$ for a suitable $\alpha \in \mathbb{R}_+^\times$, we assume that y satisfies $y'' \geq \frac{1}{2} e^{\frac{2y}{L}}$. Since in particular y is strictly convex, we may assume up to changing t into $\pm t + t_0$ for a constant $t_0 \in \mathbb{R}$ that $y' \geq 0$ on $[0, \infty[$. Multiplying with y' yields $y''y' \geq \frac{y'}{2} e^{\frac{2y}{L}}$, so that $(y')^2(t) - (y')^2(0) \geq \frac{L}{2}(e^{\frac{2y(t)}{L}} - e^{\frac{2y(0)}{L}})$ for every $t \geq 0$, which in turn gives

$$\int_{y(0)}^{y(t)} \frac{dz}{\sqrt{e^{\frac{2z}{L}} - e^{\frac{2y(0)}{L}}}} \geq \frac{L}{2} t$$

for every $t \geq 0$. Because of $\int_{y(0)}^{\infty} \frac{dz}{\sqrt{e^{\frac{2z}{L}} - e^{\frac{2y(0)}{L}}}} < \infty$, the existence interval of y is bounded above, or in other words $y(t) \to \infty$ in finite time. In particular, y is not defined on $\mathbb{R}$. The case where $S_{\overline{g}} < 0$ is analogous (this time y is concave and goes to $-\infty$ in finite time). This shows 2) and concludes the proof. $\Box$

Notes 5.1.5

1. Since the Cauchy data for (5.1) along a given Cauchy hypersurface may be prescribed arbitrarily, there are actually infinitely many conformal flat metrics which are non homothetic to each other on a given globally hyperbolic 2-dimensional spacetime. Alternatively – and as is well-known – all solutions to $\Box u = 0$ on $(M^2, g) = (I \times \Sigma, -dt^2 \oplus ds^2)$ are of the form $u(t,s) = v(t+s) + w(t-s)$, with arbitrary (and periodic if $\Sigma = \mathbb{S}^1$) smooth functions v, w on $\mathbb{R}$, see also Note 5.2.7.2 below.

2. For $S_{\overline{g}} \in \mathbb{R}^\times$, Theorem 5.1.4 states that there is no solution to (5.1) on $M^2 = I \times \mathbb{S}^1$ when the time interval I is long enough. But solutions exist for short I, as we know anyway from the local theory mentioned above. For example, the 2-dimensional de Sitter spacetime, which can be described as the warped product $(\mathbb{R} \times \mathbb{S}^1, -dt^2 \oplus \cosh(t)^2 ds^2)$, is conformally equivalent to the flat cylinder $(]-\frac{\pi}{2}, \frac{\pi}{2}[\times \mathbb{S}^1, -dt^2 \oplus ds^2)$, see Corollary 5.2.10 below. In particular, there exists a conformal metric with scalar curvature 2 on $(]-\frac{\pi}{2}, \frac{\pi}{2}[\times \mathbb{S}^1, -dt^2 \oplus ds^2)$.

In the non globally hyperbolic setting, conformal flatness may or may not hold. For instance, the 2-dimensional anti de Sitter spacetime $(\mathbb{S}^1 \times \mathbb{S}^1_+, \frac{1}{x_2^2}(-dt^2 \oplus ds^2))$ (where (x_1, x_2) are the cartesian coordinates for the second factor $\mathbb{S}^1_+ := \{(x_1, x_2) \in \mathbb{S}^1 \,|\, x_2 > 0\})$ is obviously conformally flat. On $M = \mathbb{R}^2$ or the 2-torus $\mathbb{T}^2$, Miguel Sánchez has shown that an arbitrary metric g is conformally flat if and only if it admits a non-zero conformal Killing vector field which is everywhere timelike or everywhere spacelike [D25, Thm. 2.3]. Moreover, he constructed whole families of metrics on $\mathbb{T}^2$ (and $\mathbb{R}^2$) without any such conformal Killing vector field and which hence are not conformally flat [D25, Sec. 3]. Note that none of those metrics on $\mathbb{R}^2$ can be globally hyperbolic by Theorem 5.1.4.

Let us mention that there is still a lot of freedom left when prescribing scalar curvature functions in 2 dimensions: generalizing previous work by John Burns [D9, Thm. 2.2], Marc Nardmann proved that any function which is either identically vanishing or sign-changing on a closed Lorentzian surface M is the scalar curvature of some Lorentzian metric on M [D22, Thm. 1.3.13].

From now on, we assume $n \geq 3$. In that case we know local solutions exist by the remarks above. One can do a bit better: as for the existence problem for solutions to the Einstein equations [D10, Thm. 3], there is a maximal domain of existence for solutions to the Yamabe problem:

Theorem 5.1.6 *Let (M^n, g) be an $n(\geq 3)$-dimensional globally hyperbolic spacetime with smooth spacelike closed Cauchy hypersurface $\Sigma \subset M$ and $S_{\overline{g}} \in \mathbb{R}$ be an arbitrary constant. Denote by $\nu \in \Gamma(T^\perp\Sigma)$ the future-directed (timelike) unit normal along Σ. Then for any $\varphi_0, \varphi_1 \in C^\infty(\Sigma, \mathbb{R})$ with $\varphi_0 > 0$, there exists a unique maximal globally hyperbolic open subset $\widehat{D}_\Sigma$ of M in which Σ is a Cauchy hypersurface and on which the Cauchy problem* (5.2) *with $\varphi_{|_\Sigma} = \varphi_0$ and $\partial_\nu \varphi = \varphi_1$ has a unique smooth positive solution.*

Proof: The proof mainly relies on local existence and (global) uniqueness for solutions to the Cauchy problem

$$\begin{cases} \Box\varphi + a_n S_g \varphi & = a_n S_{\overline{g}} \varphi^{\frac{n+2}{n-2}} \\ \varphi_{|_\Sigma} & = \varphi_0 \\ \partial_\nu \varphi & = \varphi_1, \end{cases} \tag{5.7}$$

which both follow from the theory of symmetric hyperbolic systems. Namely for any $\varphi_0, \varphi_1 \in C^\infty(\Sigma, \mathbb{R})$ with $\varphi_0 > 0$ consider the set

$$\begin{aligned} \mathscr{M}_{\Sigma,\varphi_0,\varphi_1} \quad := \quad & \Big\{ D_\Sigma \subset M, D_\Sigma \text{ open}, \Sigma \text{ Cauchy hypersurface of } D_\Sigma, \\ & \exists\, \varphi \in C^\infty(D_\Sigma, \mathbb{R}_+^\times) \text{ solving (5.7) on } D_\Sigma \Big\}. \end{aligned}$$

Note that, by uniqueness of solutions to symmetric hyperbolic systems, for each $D_\Sigma \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$, there is a *unique* positive smooth solution φ to (5.2) on D_Σ with Cauchy data φ_0, φ_1. Local existence for the Cauchy problem along the compact Cauchy hypersurface Σ already ensures $\mathscr{M}_{\Sigma,\varphi_0,\varphi_1} \neq \emptyset$: if $(M^n, g) = (\mathbb{R} \times \Sigma, -\beta dt^2 \oplus g_t)$ is split as in Theorem 5.1.2, where say $\Sigma \simeq \{0\} \times \Sigma$, then there is a nonempty open interval $J \subset \mathbb{R}$ about 0 for which a smooth positive solution to the Cauchy problem (5.7) exists on the open subset $J \times \Sigma$ of M; but with the induced metric and time orientation, $J \times \Sigma$ is clearly globally hyperbolic with Σ as a Cauchy hypersurface, therefore $J \times \Sigma \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$.
Next define $\widehat{D}_\Sigma := \bigcup\limits_{D_\Sigma \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}} D_\Sigma \subset M$, which is open in M and contains Σ. We claim that $\widehat{D}_\Sigma \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$. First, we show that Σ is a Cauchy hypersurface of $\widehat{D}_\Sigma$ (hence $\widehat{D}_\Sigma$ is globally hyperbolic). The proof of this is based on the following two claims.
Claim 1: *Let $\Omega \subset M$ be any nonempty open subset which is* causally compatible *in M (for any $p \in \Omega$, $J_\pm^M(p) \cap \Omega = J_\pm^\Omega(p)$). Then Ω itself – with the induced metric and time orientation – is globally hyperbolic if and only if $J_+^M(p) \cap J_-^M(q) \subset \Omega$ for all $p, q \in \Omega$.*
Proof of Claim 1: There exists no closed causal curve in Ω since there is already none in M. If Ω is globally hyperbolic, then for all $p, q \in \Omega$ the subset $J_+^\Omega(p) \cap J_-^\Omega(q)$ is compact; but by causal compatibility of Ω, $J_+^\Omega(p) \cap J_-^\Omega(q) = J_+^M(p) \cap J_-^M(q) \cap \Omega$; now $J_+^M(p) \cap J_-^M(q)$ is by construction (path-)connected, so that the intersection $J_+^M(p) \cap J_-^M(q) \cap \Omega$, being open and closed in $J_+^M(p) \cap J_-^M(q)$, is either empty or the whole subset $J_+^M(p) \cap J_-^M(q)$; in the first case, necessarily $J_+^M(p) \cap J_-^M(q) = \emptyset$ (otherwise $q \in J_+^M(p) \cap J_-^M(q) \cap \Omega$) and hence $J_+^\Omega(p) \cap J_-^\Omega(q) = J_+^M(p) \cap J_-^M(q)$; in the second case, we also obtain $J_+^\Omega(p) \cap J_-^\Omega(q) = J_+^M(p) \cap J_-^M(q)$. In both cases $J_+^M(p) \cap J_-^M(q) \subset \Omega$. Conversely, if $J_+^M(p) \cap J_-^M(q) \subset \Omega$ for all $p, q \in \Omega$, then $J_+^\Omega(p) \cap J_-^\Omega(q) = J_+^M(p) \cap J_-^M(q) \cap \Omega = J_+^M(p) \cap J_-^M(q)$ is compact for all $p, q \in \Omega$ and thus Ω is globally hyperbolic. $\surd$
Claim 2: *If Σ is a Cauchy hypersurface of an open subset $\Omega \subset M$, then Ω is automatically causally compatible in M.*
Proof of Claim 2: Let $p \in \Omega$ and $q \in J_+^M(p) \cap \Omega$ be arbitrary. Pick a future-directed causal curve $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$ in M and extend it to an inextendible future-directed causal curve $\widetilde{c} : \mathbb{R} \to M$. We consider the following cases. First, let $p \in J_+^\Omega(\Sigma)$. Since Σ is a spacelike Cauchy hypersurface of M, there exists a unique $t_0 \in \mathbb{R}$ with $\widetilde{c}(t_0) \in \Sigma$; note that $t_0 \leq 0$ because of $p \in J_+^\Omega(\Sigma) \subset J_+^M(\Sigma) \cap \Omega$. Define $t_{\min} := \inf\{t < 1 \,|\, \widetilde{c}(s) \in \Omega \,\forall s \in [t,1]\}$ and $t_{\max} := \sup\{t > 1 \,|\, \widetilde{c}(s) \in \Omega \,\forall s \in [1,t]\}$. Note that $t_{\min} \in [-\infty, 1[$ and $t_{\max} \in]1, \infty]$ are well-defined and that $\widetilde{c}(]t_{\min}, t_{\max}[) \subset \Omega$. The curve $\widetilde{c}_{|]t_{\min},t_{\max}[} :]t_{\min}, t_{\max}[\to \Omega$ is future-directed causal and inextendible as a curve in Ω by construction of $t_{\min}$ and $t_{\max}$, therefore it meets the Cauchy hypersurface Σ of Ω in exactly one point. But since t_0 is the unique $t \in \mathbb{R}$ with $\widetilde{c}(t) \in \Sigma$, one necessarily has $t_{\min} < t_0$, in particular $t_{\min} < 0$, from which $\widetilde{c}(s) = c(s) \in \Omega$ for all $s \in [0,1] \subset]t_{\min}, t_{\max}[$ follows. This implies $q \in J_+^\Omega(p)$. The case where $q \in J_-^\Omega(\Sigma)$ is analogous (just "reverse" time). The last case where p and q are on two different sides of Σ (i.e., $p \in I_-^\Omega(\Sigma)$ and $q \in I_+^\Omega(\Sigma)$) is also similar: one may assume $c(\frac{1}{2}) \in \Sigma$

and then one shows as above that both restrictions $c_{|_{[0,\frac{1}{2}]}}$ and $c_{|_{[\frac{1}{2},1]}}$ run entirely in Ω. Therefore $q \in J_+^{\Omega}(p)$ in all three cases. Obviously $J_+^{\Omega}(p) \subset J_+^M(p) \cap \Omega$ always holds true, thus we have shown $J_+^{\Omega}(p) = J_+^M(p) \cap \Omega$ for all $p \in \Omega$. Reversing time we also show $J_-^{\Omega}(p) = J_-^M(p) \cap \Omega$ for all $p \in \Omega$ and hence Ω is causally compatible. $\surd$
To show that Σ is a Cauchy hypersurface of $\widehat{D}_\Sigma$, let $c : \mathbb{R} \to \widehat{D}_\Sigma$ be any inextendible future-directed timelike curve. Then its intersection with each D_Σ – that we denote by $c \cap D_\Sigma$ – is again a curve (and remains inextendible, timelike and future-directed): for any $s \le t \in \mathbb{R}$ with $c(s), c(t) \in D_\Sigma$, one has $c(u) \in J_+^M(c(s)) \cap J_-^M(c(t))$ for all $u \in [s,t]$ and, because D_Σ is causally compatible by Claim 2, we have $J_+^M(c(s)) \cap J_-^M(c(t)) \subset D_\Sigma$ by Claim 1 and hence $c(u) \in D_\Sigma$. Therefore $c \cap D_\Sigma$ meets Σ in (exactly) one point, from which follows that c meets Σ in one point, which must be unique since Σ can anyway be met only once by causal curves. Therefore Σ is a Cauchy hypersurface of $\widehat{D}_\Sigma$.
It remains to show the existence of a $\varphi \in C^\infty(\widehat{D}_\Sigma, \mathbb{R}_+^\times)$ solving (5.7) on $\widehat{D}_\Sigma$. For this, we first show that $\mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$ is stable under finite intersection. For any $D_\Sigma^1, D_\Sigma^2 \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$, consider any inextendible timelike curve c in $D_\Sigma^1 \cap D_\Sigma^2$. Then one can extend c to inextendible causal curves $\widetilde{c}^i$ in D_Σ^i, $i = 1,2$ (of course it may happen that one – or both – extension already coincides with c itself), each of which meets Σ in exactly one point. Gluing $\widetilde{c}^1$ with $\widetilde{c}^2$ along c one obtains a future-directed causal curve $\widetilde{c}$ in $D_\Sigma^1 \cup D_\Sigma^2$ – this is a (piecewise smooth) curve since no two extensions can come out of the same end of c unless c is already extendible – which is also inextendible in $D_\Sigma^1 \cup D_\Sigma^2$. By the above argument (applicable to any union of elements of $\mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$), Σ is a Cauchy hypersurface of $D_\Sigma^1 \cup D_\Sigma^2$, therefore $\widetilde{c}$ meets Σ in exactly one point, which by uniqueness must lie in both D_Σ^1 and D_Σ^2; in turn this implies that c meets Σ in exactly one point. Therefore Σ is a Cauchy hypersurface in $D_\Sigma^1 \cap D_\Sigma^2$. It remains to notice that the solutions φ^1 and φ^2 to (5.7) on D_Σ^1 and D_Σ^2 respectively have to coincide on $D_\Sigma^1 \cap D_\Sigma^2$ by uniqueness of solutions to (5.7) on the globally hyperbolic spacetime $D_\Sigma^1 \cap D_\Sigma^2$. Therefore $D_\Sigma^1 \cap D_\Sigma^2 \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$.
Coming back to the Cauchy problem on $\widehat{D}_\Sigma$, define φ on $\widehat{D}_\Sigma$ via $\varphi(p) := \varphi^i(p)$ for $p \in D_\Sigma^i$, where $\varphi^i \in C^\infty(D_\Sigma^i, \mathbb{R}_+^\times)$ solves (5.7) on D_Σ^i; since $D_\Sigma^i \cap D_\Sigma^j \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$ for any $D_\Sigma^i, D_\Sigma^j \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$, we have $\varphi^i_{|_{D_\Sigma^i \cap D_\Sigma^j}} = \varphi^j_{|_{D_\Sigma^i \cap D_\Sigma^j}}$, so that the function φ is well-defined, positive, smooth and solves (5.7) on $\widehat{D}_\Sigma$. This shows $\widehat{D}_\Sigma \in \mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$. By construction, $\widehat{D}_\Sigma$ is maximal and is unique since it contains any element of $\mathscr{M}_{\Sigma,\varphi_0,\varphi_1}$. This concludes the proof of Theorem 5.1.6. $\square$

Of course, the maximal domain $\widehat{D}_\Sigma$ of Theorem 5.1.6 depends on Σ, on the metric g, on $S_{\overline{g}}$ and on the Cauchy data φ_0, φ_1. The same statement as in Theorem 5.1.6 also holds true in dimension 2 for the Cauchy problem corresponding to (5.1). In the next sections, we discuss when $\widehat{D}_\Sigma = M$ for M in a particular subcategory of spacetimes.

5.2 Conformally standard static spacetimes

In this section, we start with the particular case where (M^n, g) is conformally equivalent to the product $(I \times \Sigma, -dt^2 \oplus g_\Sigma)$ of an open interval $I \subset \mathbb{R}$ with a closed Riemannian manifold (Σ^{n-1}, g_Σ). Note that such a product is automatically globally hyperbolic. Following the literature, products are a particular case of so-called *standard static* spacetimes:

Definition 5.2.1 *A spacetime (M^n, g) is called*

i) static *if and only if it admits a timelike Killing vector field whose orthogonal distribution is integrable.*

ii) standard static *if and only if it is isometric to a product $(I\times\Sigma, -\beta dt^2\oplus g_\Sigma)$ for some open interval $I\subset\mathbb{R}$, some Riemannian manifold (Σ^{n-1}, g_Σ) and some $\beta\in C^\infty(\Sigma,\mathbb{R}_+^\times)$.*

Any standard static spacetime is static (take e.g. $\frac{\partial}{\partial t}$ as timelike Killing vector field with integrable orthogonal distribution) and any static spacetime is locally standard static. A simply connected static spacetime (M^n, g) is standard static if and only if at least one of its static vector fields (Killing, timelike, with integrable orthogonal distribution) is complete [D27, Thm. 2.2]. Note that a standard static spacetime $(I\times\Sigma, -\beta dt^2\oplus g_\Sigma)$ is globally hyperbolic if and only if the metric $\frac{1}{\beta}g_\Sigma$ is complete, in particular any standard static spacetime with closed Σ is globally hyperbolic. We refer to the excellent survey [D27] for further geometric and causal aspects of standard static spacetimes.

Thus, we shall consider in this section spacetimes that are conformally equivalent to standard static ones. Since we may first want a conformal characterisation of such spacetimes, we give the following

Proposition 5.2.2 *A spacetime (M^n, g) is conformally equivalent to a standard static spacetime if and only if there exists a smooth function $t : M \longrightarrow \mathbb{R}$ such that* $\mathrm{grad}_g(t)$ *is everywhere past-directed timelike and for the induced splitting $(M^n, g) = (I\times\Sigma, -\beta dt^2\oplus g_t)$ via the flow of $\frac{\mathrm{grad}_g(t)}{|\mathrm{grad}_g(t)|_g^2}$, the Riemannian metric $\frac{1}{\beta}g_t$ on Σ does not depend on t.*

A smooth function $t : M \longrightarrow \mathbb{R}$ whose gradient is everywhere past-directed timelike is called *temporal*, see e.g. [D18, Def. 3.48]; a temporal function is in particular a *time* function, i.e., it is monotonously increasing on any future-directed causal curve in (M^n, g). Note that the vector field – and hence the induced flow – $\frac{\mathrm{grad}_g(t)}{|\mathrm{grad}_g(t)|_g^2}$, the conditions t be a temporal function and $\frac{\partial}{\partial t}\left(\frac{1}{\beta}g_t\right) = 0$ all only depend on the conformal class of g.

Clearly, a spacetime (M^n, g) that is conformally equivalent to a standard static one has a (future-directed) timelike conformal Killing vector field, the converse being wrong in general (though a *globally hyperbolic* spacetime with complete timelike conformal Killing vector field is conformally equivalent to a so-called *standard stationary* spacetime [D27, Prop. 3.3]). In particular, globally hyperbolic spacetimes with trivial or even discrete conformal group cannot be conformally equivalent to a standard static one.

For instance, any *warped product* spacetime $(M^n, g) = (I\times\Sigma, -dt^2\oplus b(t)^2 g_\Sigma)$, where $b\in C^\infty(I,\mathbb{R}_+^\times)$, admits such a temporal function (fix $s_0\in I$ and set $t(s,x) := \int_{s_0}^s \frac{d\tau}{b(\tau)}$) and hence is conformally equivalent to a standard static spacetime. More concretely, if $(M^n, g) = (]\alpha_-,\alpha_+[\times\Sigma, -dt^2\oplus b(t)^2 g_\Sigma)$ for some $b\in C^\infty(]\alpha_-,\alpha_+[,\mathbb{R}_+^\times)$, then fixing $t_0\in]\alpha_-,\alpha_+[$, the map

$$\begin{array}{rcl}\Phi :]\alpha_-,\alpha_+[\times\Sigma & \longrightarrow &]a_-,a_+[\times\Sigma \\ (t,x) & \longmapsto & (\psi(t),x),\end{array}$$

where $a_\pm := \int_{t_0}^{\alpha_\pm} \frac{ds}{b(s)}$ and $\psi(t) := \int_{t_0}^{t} \frac{ds}{b(s)}$, is a smooth diffeomorphism with $\Phi^*(-dt^2 \oplus g_\Sigma) = -b^{-2}dt^2 \oplus g_\Sigma = b^{-2}g$.

5.2.1 Existence of solutions to the Yamabe problem

The first and most natural ansatz to solve the Yamabe problem in a product spacetime consists in separating variables.

Proposition 5.2.3 *Let $(M^n, g) = (I \times \Sigma, -dt^2 \oplus g_\Sigma)$, where $I \subset \mathbb{R}$ is an open interval, (Σ^{n-1}, g_Σ) is a closed Riemannian manifold and $n \geq 3$. Let $S_{\overline{g}} \in \mathbb{R}$, $y \in C^\infty(I, \mathbb{R}_+^\times)$ and $u \in C^\infty(\Sigma, \mathbb{R}_+^\times)$ be arbitrary. Then the function $\varphi \in C^\infty(M, \mathbb{R}_+^\times)$, $\varphi(t,x) := y(t) \cdot u(x)$, solves* (5.2) *if and only if*

i) either y or u is constant in case $S_{\overline{g}} \neq 0$; if y is constant, then u solves $L_{g_\Sigma} u = a_n S_{\overline{g}} y^{p-2} u^{p-1}$ where $p := \frac{2n}{n-2}$; if u is constant, then S_{g_Σ} is constant and y solves $y'' + a_n S_{g_\Sigma} y = a_n S_{\overline{g}} u^{p-2} y^{p-1}$.

ii) the functions y and u satisfy $y'' + \mu_1(L_{g_\Sigma})y = 0$ and $L_{g_\Sigma} u = \mu_1(L_{g_\Sigma})u$ respectively in case $S_{\overline{g}} = 0$, where $\mu_1(L_{g_\Sigma}) \in \mathbb{R}$ is the smallest eigenvalue of L_{g_Σ}.

Proof: By (5.6), the Yamabe equation (5.2) reads $\frac{\partial^2 \varphi}{\partial t^2} + L_{g_\Sigma}\varphi = a_n S_{\overline{g}} \varphi^{p-1}$. For φ of the form $\varphi(t,x) := y(t) \cdot u(x)$, this becomes $y'' \cdot u + y \cdot L_{g_\Sigma} u = a_n S_{\overline{g}} (y \cdot u)^{p-1}$. Dividing out by $y \cdot u$, this identity is equivalent to

$$\frac{y''}{y} + \frac{L_{g_\Sigma} u}{u} = a_n S_{\overline{g}} (y \cdot u)^{p-2}.$$

In case $S_{\overline{g}} \neq 0$, the first t-derivative of that identity gives $\left(\frac{y''}{y}\right)' = (p-2) a_n S_{\overline{g}} u^{p-2} y^{p-3} y'$, whose l.h.s. hence does not depend on $x \in \Sigma$, so that either $y' = 0$ on I or u is constant on Σ. If y is constant on I, then u solves $y \cdot L_{g_\Sigma} u = a_n S_{\overline{g}} (y \cdot u)^{p-1}$, that is, $L_{g_\Sigma} u = a_n S_{\overline{g}} y^{p-2} u^{p-1}$. If u is constant on Σ, then by the identity just above S_{g_Σ} must be constant and y solves the ODE $y'' + a_n S_{g_\Sigma} y = a_n S_{\overline{g}} u^{p-2} y^{p-1}$. This proves i).
In case $S_{\overline{g}} = 0$, we obtain after differentiating w.r.t. t the existence of a constant $\lambda \in \mathbb{R}$ with $\frac{y''}{y} = \lambda$ and hence also $\frac{L_{g_\Sigma} u}{u} = -\lambda$. In particular, $-\lambda$ is an eigenvalue with associated eigenfunction u for the elliptic self-adjoint linear operator L_{g_Σ} on Σ; but since we require $u > 0$, the eigenvalue $-\lambda$ can only be the smallest one $\mu_1(L_{g_\Sigma})$ by Courant's nodal domain theorem. This shows ii) and concludes the proof. □

We concentrate on the equation $L_{g_\Sigma} u = \lambda u^{p-1}$ on Σ, for which existence results are well-known, see e.g. [D17, Sec. 4] or [D2, Sec. 2.3]:

Theorem 5.2.4 (H. Yamabe [D31]) *For $n \geq 3$ let (Σ^{n-1}, g_Σ) be any closed Riemannian manifold. As above, let $L_{g_\Sigma} : C^\infty(\Sigma, \mathbb{R}) \longrightarrow C^\infty(\Sigma, \mathbb{R})$ be defined by $L_{g_\Sigma}\varphi := \Delta_{g_\Sigma}\varphi + a_n S_{g_\Sigma}\varphi$, where $a_n := \frac{n-2}{4(n-1)}$ and S_{g_Σ} is the scalar curvature of (Σ, g_Σ). For $p \in [2, \infty[$ consider the functional*

$$H^{1,2}(\Sigma) \setminus \{0\} \xrightarrow{E} \mathbb{R}, \qquad E(f) := \frac{\int_\Sigma f L_{g_\Sigma} f \, d\sigma}{\|f\|^2_{L^p(\Sigma)}},$$

where $d\sigma$ is the Riemannian density associated to g_Σ on Σ. Then we have the following:

i) An $f \in H^{1,2}(\Sigma) \setminus \{0\}$ is a critical point of E if and only if it satisfies $L_{g_\Sigma} f = \frac{E(f)}{\|f\|^{p-2}_{L^p(\Sigma)}} \cdot f^{p-1}$.

ii) If $p \in [2, p^*[$, where $p^* := \frac{2(n-1)}{n-3} \in]2, \infty]$, then there exists a minimizer of E on $H^{1,2}(\Sigma) \setminus \{0\}$.

In particular, there exists a $\varphi \in C^\infty(\Sigma, \mathbb{R}_+^\times)$ with (w.l.o.g.) $\|\varphi\|_{L^p(\Sigma)} = 1$ satisfying $L_{g_\Sigma}\varphi = \lambda_p(\Sigma, g_\Sigma) \cdot \varphi^{p-1}$ on Σ, where $\lambda_p(\Sigma, g_\Sigma) := \inf\limits_{H^{1,2}(\Sigma)\setminus\{0\}} (E) \in \mathbb{R}$.

The sign of $\lambda_p(\Sigma, g_\Sigma)$ turns out to be that of the smallest eigenvalue of the elliptic self-adjoint operator L_{g_Σ}:

Lemma 5.2.5 *With the notations of* Theorem 5.2.4 *and $p \in [2, p^*[$, the constant $\lambda_p(\Sigma, g_\Sigma)$ and the smallest eigenvalue μ_1 of L_{g_Σ} have the same sign: the one is positive (resp.* 0*, negative) if and only if the other is positive (resp.* 0*, negative).*

Proof: The negative case is clear: by definition of the constant $\lambda_p(\Sigma, g_\Sigma)$, it is negative if and only if there exists an $f \in H^{1,2}(\Sigma) \setminus \{0\}$ with $\int_\Sigma f L_{g_\Sigma} f d\sigma < 0$, which, by the min-max principle, is equivalent to $\mu_1 < 0$. Now the condition $p \geq 2$ provides a trivial inequality between $\lambda_p(\Sigma, g_\Sigma)$ and μ_1: since Σ is closed, we have, using Hölder's inequality, $\|\cdot\|_2 \leq C \cdot \|\cdot\|_p$ for some constant $C = C(\Sigma, g_\Sigma)$, hence

$$\frac{\int_\Sigma f L_{g_\Sigma} f d\sigma}{\|f\|_2^2} \geq C' \cdot \frac{\int_\Sigma f L_{g_\Sigma} f d\sigma}{\|f\|_p^2} \geq C' \cdot \lambda_p(\Sigma, g_\Sigma)$$

for some constant $C' = C'(\Sigma, g_\Sigma)$ and for every $f \in H^{1,2}(\Sigma) \setminus \{0\}$; the min-max principle yields $\mu_1 \geq C' \cdot \lambda_p(\Sigma, g_\Sigma)$. So, if $\lambda_p(\Sigma, g_\Sigma) = 0$, then this inequality implies $\mu_1 \geq 0$; on the other hand, Theorem 5.2.4 provides the existence of an $f \in C^\infty(\Sigma, \mathbb{R}_+^\times)$ with $L_{g_\Sigma} f = 0$, in particular 0 is an eigenvalue of L_{g_Σ} and hence $\mu_1 \leq 0$, so $\mu_1 = 0$. Conversely, if $\mu_1 = 0$, then the above inequality provides $\lambda_p(\Sigma, g_\Sigma) \leq 0$; on the other hand, $\int_\Sigma f L_{g_\Sigma} f d\sigma \geq 0$ holds by the min-max principle, so that $\lambda_p(\Sigma, g_\Sigma) \geq 0$ and therefore $\lambda_p(\Sigma, g_\Sigma) = 0$. This concludes the proof. □

For instance, if $S_{g_\Sigma} = 0$, then it is clear that $\mu_1 = \lambda_p(\Sigma, g_\Sigma) = 0$ (take φ to be constant on Σ). If $S_{g_\Sigma} > 0$ on Σ, then $\lambda_p(\Sigma, g_\Sigma) > 0$, as one can deduce from the bounded Sobolev embedding $H^{1,2}(\Sigma) \hookrightarrow L^p(\Sigma)$ (recall that $p \leq \frac{2(n-1)}{n-3}$): there exists a constant $C = C(\Sigma, g_\Sigma) > 0$ such that, for every $f \in H^{1,2}(\Sigma) \setminus \{0\}$,

$$\int_\Sigma f L_{g_\Sigma} f d\sigma \geq \min(1, a_n \min_\Sigma(S_{g_\Sigma})) \cdot \underbrace{\int_\Sigma |df|^2 + f^2 d\sigma}_{\|f\|^2_{H^{1,2}(\Sigma)}} \geq C \cdot \min(1, a_n \min_\Sigma(S_{g_\Sigma})) \cdot \|f\|_p^2,$$

from which we deduce $\lambda_p(\Sigma, g_\Sigma) \geq C \cdot \min(1, a_n \min_\Sigma(S_{g_\Sigma}))$. In particular, $\lambda_p(\Sigma, g_\Sigma) > 0$ as soon as $\min_\Sigma(S_{g_\Sigma}) > 0$. More generally, if $S_{g_\Sigma} \geq 0$ and does not identically vanish on Σ, then $\int_\Sigma u_1(L_{g_\Sigma} u_1) d\sigma > 0$ for any (non-zero) eigenfunction u_1 associated to the smallest eigenvalue μ_1, in particular $\mu_1 > 0$ and hence $\lambda_p(\Sigma, g_\Sigma) > 0$. Note that, if $\mu_1 < 0$ - or, equivalently, $\lambda_p(\Sigma, g_\Sigma) < 0$ - implies $\min_\Sigma(S_{g_\Sigma}) < 0$, however the other implication is wrong (use e.g. a continuity argument: perturb appropriately the standard metric on $\mathbb{S}^n$ so as to make the scalar curvature negative somewhere while keeping λ_p positive). Beware also that $\lambda_p(\Sigma, g_\Sigma)$ is *not* a conformal invariant - it is in particular not the infimum of the standard Yamabe functional.

The first global existence result of that section is the following

Theorem 5.2.6 *Let a spacetime (M^n, g) be conformally equivalent to the Lorentzian product $(I \times \Sigma, -dt^2 \oplus g_\Sigma)$ of an open interval $I \subset \mathbb{R}$ with a closed Riemannian manifold Σ^{n-1}, where $n \geq 3$. Let $\lambda_p(\Sigma, g_\Sigma) := \inf_{H^{1,2}(\Sigma)\setminus\{0\}}(E) \in \mathbb{R}$ (see* Theorem 5.2.4*) and $p := \frac{2n}{n-2}$. Then for $S_{\overline{g}} := \frac{\lambda_p(\Sigma, g_\Sigma)}{a_n}$ there exists a $\varphi \in C^\infty(M, \mathbb{R}_+^\times)$ solving* (5.2).

Proof: By conformal invariance of the Yamabe problem, we may assume that $(M^n, g) = (I \times \Sigma, -dt^2 \oplus g_\Sigma)$. In that case, (5.2) becomes $\frac{\partial^2 \varphi}{\partial t^2} + L_{g_\Sigma}\varphi = a_n S_{\overline{g}} \varphi^{p-1}$ by Lemma 5.1.3. Since $p \in [2, \frac{2(n-1)}{n-3}[$, Theorem 5.2.4 provides the existence of a smooth positive solution φ on Σ of $L_{g_\Sigma}\varphi = \lambda_p(\Sigma, g_\Sigma) \cdot \varphi^{p-1}$. This φ does not depend on t, hence solves (5.2). $\square$

As a consequence, every warped product spacetime admits at least one solution to the Yamabe problem.

Notes 5.2.7

1. The proof of Theorem 5.2.6 actually shows that the same statement as in Theorem 5.2.6 holds true for any (necessarily non globally hyperbolic) spacetime conformally equivalent to $(\mathbb{S}^1 \times \Sigma^{n-1}, -dt^2 \oplus g_\Sigma)$ with closed Σ, where $\mathbb{S}^1$ is a circle of arbitrary length: the solution we construct does not depend on time and is therefore periodic.

2. One need not have uniqueness (up to scaling by a positive constant) of a conformal metric with constant scalar curvature. Take e.g. $(M^n, g) := (\mathbb{R} \times \mathbb{T}^{n-1}, -dt^2 \oplus \mathrm{can})$, where $\mathbb{T}^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ is the $n-1$-dimensional torus obtained by modding out $\mathbb{R}^{n-1}$ by the canonically embedded lattice $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$ and can is the induced flat metric on $\mathbb{T}^{n-1}$. Taking any two 1-periodic functions $v, w \in C^\infty(\mathbb{R}, \mathbb{R}_+^\times)$, the function $\varphi \in C^\infty(\mathbb{R} \times \mathbb{R}^{n-1}, \mathbb{R}_+^\times)$ defined by

$$\varphi(t, x) := v(t + x_1) + w(t - x_1),$$

   for all $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, satisfies $\Box\varphi = 0$ and induces a smooth function (also denoted by φ) on $\mathbb{R} \times \mathbb{T}^{n-1}$ satisfying the same equation. Therefore, one obtains a whole family of non-trivial conformal metrics with vanishing scalar curvature on M^n. This also shows a big difference with the Riemannian setting, where every conformal metric with vanishing scalar curvature on $\mathbb{R} \times \mathbb{T}^{n-1}$ must be a constant positive multiple of the metric $dt^2 \oplus \mathrm{can}$ by Liouville's theorem (implying that every positive harmonic function on $\mathbb{R}^n$ must be constant). Uniqueness of the solutions is further discussed in Section 5.2.2 below.

Theorem 5.2.6 shows the existence of at least one conformal metric with constant scalar curvature on any conformally standard static spacetime. However, we notice that the sign of that conformal scalar curvature is given by that of the conformal invariant $\lambda_p(\Sigma, g_\Sigma)$ defined in Theorem 5.2.4. Therefore, we are led to asking whether *any* constant scalar curvature may be prescribed in *any* conformal class, and if not, how "large" the maximal domain of existence for solutions is. For this, the following lemma is useful.

Lemma 5.2.8 (Grönwall) *Let $\alpha, \beta : I \longrightarrow \mathbb{R}$ be continuous functions and $t_0 \in I$ be arbitrary.*

1) *If $y' + \alpha(t)y \leq 0$, then $y(t) - y(t_0)e^{-\int_{t_0}^{t}\alpha(s)ds} \begin{cases} \leq 0 & \text{if } t \geq t_0 \\ \geq 0 & \text{if } t \leq t_0 \end{cases}$.*

2) *If $y'' + \alpha(t)y' + \beta(t)y \leq 0$, then $y(t) \leq y(t_0)y_0 + y'(t_0)z_0$ for every $t \in I$, where y_0, z_0 solve the differential equation $w'' + \alpha(t)w' + \beta(t)w = 0$ on I with initial conditions $y_0(t_0) = 1 = z_0'(t_0)$ and $y_0'(t_0) = 0 = z_0(t_0)$. In other words, y must be lower than or equal to the solution of the corresponding differential equation with the* same initial conditions *at t_0.*

We come to the main existence result of this section.

Theorem 5.2.9 *Let a spacetime (M^n, g) be conformally equivalent to the Lorentzian product $(I \times \Sigma, -dt^2 \oplus g_\Sigma)$ of an open interval $I \subset \mathbb{R}$ with a closed Riemannian manifold (Σ^{n-1}, g_Σ), where $n \geq 3$. Let $\mu_1(L_{g_\Sigma}) \in \mathbb{R}$ be the smallest eigenvalue of L_{g_Σ} and let $S_{\overline{g}} \in \mathbb{R}$ be an arbitrary constant.*

1) *If $\mu_1(L_{g_\Sigma}) < 0$, then*

   1a) *either $S_{\overline{g}} \leq 0$ and then* (5.2) *has a globally defined smooth positive solution on M^n,*

   1b) *or $S_{\overline{g}} > 0$ and then* (5.2) *has no globally defined smooth positive solution on $M^n = I \times \Sigma$ if $I = \mathbb{R}$.*

2) *If $\mu_1(L_{g_\Sigma}) = 0$, then*

   2a) *either $S_{\overline{g}} < 0$ and then* (5.2) *has no globally defined smooth positive solution on $M^n = I \times \Sigma$ if $I = \mathbb{R}$.*

   2b) *or $S_{\overline{g}} = 0$ and then* (5.2) *has a globally defined smooth positive solution on M^n,*

   2c) *or $S_{\overline{g}} > 0$ and then* (5.2) *has no globally defined smooth positive solution on $M^n = I \times \Sigma$ if $I = \mathbb{R}$.*

3) *If $\mu_1(L_{g_\Sigma}) > 0$, then*

   3a) *either $S_{\overline{g}} < 0$ and then* (5.2) *has a globally defined smooth positive solution on $M^n = I \times \Sigma$ only if $|I| \leq \frac{\pi}{\sqrt{\mu_1(L_{g_\Sigma})}}$,*

   3b) *or $S_{\overline{g}} = 0$ and then* (5.2) *has a globally defined smooth positive solution on $M^n = I \times \Sigma$ if and only if $|I| \leq \frac{\pi}{\sqrt{\mu_1(L_{g_\Sigma})}}$,*

   3c) *or $S_{\overline{g}} > 0$ and then* (5.2) *has a globally defined smooth positive solution on M^n.*

Proof of Theorem 5.2.9: Note that the statements 1a) for the subcase $S_{\overline{g}} < 0$, 2b) and 3c) are already contained in Theorem 5.2.6 via Lemma 5.2.5 and after possibly rescaling the solution so as to adjust the constant on the r.h.s.
We show how to obtain in all cases a necessary condition for the existence of a global solution to (5.2). Given any constant $S_{\overline{g}} \in \mathbb{R}$, assume $\varphi \in C^\infty(M, \mathbb{R}_+^\times)$ is a solution to (5.2). Again, we may assume that $(M, g) = (I \times \Sigma, -dt^2 \oplus g_\Sigma)$. Let u be any positive (necessarily smooth) eigenfunction associated to the smallest eigenvalue $\mu_1(L_{g_\Sigma})$ of

L_{g_Σ}. Multiplying (5.2) with u and integrating w.r.t. the Riemannian measure $d\sigma$ associated to g_Σ on Σ, we obtain, using the formal self-adjointness of L_{g_Σ}:

$$\begin{aligned}
a_n S_{\overline{g}} \int_\Sigma \varphi^{p-1}(t,x)u(x)d\sigma(x) &\overset{(5.2)}{=} \int_\Sigma (\Box\varphi + a_n S_g \varphi)(t,x)u(x)d\sigma(x) \\
&\overset{(5.6)}{=} \int_\Sigma \left\{ \frac{\partial^2\varphi}{\partial t^2}(t,x)u(x) + (L_{g_\Sigma}\varphi)(t,x)u(x) \right\} d\sigma(x) \\
&= \frac{d^2}{dt^2}\left(\int_\Sigma \varphi(t,\cdot)ud\sigma\right) + \int_\Sigma \varphi(t,\cdot)L_{g_\Sigma}ud\sigma \\
&= \frac{d^2}{dt^2}\left(\int_\Sigma \varphi(t,\cdot)ud\sigma\right) + \mu_1(L_{g_\Sigma})\int_\Sigma \varphi(t,\cdot)ud\sigma,
\end{aligned}$$

where $p = \frac{2n}{n-2}$. As a consequence, the smooth positive function $y : I \to \mathbb{R}_+^\times$, $t \mapsto \int_\Sigma \varphi(t,\cdot)ud\sigma$, satisfies

$$y'' + \mu_1(L_{g_\Sigma})y = a_n S_{\overline{g}} \int_\Sigma \varphi^{p-1}(t,\cdot)ud\sigma \tag{5.8}$$

on I. An immediate consequence of this is that, if $S_{\overline{g}} = 0$, then the existence of a smooth positive solution to (5.2) is actually *equivalent* to that of a smooth positive solution to (5.8): it is necessary by the above argument and, conversely, if some $y \in C^\infty(I,\mathbb{R}_+^\times)$ solves (5.8), then Proposition 5.2.3 implies that, for any positive (smooth) eigenfunction u associated to the smallest eigenvalue $\mu_1(L_{g_\Sigma})$ of L_{g_Σ}, the function $\varphi(t,x) := y(t)\cdot u(x) > 0$ solves $\Box\varphi + a_n S_g \varphi = 0$ on M. Since obviously a positive smooth solution to the ODE (5.8) with $S_{\overline{g}} = 0$ exists for $\mu_1(L_{g_\Sigma}) \leq 0$, we obtain $1a)$ for the subcase $S_{\overline{g}} = 0$ (as well as $2b)$). For $\mu_1(L_{g_\Sigma}) > 0$, any solution to (5.8) with $S_{\overline{g}} = 0$ is of the form $t \mapsto A\cos(\sqrt{\mu_1(L_{g_\Sigma})}t + c)$, $A, c \in \mathbb{R}$, so that the existence of (at least) a positive solution (5.8) is equivalent to the length of I being no greater than the half of the period of $t \mapsto \cos(\sqrt{\mu_1(L_{g_\Sigma})}t)$, i.e., to $|I| \leq \frac{\pi}{\sqrt{\mu_1(L_{g_\Sigma})}}$. This proves $3b)$.
Assume now $S_{\overline{g}} < 0$ and $\mu_1(L_{g_\Sigma}) \geq 0$. If $\varphi \in C^\infty(M,\mathbb{R}_+^\times)$ solves (5.2), then by (5.8) the function y defined as above from φ satisfies $y'' + \mu_1(L_{g_\Sigma})y < 0$ on I. If $\mu_1(L_{g_\Sigma}) = 0$, then $y'' < 0$ on I, so that y is strictly concave and hence has to change sign if $I = \mathbb{R}$. This shows $2a)$. If $\mu_1(L_{g_\Sigma}) > 0$, then fix any $t_0 \in I$. By Lemma 5.2.8 the function y must satisfy $y \leq z$, where $z \in C^\infty(I,\mathbb{R})$ solves $z'' + \mu_1(L_{g_\Sigma})z = 0$ on I with $z(t_0) = y(t_0)$ as well as $z'(t_0) = y'(t_0)$. Since z – and hence also y – can remain positive only on an interval of length at most $\frac{\pi}{\sqrt{\mu_1(L_{g_\Sigma})}}$ (see just above), the length $|I|$ of I must satisfy $|I| \leq \frac{\pi}{\sqrt{\mu_1(L_{g_\Sigma})}}$. This shows $3a)$.
In the remaining case where $S_{\overline{g}} > 0$ and $\mu_1(L_{g_\Sigma}) \leq 0$, the identity (5.8) implies that, if $\varphi \in C^\infty(M,\mathbb{R}_+^\times)$ solves (5.2), then for any smooth positive $u \in \mathrm{Ker}(L_{g_\Sigma} - \mu_1(L_{g_\Sigma}))$, the smooth positive function $y(t) := \int_\Sigma \varphi(t,\cdot)ud\sigma$ satisfies

$$y'' \geq y'' + \mu_1(L_{g_\Sigma})y = a_n S_{\overline{g}} \int_\Sigma (\varphi(t,\cdot)u^{\frac{1}{p-1}})^{p-1}d\sigma$$

on I. But since u is continuous and positive on the compact space Σ, there is a positive constant C (depending on $p = \frac{2n}{n-2}$ and u) such that $u^{\frac{1}{p-1}} \geq Cu$, so that, by Hölder inequality,

$$y'' \geq a_n S_{\overline{g}} C^{p-1} \int_\Sigma (\varphi(t,\cdot)u)^{p-1}d\sigma \geq \frac{a_n S_{\overline{g}} C^{p-1}}{\mathrm{Vol}(\Sigma,g_\Sigma)^{p-2}} y^{p-1}$$

on I. This leads to an explosion of y in finite time and hence to a contradiction in case $I=\mathbb{R}$. Namely we may first assume, up to changing y into $t\mapsto y(\alpha t)$ for some $\alpha>0$, that

$$y''\geq\frac{p}{2}y^{p-1} \tag{5.9}$$

on $\mathbb{R}$. Since then y is strictly convex, only two (non disjoint) situations can occur: there is an interval of the form $[t_0,\infty[$ on which $y'\geq 0$ or there is an interval of the form $]-\infty,t_0]$ on which $y'\leq 0$. In the latter case, up to changing t into $-t$ – which does not modify (5.9) – we can again assume that $y'\geq 0$ on some interval of the form $[t_0,\infty[$. Up to translating by t_0, we can also assume that $t_0=0$. Since $y'\geq 0$ on $[0,\infty[$, the identity (5.9) yields $2y''y'\geq py^{p-1}y'$ on $[0,\infty[$, hence $(y')^2(t)-(y')^2(0)\geq y^p(t)-y^p(0)$ for any $t\geq 0$, in particular $y'\geq\sqrt{y^p-y^p(0)}$ on $[0,\infty[$. The latter inequality gives

$$\int_{y(0)}^{y(t)}\frac{dz}{\sqrt{z^p-y^p(0)}}\geq t$$

for any $t\geq 0$. Now since $p>2$ the integral $\int_{y(0)}^{\infty}\frac{dz}{\sqrt{z^p-y^p(0)}}$ converges, that is, the domain where $y(t)$ is defined is bounded above, or, equivalently, $y(t)\to\infty$ as $t\to T$ for some $T<\infty$. This shows $1b)$ and $2c)$ and concludes the proof of Theorem 5.2.9. $\square$

Note that, in the cases $1b)$, $2a)$, $2c)$ and $3a)$, local existence of solutions to (5.2) implies anyway the existence of a smooth positive solution φ to (5.2) on $I\times\Sigma$ for sufficiently short $|I|$. Even if it looks like it, global existence of solutions has nothing to do with timelike geodesic completeness of the product metric (which is anyway not a conformal invariant), see de Sitter spacetime below. For further ODE-like obstructions to the existence of particular metrics in (pseudo-)Riemannian conformal classes, we refer to [D20].

A first application of Theorem 5.2.9 is the following surprising example, where we see there exist spacetimes with *positive* scalar curvature admitting conformal metrics with *vanishing* scalar curvature – and this only in low dimensions.

Corollary 5.2.10 *Let a spacetime (M^n,g) be conformally equivalent to the warped product $(\mathbb{R}\times\Sigma^{n-1},-dt^2\oplus\cosh(t)^2g_\Sigma)$ of $\mathbb{R}$ with a closed Riemannian manifold (Σ^{n-1},g_Σ) of constant scalar curvature $(n-1)(n-2)$ and with warping function $b=\cosh$. Then there exists a conformal metric with vanishing scalar curvature on (M^n,g) if and only if $n\leq 4$.*

Proof: Note that, by (5.5), the scalar curvature of (M^n,g) is $S_g=n(n-1)>0$. We have already constructed an explicit isometry between $(M^n,b^{-2}g)$ (which is conformally equivalent to (M^n,g)) and $(]a_-,a_+[\times\Sigma^{n-1},-dt^2\oplus g_\Sigma)$, where $b(t):=\cosh(t)$ and $a_\pm:=\int_0^{\pm\infty}\frac{ds}{b(s)}$: set $\Phi(t,x):=(\psi(t),x)$ with $\psi(t):=\int_0^t\frac{ds}{b(s)}$. It is elementary to compute $\psi(t)=2\int_0^t\frac{e^{-s}ds}{1+e^{-2s}}=\frac{\pi}{2}-2\arctan(e^{-t})$, so that $a_\pm=\pm\frac{\pi}{2}$. Now since $S_{g_\Sigma}=(n-1)(n-2)$ is constant, $\mu_1(L_{g_\Sigma})=a_nS_{g_\Sigma}=\frac{(n-2)^2}{4}$, so that, by Theorem 5.2.9, there exists a positive solution to (5.2) with $S_{\overline{g}}=0$ if and only if $a_+-a_-\leq\frac{\pi}{\sqrt{\mu_1}}$, that is, if and only if $\pi\leq\frac{2\pi}{n-2}$, that is, if and only if $n\leq 4$. $\square$

For instance, if $(M^n,g):=(\mathbb{R}\times\mathbb{S}^{n-1},-dt^2\oplus\cosh(t)^2\mathrm{can})$ is the de Sitter spacetime of constant sectional curvature 1, where $(\mathbb{S}^{n-1},\mathrm{can})$ is the round sphere (of constant sectional curvature 1 if $n \geq 3$), then Corollary 5.2.10 shows that the existence of a conformal metric with vanishing scalar curvature is equivalent to $n \leq 4$.

Note 5.2.11 There is something deeply unsatisfying about Theorem 5.2.9: although the results we obtain are by nature conformally invariant, the assumptions we work with are not. For recall that we have first chosen a foliation by spacelike hypersurfaces – or, equivalently, a temporal function on the spacetime. Even more disturbing is the fact that the sign of the first eigenvalue of the Laplace-type operator L_{g_Σ} on each leaf Σ can change when fixing the foliation but changing the metric conformally on the spacetime. This remark is crucial when wanting to generalise the existence results to arbitrary globally hyperbolic spacetimes.

5.2.2 Uniqueness of solutions to the Yamabe problem

Next we turn to the uniqueness issue for the Yamabe problem. As we already noticed, given a globally hyperbolic spacetime M^n with closed spacelike Cauchy hypersurface Σ having future unit normal ν, the local well-posedness of the Cauchy problem $\Box\varphi + a_n S_g \varphi = a_n S_{\overline{g}} \varphi^{\frac{n+2}{n-2}}$ on M, $\varphi_{|_\Sigma} = \varphi_0$ and $\partial_\nu \varphi = \varphi_1$ on Σ, ensures - at least in a neighbourhood of Σ - the existence of infinitely many "independent" local solutions to the Yamabe problem. Therefore the only interesting question in this respect deals with the global aspects of uniqueness.

We start with looking at the ODE $y'' + a_n S_{g_\Sigma} y = a_n S_{\overline{g}} y^{p-1}$ from Proposition 5.2.3 on $I \subset \mathbb{R}$ and under the assumption that the scalar curvature S_{g_Σ} of (Σ, g_Σ) is constant. It is easy to see what happens for $S_{\overline{g}} = 0$: if $S_{g_\Sigma} < 0$, then there always exists a 2-parameter-family of positive solutions to $y'' + a_n S_{g_\Sigma} y = 0$ on I; if $S_{g_\Sigma} = 0$, then only constant solutions $y > 0$ to $y'' = 0$ can remain positive on $\mathbb{R}$; in case $S_{g_\Sigma} > 0$, there is no positive solution to $y'' + a_n S_{g_\Sigma} y = 0$ on $\mathbb{R}$ (but obviously there is one and even a 2-parameter-family of solutions on a sufficiently small interval). In the case $S_{\overline{g}} \neq 0$, we may assume, up to multiplying y by a positive constant, that $a_n S_{\overline{g}} = \varepsilon \frac{p}{2}$ with $\varepsilon \in \{\pm 1\}$.

Lemma 5.2.12 *Given $s \in \mathbb{R}_+^\times$, $p \in]2, \infty[$ and $\varepsilon \in \{\pm 1\}$, consider the ODE $y'' = \varepsilon(\frac{p}{2} y^{p-1} - sy)$ on some open interval $I \subset \mathbb{R}$.*

1) *If $\varepsilon = 1$, then the only positive solution to that ODE on $\mathbb{R}$ is the constant one $y = (\frac{2s}{p})^{\frac{1}{p-2}}$.*

2) *If $\varepsilon = -1$, then there are infinitely many non-constant positive solutions to that ODE on I. More precisely, for any $T \in]\frac{2\pi}{\sqrt{(p-2)s}}, \infty[$, there exists a T-periodic positive solution to $y'' = -\frac{p}{2} y^{p-1} + sy$ on $\mathbb{R}$.*

Proof: If y solves $y'' = \varepsilon(\frac{p}{2} y^{p-1} - sy)$, then multiplying with y' and integrating one obtains

$$(y')^2 = \varepsilon F(y) - \lambda$$

for some $\lambda \in \mathbb{R}$, where $F : \mathbb{R}_+ \to \mathbb{R}$, $F(y) := y^p - sy^2$. Therefore, we just have to investigate the qualitative behaviour of solutions to the first-order ODE $(y')^2 = \varepsilon F(y) - \lambda$ according to the value of λ. This equation can be solved in the form $t = t(y) = \pm \int^y \frac{dz}{\sqrt{\varepsilon F(z) - \lambda}}$ according to the sign of y' on the interval under consideration. Moreover, any solution to $(y')^2 = \varepsilon F(y) - \lambda$ which is not a critical point of F is a solution to the original equation $y'' = \varepsilon(\frac{p}{2}y^{p-1} - sy)$. Hence we first have to determine the regular and critical values of F. A short computation gives the two critical values 0 and $-\frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}$ for F, with corresponding critical points 0 and $(\frac{2s}{p})^{\frac{1}{p-2}}$ respectively. We start with the case $\varepsilon = 1$:

- Any $\lambda \in \mathbb{R}_+^\times$ is a regular value of F and $F^{-1}(\{\lambda\}) = \{x_\lambda\}$ with $x_\lambda \in]s^{\frac{1}{p-2}}, \infty[$. Because $p > 2$ we have $\int_{x_\lambda+1}^\infty \frac{dy}{\sqrt{F(y)-\lambda}} < \infty$, so that any solution y corresponding to $\lambda > 0$ explodes in finite time and therefore cannot exist on $\mathbb{R}$.
- For $\lambda = 0$, apart from the trivial solution $y = 0$ (we exclude anyway), the only solution shows exactly the same behaviour as before.
- For $\lambda \in]-\frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}, 0[$, the preimage $F^{-1}([\lambda, \infty[)$ consists of two intervals of the form $[0, x_\lambda^-]$ and $[x_\lambda^+, \infty[$ respectively, with $0 < x_\lambda^- < (\frac{2s}{p})^{\frac{1}{p-2}} < x_\lambda^+ < s^{\frac{1}{p-2}}$. Since λ is a regular value of F, the behaviour of the solution taking its values in $[x_\lambda^+, \infty[$ is the same as before (explosion in finite time); for $[0, x_\lambda^-]$ the solution vanishes in finite time because of $\int_0^{\frac{x_\lambda^-}{2}} \frac{dy}{\sqrt{F(y)-\lambda}} < \infty$. In both cases, y is not everywhere positive or is not defined on $\mathbb{R}$.
- For $\lambda = -\frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}$, apart from the constant solution $(\frac{2s}{p})^{\frac{1}{p-2}}$, we have two kinds of behaviour for y according to one value $y(t_0)$ of y lying in $]0, (\frac{2s}{p})^{\frac{1}{p-2}}[$ or in $](\frac{2s}{p})^{\frac{1}{p-2}}, \infty[$. If $y(t_0) \in](\frac{2s}{p})^{\frac{1}{p-2}}, \infty[$, then y explodes in finite time on one side and attains the critical point $(\frac{2s}{p})^{\frac{1}{p-2}}$ in infinite time on the other. If $y(t_0) \in]0, (\frac{2s}{p})^{\frac{1}{p-2}}[$, then y vanishes in finite time on the one side and attains the critical point $(\frac{2s}{p})^{\frac{1}{p-2}}$ in infinite time on the other. Again, no non-constant positive solution is defined on $\mathbb{R}$.
- For $\lambda \in]-\infty, -\frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}[$ the function y' cannot change sign; the solution y must vanish in finite time on the one side and explode in finite time on the other.

This shows 1). The case $\varepsilon = -1$ can also be divided in different subcases, compare [D28, pp. 132-135]:

- For $\lambda \in]\frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}, \infty[$, there is of course no solution to $(y')^2 = -F(y) - \lambda$.
- For $\lambda = \frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}$, the only solution to $(y')^2 = -F(y) - \lambda$ is the constant one $y = (\frac{2s}{p})^{\frac{1}{p-2}}$.
- For $\lambda \in]0, \frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}[$, the preimage $(-F)^{-1}([\lambda, \infty[) = [x_\lambda^-, x_\lambda^+]$, where $0 < x_\lambda^- < (\frac{2s}{p})^{\frac{1}{p-2}} < x_\lambda^+ < s^{\frac{1}{p-2}}$. This time, y is periodic (in particular defined on $\mathbb{R}$) and oscillates between the values x_λ^- and x_λ^+. Its period T_λ (depending on λ) is given by $T_\lambda = 2\int_{x_\lambda^-}^{x_\lambda^+} \frac{dy}{\sqrt{-F(y)-\lambda}}$, which can be easily seen to depend continuously on λ (since $x_\lambda^\pm$ do) with $T_\lambda \underset{\lambda\to 0^+}{\longrightarrow} \infty$ as well as $T_\lambda \underset{\lambda \to \frac{p-2}{p}(\frac{2s}{p})^{\frac{p}{p-2}}-}{\longrightarrow} \frac{2\pi}{\sqrt{(p-2)s}} > 0$, which is the period for the linearized equation $y'' = -(p-2)sy$ at $(\frac{2s}{p})^{\frac{1}{p-2}}$.

- For $\lambda = 0$, apart from the trivial solution $y = 0$, we obtain the solutions $t \mapsto s^{\frac{1}{p-2}} \cosh(\frac{2\sqrt{s}}{n-2}(t+c))^{-\frac{n-2}{2}}$, $c \in \mathbb{R}$, which are positive solutions defined on $\mathbb{R}$, symmetric about their maximum $t = -c$, with $s^{\frac{1}{p-2}}$ as maximum value, and which tend to 0 at infinity.

- For $\lambda \in \mathbb{R}_-^\times$, we obtain as above a solution which explodes on both sides in finite time.

This shows 2) and concludes the proof. □

Corollary 5.2.13 *Let a spacetime (M^n, g) be conformally equivalent to the product $(I \times \Sigma, -dt^2 \oplus g_\Sigma)$ of an open interval $I \subset \mathbb{R}$ with a closed Riemannian manifold (Σ, g_Σ) of constant negative scalar curvature. Then there exist infinitely many non-homothetic conformal metrics with constant negative scalar curvature on (M^n, g).*

Proof: Immediate consequence of Proposition 5.2.3 and Lemma 5.2.12. □

We turn to the subcritical equation $L_{g_\Sigma} u = \lambda u^{p-1}$ on Σ. First notice that, if $u, v \in C^\infty(\Sigma, \mathbb{R}_+^\times)$ solve $L_{g_\Sigma} u = \lambda u^{p-1}$ and $L_{g_\Sigma} v = \mu v^{p-1}$ on Σ respectively, for some $\lambda, \mu \in \mathbb{R}$, then λ and μ have the same sign ($\lambda\mu \geq 0$ and vanishes if and only if $\lambda = \mu = 0$): by formal self-adjointness of L_{g_Σ},

$$\lambda \int_\Sigma u^{p-1} v d\sigma = \int_\Sigma (L_{g_\Sigma} u) v d\sigma = \int_\Sigma u (L_{g_\Sigma} v) d\sigma = \mu \int_\Sigma u v^{p-1} d\sigma.$$

In particular, we only need consider uniqueness of solutions to $L_{g_\Sigma} u = \lambda u^{p-1}$ with constant λ of the same sign as $\mu_1(L_{g_\Sigma})$.

Theorem 5.2.14 *Let (Σ^{n-1}, g_Σ) be a closed connected Riemannian manifold, where $n \geq 3$. Let $\mu_1(L_{g_\Sigma}) \in \mathbb{R}$ be the smallest eigenvalue of L_{g_Σ} and $p := \frac{2n}{n-2}$.*

1) *If $\mu_1(L_{g_\Sigma}) < 0$, then for any $S_{\overline{g}} \in \mathbb{R}_-^\times$ the equation $L_{g_\Sigma} \varphi = a_n S_{\overline{g}} \varphi^{p-1}$ admits a unique smooth positive solution on Σ.*

2) *If $\mu_1(L_{g_\Sigma}) = 0$, then the equation $L_{g_\Sigma} \varphi = 0$ admits a unique smooth positive solution up to scale on Σ.*

3) *For any $\Lambda \in \mathbb{R}_+^\times$ the set*

$$S_\Lambda := \left\{ u \in C^\infty(\Sigma, \mathbb{R}_+) \,|\, L_{g_\Sigma} u = \lambda u^{p-1}, |\lambda| \leq \Lambda, \|u\|_{L^p(\Sigma)} \leq \Lambda \right\}$$

is compact in $C^2(\Sigma, \mathbb{R})$.

Proof: By Courant's nodal domain theorem, $\mathrm{Ker}(L_{g_\Sigma} - \mu_1(L_{g_\Sigma}))$ is a real line generated by a positive smooth function on Σ. This already implies 2). Statement 1) relies on the method of sub- and super-solutions developed in [D15, D16] and further in [D24, D23]. We briefly recall the concepts and statements we need for the proof. Given a C^1 function $f : \Sigma \times \mathbb{R} \to \mathbb{R}$, a strong *sub-* (resp. *super-*) *solution* for the equation $\Delta u = f(x, u)$ is a C^2-function v on Σ with $\Delta v \leq f(x, v)$ (resp. $\Delta v \geq f(x, v)$) on Σ. A weak *sub-* (resp. *super-*) *solution* for the equation $\Delta u = f(x, u)$ is a $v \in H^{1,2}(\Sigma) \cap C^0(\Sigma, \mathbb{R})$ satisfying $\int_\Sigma (g_\Sigma(dv, d\varphi) - f(x, v)\varphi)\, d\sigma \leq 0$ (resp. $\int_\Sigma (g_\Sigma(dv, d\varphi) - f(x, v)\varphi)\, d\sigma \geq 0$) for all $\varphi \in C^\infty(\Sigma, \mathbb{R}_+)$. Of course, every strong sub- or super-solution is a weak one. The steps in the proof of statement 1) are the following:

a) If $v_1, v_2 \in C^2(\Sigma,\mathbb{R})$ are strong super-solutions to $\Delta u = f(x,u)$, then $\min(v_1,v_2) \in H^{1,2}(\Sigma)\cap C^0(\Sigma,\mathbb{R})$ is a weak super-solution to the same equation [D23, Prop. 1].

b) Let $v_1, v_2 \in C^2(\Sigma,\mathbb{R})$ (resp. $v_- \in C^2(\Sigma,\mathbb{R})$) be strong super-solutions (resp. a strong sub-solution) to $\Delta u = f(x,u)$ with $v_- \le \min(v_1,v_2)$. Then there exists a strong solution $v \in C^2(\Sigma,\mathbb{R})$ to the same equation with $v_- \le v \le \min(v_1,v_2)$, compare e.g. [D13, Thm. 7.4.1] or [D15, Lemma 2.6] and references therein.

Now let $u_1, u_2 \in C^\infty(\Sigma,\mathbb{R}_+^\times)$ both solve $L_{g_\Sigma} u_i = \lambda u_i^{p-1}$ for some $\lambda \in \mathbb{R}_-^\times$. Up to multiplying u_1 and u_2 by a positive constant, we may assume that $\lambda = -1$. We construct suitable sub- and super-solutions for $L_{g_\Sigma} w = -w^{p-1}$ in order to be able to assume $u_1 \le u_2$, compare [D23, Lemma 1]. First, if $u \in \mathrm{Ker}(L_{g_\Sigma} - \mu_1(L_{g_\Sigma}))$ is positive, then there is a strong sub-solution to $L_{g_\Sigma} w = -w^{p-1}$ of the form $u_- := \alpha u$ with appropriate $\alpha \in \mathbb{R}_+^\times$: namely $L_{g_\Sigma} u_- \le -u_-^{p-1}$ if and only if $\alpha\mu_1(L_{g_\Sigma})u \le -\alpha^{p-1}u^{p-1}$, i.e., if and only if $\alpha \le \frac{1}{\max_\Sigma(u)}(-\mu_1(L_{g_\Sigma}))^{\frac{1}{p-2}}$ (recall that $\mu_1(L_{g_\Sigma}) < 0$), whose r.h.s. is positive since Σ is compact. Therefore $u_- = \alpha u$ is a strong sub-solution to $L_{g_\Sigma} w = -w^{p-1}$ for $\alpha > 0$ sufficiently small. Again, by compactness of Σ and continuity of u_1, u_2, one may choose $\alpha > 0$ small enough such that $u_- \le u_i$, $i = 1,2$. So we are in the situation where u_- is a strong sub-solution and u_1, u_2 are strong (super-)solutions to $L_{g_\Sigma} w = -w^{p-1}$ with $u_- \le \min(u_1,u_2)$. By b) just above, there exists a strong solution $v \in C^2(\Sigma,\mathbb{R})$ to $L_{g_\Sigma} w = -w^{p-1}$ with $u_- \le v \le \min(u_1,u_2)$, in particular $v > 0$ on Σ. Actually, classical elliptic regularity yields $v \in C^\infty(\Sigma,\mathbb{R}_+^\times)$. As a consequence, for both $i = 1,2$,

$$-\int_\Sigma u_i^{p-1} v\, d\sigma = \int_\Sigma (L_{g_\Sigma} u_i) v\, d\sigma = \int_\Sigma u_i (L_{g_\Sigma} v)\, d\sigma = -\int_\Sigma u_i v^{p-1} d\sigma,$$

so that $\int_\Sigma u_i v(u_i^{p-2} - v^{p-2})d\sigma = 0$. Because of $p-2 > 0$, we have $u_i^{p-2} - v^{p-2} \ge 0$ and therefore $u_i^{p-2} - v^{p-2} = 0$, that is, $u_i = v$ for $i = 1,2$, in particular $u_1 = u_2$. This proves statement 1).

The compactness of the set S_Λ relies mainly on the following so-called *regularity theorem* (actually needed for the proof of Theorem 5.2.4), see e.g. [D17, Thm. 4.1] or [D2, Satz 2.3.3]:

Let (Σ^{n-1}, g_Σ) be a closed Riemannian manifold with $n \ge 3$, $p \in [2,\infty[$, $h \in C^\infty(\Sigma,\mathbb{R})$ and $L := \Delta + h$. Then for any $\Lambda_1, \Lambda_2 \ge 0$ and $r \in]\frac{n-1}{2}(p-2),\infty[$, there exists a constant $C = C(\Sigma, g_\Sigma, \|h\|_{L^\infty(\Sigma)}, \Lambda_1, \Lambda_2, r) \ge 0$ and $\alpha = \alpha(r) \in]0,1[$ such that for all almost everywhere nonnegative $\varphi \in H^{1,2}(\Sigma)\cap L^r(\Sigma)$ solving (weakly) $L\varphi = \lambda\varphi^{p-1}$ with $|\lambda| \le \Lambda_1$ and $\|\varphi\|_{L^r(\Sigma)} \le \Lambda_2$, we have: $\varphi \in C^\infty(\Sigma,\mathbb{R})$, either $\varphi > 0$ or $\varphi = 0$ everywhere on Σ and $\|\varphi\|_{C^{2,\alpha}(\Sigma)} \le C$.

Fixing $r = p = \frac{2n}{n-2}$ and noticing that $p > \frac{n-1}{2}(p-2)$, the regularity theorem provides, for any $\Lambda \in]0,\infty[$, the existence of an $\alpha \in]0,1[$ and of a constant $C = C(\Sigma, g_\Sigma, \Lambda) > 0$ with $\|\varphi\|_{C^{2,\alpha}(\Sigma)} \le C$ for all $\varphi \in S_\Lambda$. With other words, S_Λ is included in the closed C-ball around the origin in $C^{2,\alpha}(\Sigma,\mathbb{R})$. But by Arzelà-Ascoli theorem, the inclusion $C^{2,\alpha}(\Sigma,\mathbb{R}) \hookrightarrow C^2(\Sigma,\mathbb{R})$ is *compact*, so that S_Λ is relatively compact in $C^2(\Sigma,\mathbb{R})$. Thus it remains to show that S_Λ is closed in $C^2(\Sigma,\mathbb{R})$. Consider the map

$$\Phi : C^2(\Sigma,\mathbb{R}_+) \times [-\Lambda,\Lambda] \to C^0(\Sigma,\mathbb{R}), \qquad (u,\lambda) \mapsto L_{g_\Sigma} u - \lambda u^{p-1}.$$

We show that Φ is continuous w.r.t. the standard topologies on both sides. Let $(u_k,\lambda_k)_{k\in\mathbb{N}}$ be a sequence of $C^2(\Sigma,\mathbb{R}_+) \times [-\Lambda,\Lambda]$ converging to some $(u,\lambda) \in$

$C^2(\Sigma,\mathbb{R}_+)\times[-\Lambda,\Lambda]$, i.e., $u_k \underset{k\to\infty}{\longrightarrow} u$ in $C^2(\Sigma)$ and $\lambda_k \underset{k\to\infty}{\longrightarrow} \lambda$ in $\mathbb{R}$. Then $\Delta u_k \underset{k\to\infty}{\longrightarrow} \Delta u$ in $C^0(\Sigma)$ and, because of $\|S_{g_\Sigma}\|_{C^0(\Sigma)} < \infty$, we have $L_{g_\Sigma}u_k \underset{k\to\infty}{\longrightarrow} L_{g_\Sigma}u$ in $C^0(\Sigma)$. Moreover, since $u_k \underset{k\to\infty}{\longrightarrow} u$ in $C^0(\Sigma)$, we can fix a small $\varepsilon>0$ and use $\sup\limits_{x\in[0,\|u\|_{C^0(\Sigma)}+\varepsilon]}(p-1)x^{p-2}<\infty$ to deduce that $\|u_k^{p-1}-u^{p-1}\|_{C^0(\Sigma)} \leq c\cdot\|u_k-u\|_{C^0(\Sigma)}$ for some constant $c>0$ (independent of k) and all sufficiently large $k\in\mathbb{N}$, in particular $\|u_k^{p-1}-u^{p-1}\|_{C^0(\Sigma)} \underset{k\to\infty}{\longrightarrow} 0$. Therefore, $L_{g_\Sigma}u_k-\lambda_k u_k^{p-1} \underset{k\to\infty}{\longrightarrow} L_{g_\Sigma}u-\lambda u^{p-1}$, i.e., $\Phi(u_k,\lambda_k) \underset{k\to\infty}{\longrightarrow} \Phi(u,\lambda)$ in $C^0(\Sigma)$. Hence Φ is continuous and thus $\Phi^{-1}(\{0\})$ is closed in $C^2(\Sigma,\mathbb{R}_+)\times[-\Lambda,\Lambda]$. But $[-\Lambda,\Lambda]$ being compact, the first projection $\mathrm{pr}_1(\Phi^{-1}(\{0\}))$ of $\Phi^{-1}(\{0\})$ is also closed in $C^2(\Sigma,\mathbb{R}_+)$. By restriction, $S_\Lambda = \mathrm{pr}_1(\Phi^{-1}(\{0\}))\cap\left\{\varphi\in C^2(\Sigma,\mathbb{R})\,|\,\|\varphi\|_{L^p(\Sigma)}\leq\Lambda\right\}$ is closed in $C^2(\Sigma,\mathbb{R}_+)\cap\left\{\varphi\in C^2(\Sigma,\mathbb{R})\,|\,\|\varphi\|_{L^p(\Sigma)}\leq\Lambda\right\}$. Now the set $C^2(\Sigma,\mathbb{R}_+)\cap\left\{\varphi\in C^2(\Sigma,\mathbb{R})\,|\,\|\varphi\|_{L^p(\Sigma)}\leq\Lambda\right\}$ is closed in $C^2(\Sigma,\mathbb{R})$: the subset $C^2(\Sigma,\mathbb{R}_+)$ is obviously closed in $C^2(\Sigma,\mathbb{R})$ and, if $u_k \underset{k\to\infty}{\longrightarrow} u$ in $C^2(\Sigma,\mathbb{R})$, then also in $C^0(\Sigma,\mathbb{R})$ and hence in $L^p(\Sigma)$, in particular $\|\cdot\|_{L^p(\Sigma)}: C^2(\Sigma,\mathbb{R})\to\mathbb{R}_+$ is continuous. On the whole, S_Λ is closed in $C^2(\Sigma,\mathbb{R})$ and therefore compact by the above argument. This shows statement 3) and concludes the proof of Theorem 5.2.14. $\square$

As for the Riemannian Yamabe problem, uniqueness need not hold in case $\mu_1(L_{g_\Sigma})>0$, as the following example shows, compare [D28, pp. 132-135].

Example 5.2.15 Let $\Sigma^{n-1} := \Sigma_1^{n-2}\times\mathbb{S}^1(L)$ be endowed with the product metric $g_\Sigma = g_1\oplus dt^2$, where (Σ_1^{n-2},g_1) is a closed Riemannian manifold of constant positive scalar curvature S_{g_1} and $\mathbb{S}^1(L)$ is the circle of length $L>0$. The subcritical equation $L_{g_\Sigma}\varphi = a_nS_{\overline{g}}\varphi^{p-1}$ with $S_{\overline{g}}\in\mathbb{R}_+^\times$ can be rewritten in the form

$$-\frac{\partial^2\varphi}{\partial t^2}+\Delta_{g_1}\varphi+a_nS_{g_1}\varphi = a_nS_{\overline{g}}\varphi^{p-1},$$

where $\Delta_{g_1}: C^\infty(\Sigma_1,\mathbb{R})\to C^\infty(\Sigma_1,\mathbb{R})$ is the scalar Laplace operator of (Σ_1,g_1). Looking for solutions of the form $\varphi = y\in C^\infty(\mathbb{S}^1,\mathbb{R}_+^\times)$, we have to find $\frac{L}{k}$-periodic solutions to the ODE $-y''+a_nS_{g_1}y = a_nS_{\overline{g}}y^{p-1}$ on $\mathbb{R}$, for any $k\in\mathbb{N}\setminus\{0\}$. Up to multiplying y with a positive constant, we may assume that $a_nS_{\overline{g}} = \frac{p}{2}$, so that the ODE becomes $y'' = sy-\frac{p}{2}y^{p-1}$, where $s := a_nS_{g_1}\in\mathbb{R}_+^\times$. Now Lemma 5.2.12 states that, for any $T\in\left]\frac{2\pi}{\sqrt{(p-2)s}},\infty\right[$, there exists a T-periodic (non-constant) positive solution to $y'' = sy-\frac{p}{2}y^{p-1}$. Hence, if $L\in\left]\frac{2\pi}{\sqrt{(p-2)s}},\infty\right[$, then there exists a non-constant L-periodic positive solution to that equation. More precisely, if $L\in\left]\frac{2k\pi}{\sqrt{(p-2)s}},\frac{2(k+1)\pi}{\sqrt{(p-2)s}}\right[$ for some $k\in\mathbb{N}\setminus\{0\}$, then there are positive solutions with periods $L,\frac{L}{2},\ldots,\frac{L}{k}$ respectively to that equation. In particular, the subcritical equation on $\Sigma_1^{n-2}\times\mathbb{S}^1(L)$ has more than one solution for $L>0$ sufficiently large. Combined with Proposition 5.2.3, this fact in turn implies the existence of non-homothetic conformal metrics with constant positive scalar curvature on any spacetime conformally equivalent to $(I\times\Sigma,-dt^2\oplus g_\Sigma)$ for Σ as above.

However, if the Ricci curvature of (Σ,g_Σ) is large enough, then uniqueness for the subcritical equation is satisfied.

Theorem 5.2.16 (M.-F. Bidaut-Véron & L. Véron [D8]) *Let (Σ^{n-1},g_Σ) be a closed Riemannian manifold with $n\geq 4$. Assume there exist $\lambda\in\mathbb{R}_+^\times$ and $q\in]2,\infty[$ such that* *i)* $\mathrm{ric}_{g_\Sigma} \geq \frac{n-2}{n-1}(q-2)\lambda \cdot g_\Sigma$ *and*

ii) $q \leq \frac{2(n-1)}{n-3}$

with strict inequality in $i)$ *or* $ii)$ *if* (Σ^{n-1}, g_Σ) *is conformally equivalent to* $(\mathbb{S}^{n-1}, \mathrm{can})$. *Then the only solution* $u > 0$ *to* $\Delta u + \lambda u = u^{q-1}$ *is the constant one* $u = \lambda^{\frac{1}{q-2}}$.

Examples 5.2.17

1. Let (Σ^{n-1}, g_Σ) be any $n-1(\geq 3)$-dimensional closed Riemannian manifold with constant positive scalar curvature S_{g_Σ} and $\mathrm{ric}_{g_\Sigma} \geq \frac{n-2}{n-1} \cdot \frac{S_{g_\Sigma}}{n-1} \cdot g_\Sigma$. For instance, any Einstein metric – or, more generally, any sufficiently small C^2-perturbation of an Einstein metric (think e.g. of small perturbations of the round metric on $\mathbb{S}^{2n+1}$ into Berger metrics) – with constant positive scalar curvature satisfies this condition. Then Theorem 5.2.16 with $\lambda = a_n S_{g_\Sigma}$ and $q = p = \frac{2n}{n-2} \in]2, \frac{2(n-1)}{n-3}[$ applies and yields in particular the uniqueness of solutions to the subcritical equation $L_{g_\Sigma} u = u^{p-1}$ on Σ.

2. Let $(\Sigma^{n-1}, g_\Sigma) := (\Sigma_1 \times \Sigma_2, g_1 \oplus g_2)$ with $n_1 + n_2 = n-1 \geq 4$ be the Riemannian product of two closed Einstein manifolds with constant positive scalar curvature S_{g_1} and S_{g_2} respectively. For $\lambda = a_n S_{g_\Sigma} = \frac{n-2}{4(n-1)}(S_{g_1} + S_{g_2}) \in \mathbb{R}_+^\times$ and $q = p = \frac{2n}{n-2} \in]2, \frac{2(n-1)}{n-3}[$, we have $\frac{n-2}{n-1}(q-2)\lambda = \frac{n-2}{(n-1)^2}(S_{g_1} + S_{g_2})$. Because of $\mathrm{ric}_{g_\Sigma} = \mathrm{ric}_{g_1} \oplus \mathrm{ric}_{g_2} = \frac{S_{g_1}}{n_1} g_1 \oplus \frac{S_{g_2}}{n_2} g_2$, a short computation shows that $\mathrm{ric}_{g_\Sigma} \geq \frac{n-2}{(n-1)^2}(S_{g_1} + S_{g_2}) \cdot g_\Sigma$ is equivalent to

$$\frac{n_2(n_1+n_2-1)}{n_1^2+n_1n_2+n_2} S_{g_1} \leq S_{g_2} \leq \frac{n_1+n_1n_2+n_2^2}{n_1(n_1+n_2-1)} S_{g_1}.$$

   In that case, Theorem 5.2.16 applies and yields the uniqueness of solutions to the subcritical equation $L_{g_\Sigma} u = u^{p-1}$ on Σ. Note that the inequality just above is in particular fulfilled if the Einstein condition $\frac{S_{g_1}}{n_1} = \frac{S_{g_2}}{n_2}$ is.

5.3 General case and outlook

In this section we come back to arbitrary globally hyperbolic spacetimes (M^n, g) with closed Cauchy hypersurface. We face several kinds of problems when looking for a smooth positive global solution to (5.2). First, we must show the existence of a solution – at least in the weak sense. We have seen that, for standard static spacetimes, we could always reduce the equation to a subcritical eigenvalue problem for the Laplace operator on a spacelike slice, whose solvability is well-known, at least in the compact setting. In general, it is possible to fix a spacelike Cauchy hypersurface in M^n and to try to solve the Cauchy problem associated to (5.2) with initial data along the hypersurface. For the case where $M = \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ with standard Minkowski metric, Konrad Jörgens could show [D14] (see also [D29, Thm. 6.5]) that, given any $p \in [2, 6[$ and any compactly supported smooth initial data on $\mathbb{R}^3 \simeq \{0\} \times \mathbb{R}^3$, there always exists a smooth solution to the Cauchy problem associated to the – slightly different – equation $\Box\varphi = -\varphi|\varphi|^{p-2}$. This works in particular for $p = \frac{2n}{n-2} = 4$.

Not much is known for arbitrary globally hyperbolic spacetimes, even with closed Cauchy hypersurface. The subcriticality of the exponent $p = \frac{2n}{n-2}$ for the embedding of the $H^{1,2}$-Sobolev space of the hypersurface is likely to provide at least weak solutions (in the distributional sense) to (5.2). The existence of those solutions is tightly connected to the choice of sign for the conformal scalar curvature: which kind of invariant could determine it? It is pointless to try to minimize the energy functional whose critical points are the solutions to the Yamabe problem, for that infimum can be shown to be minus infinity. The regularity of solutions is also an issue in itself, but the really delicate point – also related to the choice of conformal scalar curvature – consists in controlling their *sign*. For we have no maximum principle available to show that a given solution must be positive. In the particular case of standard static spacetimes, the integration of a given solution (possibly against a particular positive function) along the leaves of the standard foliation by Cauchy hypersurfaces leads to an ordinary differential equation or inequation, that straightforwardly provides obstructions for the existence of positive solutions: if the leafwise integral of a function is negative, then the function itself is negative somewhere.

In general, we cannot expect such an elementary obstruction to the existence of positive solutions, already because no separation of variables is possible. In fact, we first of all have to split the spacetime appropriately, or equivalently, choose a "good" temporal function. There is no canonical choice of temporal function on a given globally hyperbolic spacetime, though some choices are better adapted than other according to the question under consideration, see e.g. [D21, D19]. Besides fixing a temporal function, we also have to choose a background metric in the given conformal class. Both choices are intimately connected.

When focussing on the Yamabe equation (5.2), one could start with an arbitrary splitting $(M^n, g) = (\mathbb{R}\times\Sigma, -\beta dt^2 \oplus g_t)$ as in Theorem 5.1.2 and, up to changing the metric g conformally, assume that $\beta = 1$. The first and superficial reason for this is that it makes the expression of the d'Alembert operator $\Box$ relatively simple, see Lemma 5.1.3. But this is not necessarily the best choice, as we have already seen: for warped product spacetimes $(I\times\Sigma, -dt^2 \oplus b(t)^2 g_\Sigma)$, the choice $b(t)^{-2}g$ of conformal metric leads to the even simpler setting of standard static spacetimes, where the Yamabe problem can be completely solved. Still fixing the splitting $(M^n, g) = (\mathbb{R}\times\Sigma, -\beta dt^2 \oplus g_t)$, it is elementary to find a metric conformal to g such that all hypersurfaces $\{t\}\times\Sigma$ are maximal, i.e., $\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t}) = 0$ – in particular $\frac{\partial}{\partial t}(d\sigma_{g_t}) = 0$, which is the case for Lorentzian products; and a conformal metric such that $\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t}) = \frac{1}{\beta}\frac{\partial \beta}{\partial t}$, which makes the first-order-$\frac{\partial}{\partial t}$-term in $\Box_g$ vanish. Each of those choices presents technical advantages as well as drawbacks and we have for the moment no clue about which one could be "best" adapted to the Yamabe equation.

Note that one could also construct for each t a metric with constant scalar curvature in the conformal class of g_t on the Cauchy hypersurface Σ – which is possible by the existence of a solution to the Riemannian Yamabe problem. But this does not help much in our setting: even assuming the existence of a *smooth*[2] $f : I\times\Sigma \longrightarrow \mathbb{R}_+^\times$ such that $f(t,\cdot)^2 g_t = \breve{g}_0$ does not depend on t and has constant scalar curvature, a metric of

[2]The smooth dependence of f in $t \in I$ is already a very delicate question, at least in the case of positive Yamabe invariants on Σ, see e.g. [D11].

the form $-f^2dt^2 \oplus \breve{g}_0$ is in general not conformally equivalent to a (standard) static one – unless f is constant.

On the whole, the Lorentzian Yamabe problem remains widely open.

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Chapter 6

Classical and quantum fields on Lorentzian manifolds

This chapter coincides (up to minor changes such as enumeration of pages, sections, theorems, references etc.) with the published article [11].

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Abstract. We construct bosonic and fermionic locally covariant quantum field theories on curved backgrounds for large classes of fields. We investigate the quantum field and n-point functions induced by suitable states.

MSC classification: 58J45,35Lxx,81T20

Keywords: Wave operator, Dirac-type operator, globally hyperbolic spacetime, Green's operator, CCR-algebra, CAR-algebra, state, representation, locally covariant quantum field theory, quantum field, n-point function

6.1 Introduction

Classical fields on spacetime are mathematically modeled by sections of a vector bundle over a Lorentzian manifold. The field equations are usually partial differential equations. We introduce a class of differential operators, called Green-hyperbolic operators, which have good analytical solubility properties. This class includes wave operators as well as Dirac type operators.

In order to quantize such a classical field theory on a curved background, we need local algebras of observables. They come in two flavors, bosonic algebras encoding the canonical commutation relations and fermionic algebras encoding the canonical anti-commutation relations. We show how such algebras can be associated to manifolds equipped with suitable Green-hyperbolic operators. We prove that we obtain locally covariant quantum field theories in the sense of [E11]. There is a large literature where such constructions are carried out for particular examples of fields, see e.g. [E14, E17, E18, E20, E26, E38]. In all these papers the well-posedness of the Cauchy problem plays an important role. We avoid using the Cauchy problem altogether and only make use of Green's operators. In this respect, our approach is similar to the one in

[E39]. This allows us to deal with larger classes of fields, see Section 6.2.7, and to treat them systematically. Much of the earlier work on constructing observable algebras for particular examples can be subsumed under this general approach.
It turns out that bosonic algebras can be obtained in much more general situations than fermionic algebras. For instance, for the classical Dirac field both constructions are possible. Hence, on the level of observable algebras, there is no spin-statistics theorem. In order to obtain results like Theorem 5.1 in [E41] one needs more structure, namely representations of the observable algebras with good properties.
In order to produce numbers out of our quantum field theory that can be compared to experiments, we need states, in addition to observables. We show how states with suitable regularity properties give rise to quantum fields and n-point functions. We check that they have the properties expected from traditional quantum field theories on a Minkowski background.
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6.2 Field equations on Lorentzian manifolds

6.2.1 Globally hyperbolic manifolds

We begin by fixing notation and recalling general facts about Lorentzian manifolds, see e.g. [E30] or [E4] for more details. Unless mentioned otherwise, the pair (M,g) will stand for a smooth m-dimensional manifold M equipped with a smooth Lorentzian metric g, where our convention for Lorentzian signature is $(-+\cdots+)$. The associated volume element will be denoted by dV. We shall also assume our Lorentzian manifold (M,g) to be time-orientable, i.e., that there exists a smooth timelike vector field on M. Time-oriented Lorentzian manifolds will be also referred to as *spacetimes*. Note that in contrast to conventions found elsewhere, we do not assume that a spacetime is connected nor do we assume that its dimension be $m=4$.
For every subset A of a spacetime M we denote the causal future and past of A in M by $J_+(A)$ and $J_-(A)$, respectively. If we want to emphasize the ambient space M in which the causal future or past of A is considered, we write $J_\pm^M(A)$ instead of $J_\pm(A)$. Causal curves will always be implicitly assumed (future or past) oriented.

Definition 6.2.1 *A* Cauchy hypersurface *in a spacetime (M,g) is a subset of M which is met exactly once by every inextensible timelike curve.*

Cauchy hypersurfaces are always topological hypersurfaces but need not be smooth. All Cauchy hypersurfaces of a spacetime are homeomorphic.

Definition 6.2.2 *A spacetime (M,g) is called* globally hyperbolic *if and only if it contains a Cauchy hypersurface.*

A classical result of R. Geroch [E21] says that a globally hyperbolic spacetime can be foliated by Cauchy hypersurfaces. It is a rather recent and very important result that this also holds in the smooth category:

Theorem 6.2.3 (A. Bernal and M. Sánchez [E6, Thm. 1.1]) *Let (M,g) be a globally hyperbolic spacetime.*

Then there exists a smooth manifold Σ, a smooth one-parameter-family of Riemannian metrics $(g_t)_t$ on Σ and a smooth positive function β on $\mathbb{R}\times\Sigma$ such that (M,g) is isometric to $(\mathbb{R}\times\Sigma, -\beta dt^2 \oplus g_t)$. Each $\{t\}\times\Sigma$ corresponds to a smooth spacelike Cauchy hypersurface in (M,g).

For our purposes, we shall need a slightly stronger version of Theorem 6.2.3 where one of the Cauchy hypersurfaces $\{t\}\times\Sigma$ can be prescribed:

Theorem 6.2.4 (A. Bernal and M. Sánchez [E7, Thm. 1.2]) *Let (M,g) be a globally hyperbolic spacetime and $\tilde{\Sigma}$ a smooth spacelike Cauchy hypersurface in (M,g). Then there exists a smooth splitting $(M,g) \cong (\mathbb{R}\times\Sigma, -\beta dt^2 \oplus g_t)$ as in Theorem 6.2.3 such that $\tilde{\Sigma}$ corresponds to $\{0\}\times\Sigma$.*

We shall also need the following result which tells us that one can extend any compact acausal spacelike submanifold to a smooth spacelike Cauchy hypersurface. Here a subset of a spacetime is called *acausal* if no causal curve meets it more than once.

Theorem 6.2.5 (A. Bernal and M. Sánchez [E7, Thm. 1.1]) *Let (M,g) be a globally hyperbolic spacetime and let $K\subset M$ be a compact acausal smooth spacelike submanifold with boundary. Then there exists a smooth spacelike Cauchy hypersurface Σ in (M,g) with $K\subset\Sigma$.*

Definition 6.2.6 *A closed subset $A\subset M$ is called* spacelike compact *if there exists a compact subset $K\subset M$ such that $A\subset J^M(K) := J^M_-(K)\cup J^M_+(K)$.*

Note that a spacelike compact subset is in general not compact, but its intersection with any Cauchy hypersurface is compact, see e.g. [E4, Cor. A.5.4].

Definition 6.2.7 *A subset Ω of a spacetime M is called* causally compatible *if and only if $J^\Omega_\pm(x) = J^M_\pm(x)\cap\Omega$ for every $x\in\Omega$.*

This means that every causal curve joining two points in Ω must be contained entirely in Ω.

6.2.2 Differential operators and Green's functions

A *differential operator* of order (at most) k on a vector bundle $S\to M$ over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ is a linear map $P: C^\infty(M,S)\to C^\infty(M,S)$ which in local coordinates $x=(x^1,\dots,x^m)$ of M and with respect to a local trivialization looks like

$$P = \sum_{|\alpha|\le k} A_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}.$$

Here $C^\infty(M,S)$ denotes the space of smooth sections of $S\to M$, $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}_0\times\cdots\times\mathbb{N}_0$ runs over multi-indices, $|\alpha|=\alpha_1+\ldots+\alpha_m$ and $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial(x^1)^{\alpha_1}\cdots\partial(x^m)^{\alpha_m}}$. The *principal symbol* σ_P of P associates to each covector $\xi\in T^*_xM$ a linear map $\sigma_P(\xi): S_x\to S_x$. Locally, it is given by

$$\sigma_P(\xi) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}$ and $\xi = \sum_j \xi_j dx^j$. If P and Q are two differential operators of order k and ℓ respectively, then $Q \circ P$ is a differential operator of order $k+\ell$ and

$$\sigma_{Q\circ P}(\xi) = \sigma_Q(\xi) \circ \sigma_P(\xi).$$

For any linear differential operator $P : C^\infty(M,S) \to C^\infty(M,S)$ there is a unique formally dual operator $P^* : C^\infty(M,S^*) \to C^\infty(M,S^*)$ of the same order characterized by

$$\int_M \langle \phi, P\psi \rangle \, \mathrm{dV} = \int_M \langle P^*\phi, \psi \rangle \, \mathrm{dV}$$

for all $\psi \in C^\infty(M,S)$ and $\phi \in C^\infty(M,S^*)$ with $\mathrm{supp}(\phi) \cap \mathrm{supp}(\psi)$ compact. Here $\langle \cdot, \cdot \rangle : S^* \otimes S \to \mathbb{K}$ denotes the canonical pairing, i.e., the evaluation of a linear form in S_x^* on an element of S_x, where $x \in M$. We have $\sigma_{P^*}(\xi) = (-1)^k \sigma_P(\xi)^*$ where k is the order of P.

Definition 6.2.8 *Let a vector bundle $S \to M$ be endowed with a non-degenerate inner product $\langle \cdot, \cdot \rangle$. A linear differential operator P on S is called* formally self-adjoint *if and only if*

$$\int_M \langle P\phi, \psi \rangle \, \mathrm{dV} = \int_M \langle \phi, P\psi \rangle \, \mathrm{dV}$$

holds for all $\phi, \psi \in C^\infty(M,S)$ with $\mathrm{supp}(\phi) \cap \mathrm{supp}(\psi)$ *compact.*
Similarly, we call P formally skew-adjoint *if instead*

$$\int_M \langle P\phi, \psi \rangle \, \mathrm{dV} = -\int_M \langle \phi, P\psi \rangle \, \mathrm{dV}.$$

We recall the definition of advanced and retarded Green's operators for a linear differential operator.

Definition 6.2.9 *Let P be a linear differential operator acting on the sections of a vector bundle S over a Lorentzian manifold M. An* advanced Green's operator *for P on M is a linear map*

$$G_+ : C_c^\infty(M,S) \to C^\infty(M,S)$$

satisfying:

(G_1) $P \circ G_+ = \mathrm{id}_{C_c^\infty(M,S)}$;

(G_2) $G_+ \circ P|_{C_c^\infty(M,S)} = \mathrm{id}_{C_c^\infty(M,S)}$;

(G_3^+) $\mathrm{supp}(G_+\phi) \subset J_+^M(\mathrm{supp}(\phi))$ *for any $\phi \in C_c^\infty(M,S)$.*

A retarded Green's operator *for P on M is a linear map $G_- : C_c^\infty(M,S) \to C^\infty(M,S)$ satisfying (G_1), (G_2), and*

(G_3^-) $\mathrm{supp}(G_-\phi) \subset J_-^M(\mathrm{supp}(\phi))$ *for any $\phi \in C_c^\infty(M,S)$.*

Here we denote by $C_c^\infty(M,S)$ the space of compactly supported smooth sections of S.

Definition 6.2.10 *Let $P : C^\infty(M,S) \to C^\infty(M,S)$ be a linear differential operator. We call P* Green-hyperbolic *if the restriction of P to any globally hyperbolic subregion of M has advanced and retarded Green's operators.*

Note 6.2.11 If the Green's operators of the restriction of P to a globally hyperbolic subregion exist, then they are necessarily unique, see Remark 6.3.7.

6.2.3 Wave operators

The most prominent class of Green-hyperbolic operators are wave operators, sometimes also called normally hyperbolic operators.

Definition 6.2.12 *A linear differential operator of second order $P : C^\infty(M,S) \to C^\infty(M,S)$ is called a* wave operator *if its principal symbol is given by the Lorentzian metric, i.e., for all $\xi \in T^*M$ we have*

$$\sigma_P(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id}.$$

In other words, if we choose local coordinates $x^1, \ldots, x^m$ on M and a local trivialization of S, then

$$P = -\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{m} A_j(x) \frac{\partial}{\partial x^j} + B(x)$$

where A_j and B are matrix-valued coefficients depending smoothly on x and (g^{ij}) is the inverse matrix of (g_{ij}) with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$. If P is a wave operator, then so is its dual operator P^*. In [E4, Cor. 3.4.3] it has been shown that wave operators are Green-hyperbolic.

Example 6.2.13 (d'Alembert operator) Let S be the trivial line bundle so that sections of S are just functions. The d'Alembert operator $P = \Box = -\mathrm{div} \circ \mathrm{grad}$ is a formally self-adjoint wave operator, see e.g. [E4, p. 26].

Example 6.2.14 (connection-d'Alembert operator) More generally, let S be a vector bundle and let ∇ be a connection on S. This connection and the Levi-Civita connection on T^*M induce a connection on $T^*M \otimes S$, again denoted ∇. We define the connection-d'Alembert operator $\Box^\nabla$ to be the composition of the following three maps

$$C^\infty(M,S) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes S) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes T^*M \otimes S) \xrightarrow{-\mathrm{tr}\otimes \mathrm{id}_S} C^\infty(M,S)$$

where $\mathrm{tr} : T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\mathrm{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$. We compute the principal symbol,

$$\sigma_{\Box^\nabla}(\xi)\phi = -(\mathrm{tr} \otimes \mathrm{id}_S) \circ \sigma_\nabla(\xi) \circ \sigma_\nabla(\xi)(\phi) = -(\mathrm{tr} \otimes \mathrm{id}_S)(\xi \otimes \xi \otimes \phi) = -\langle \xi, \xi \rangle \, \phi.$$

Hence $\Box^\nabla$ is a wave operator.

Example 6.2.15 (Hodge-d'Alembert operator) Let $S = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ increases the degree by one while the codifferential $\delta = d^* : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ decreases the degree by one. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = -d\delta - \delta d$ is a formally self-adjoint wave operator.

6.2.4 The Proca equation

The Proca operator is an example of a Green-hyperbolic operator of second order which is not a wave operator. First we need the following observation:

Lemma 6.2.16 *Let M be globally hyperbolic, let $S \to M$ be a vector bundle and let P and Q be differential operators acting on sections of S. Suppose P has advanced and retarded Green's operators G_+ and G_-.*
If Q commutes with P, then it also commutes with G_+ and with G_-.

Proof: Assume $[P,Q]=0$. We consider

$$\tilde{G}_\pm := G_\pm + [G_\pm, Q] : \, C_c^\infty(M,s) \to C_{sc}^\infty(M,S).$$

We compute on $C_c^\infty(M,S)$:

$$\tilde{G}_\pm P = G_\pm P + G_\pm QP - QG_\pm P = \mathrm{id} + G_\pm PQ - Q = \mathrm{id} + Q - Q = \mathrm{id}$$

and similarly $P\tilde{G}_\pm = \mathrm{id}$. Hence $\tilde{G}_\pm$ are also advanced and retarded Green's operators, respectively. By Remark 6.2.11, Green's operators are unique, hence $\tilde{G}_\pm = G_\pm$ and therefore $[G_\pm, Q] = 0$. □

Example 6.2.17 (Proca operator) The discussion of this example follows [E39, p. 116f], see also [E20] where is the discussion is based on the Cauchy problem. The Proca equation describes massive vector bosons. We take $S = T^*M$ and let $m_0 > 0$. The Proca equation is

$$P\phi := \delta d\phi + m_0^2\phi = 0 \tag{6.1}$$

where $\phi \in C^\infty(M,S)$. Applying δ to (6.1) we obtain, using $\delta^2 = 0$ and $m_0 \neq 0$,

$$\delta\phi = 0 \tag{6.2}$$

and hence

$$(d\delta + \delta d)\phi + m_0^2\phi = 0. \tag{6.3}$$

Conversely, (6.2) and (6.3) clearly imply (6.1).
Since $\tilde{P} := d\delta + \delta d + m_0^2$ is minus a wave operator, it has Green's operators $\tilde{G}_\pm$. We define

$$G_\pm : C_c^\infty(M,S) \to C_{sc}^\infty(M,S), \quad G_\pm := (m_0^{-2} d\delta + \mathrm{id}) \circ \tilde{G}_\pm = \tilde{G}_\pm \circ (m_0^{-2} d\delta + \mathrm{id}).$$

The last equality holds because d and δ commute with $\tilde{P}$. For $\phi \in C_c^\infty(M,S)$ we compute

$$G_\pm P\phi = \tilde{G}_\pm (m_0^{-2} d\delta + \mathrm{id})(\delta d + m_0^2)\phi = \tilde{G}_\pm \tilde{P}\phi = \phi$$

and similarly $PG_\pm\phi = \phi$. Since the differential operator $m_0^{-2} d\delta + \mathrm{id}$ does not increase supports, the third axiom in the definition of advanced and retarded Green's operators holds as well.
This shows that G_+ and G_- are advanced and retarded Green's operators for P, respectively. Thus P is not a wave operator but Green-hyperbolic.

6.2.5 Dirac type operators

The most important Green-hyperbolic operators of first order are the so-called Dirac type operators.

Definition 6.2.18 *A linear differential operator $D : C^\infty(M,S) \to C^\infty(M,S)$ of first order is called* of Dirac type*, if $-D^2$ is a wave operator.*

Note 6.2.19 If D is of Dirac type, then i times its principal symbol satisfies the Clifford relations

$$(i\sigma_D(\xi))^2 = -\sigma_{D^2}(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id},$$

hence by polarization

$$(i\sigma_D(\xi))(i\sigma_D(\eta)) + (i\sigma_D(\eta))(i\sigma_D(\xi)) = -2\langle \xi, \eta \rangle \cdot \mathrm{id}.$$

The bundle S thus becomes a module over the bundle of Clifford algebras $\mathrm{Cl}(TM)$ associated with $(TM, \langle \cdot, \cdot \rangle)$. See [E5, Sec. 1.1] or [E27, Ch. I] for the definition and properties of the Clifford algebra $\mathrm{Cl}(V)$ associated with a vector space V with inner product.

Note 6.2.20 If D is of Dirac type, then so is its dual operator D^*. On a globally hyperbolic region let G_+ be the advanced Green's operator for D^2 which exists since $-D^2$ is a wave operator. Then it is not hard to check that $D \circ G_+$ is an advanced Green's operator for D, see e.g. the proof of Theorem 2.3 in [E14] or [E29, Thm. 3.2]. The same discussion applies to the retarded Green's operator. Hence any Dirac type operator is Green-hyperbolic.

Example 6.2.21 (Classical Dirac operator) If the spacetime M carries a spin structure, then one can define the spinor bundle $S = \Sigma M$ and the classical Dirac operator

$$D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M), \quad D\phi := i \sum_{j=1}^{m} \varepsilon_j e_j \cdot \nabla_{e_j} \phi.$$

Here $(e_j)_{1 \le j \le m}$ is a local orthonormal basis of the tangent bundle, $\varepsilon_j = \langle e_j, e_j \rangle = \pm 1$ and "$\cdot$" denotes the Clifford multiplication, see e.g. [E5] or [E3, Sec. 2]. The principal symbol of D is given by

$$\sigma_D(\xi)\psi = i\xi^\sharp \cdot \psi.$$

Here $\xi^\sharp$ denotes the tangent vector dual to the 1-form ξ via the Lorentzian metric, i.e., $\langle \xi^\sharp, Y \rangle = \xi(Y)$ for all tangent vectors Y over the same point of the manifold. Hence

$$\sigma_{D^2}(\xi)\psi = \sigma_D(\xi)\sigma_D(\xi)\psi = -\xi^\sharp \cdot \xi^\sharp \cdot \psi = \langle \xi, \xi \rangle \, \psi.$$

Thus $P = -D^2$ is a wave operator. Moreover, D is formally self-adjoint, see e.g. [E3, p. 552].

Example 6.2.22 (Twisted Dirac operators) More generally, let $E \to M$ be a complex vector bundle equipped with a non-degenerate Hermitian inner product and a metric connection ∇^E over a spin spacetime M. In the notation of Example 6.2.21, one may define the Dirac operator of M twisted with E by

$$D^E := i \sum_{j=1}^{m} \varepsilon_j e_j \cdot \nabla^{\Sigma M \otimes E}_{e_j} : C^\infty(M, \Sigma M \otimes E) \to C^\infty(M, \Sigma M \otimes E),$$

where $\nabla^{\Sigma M \otimes E}$ is the tensor product connection on $\Sigma M \otimes E$. Again, D^E is a formally self-adjoint Dirac type operator.

Example 6.2.23 (Euler operator) In Example 6.2.15, replacing $\Lambda^k T^*M$ by $S := \Lambda T^*M \otimes \mathbb{C} = \oplus_{k=0}^{n} \Lambda^k T^*M \otimes \mathbb{C}$, the Euler operator $D = i(d - \delta)$ defines a formally self-adjoint Dirac type operator. In case M is spin, the Euler operator coincides with the Dirac operator of M twisted with ΣM if m is even and with $\Sigma M \oplus \Sigma M$ if m is odd.

Example 6.2.24 (Buchdahl operators) On a 4-dimensional spin spacetime M, consider the standard orthogonal and parallel splitting $\Sigma M = \Sigma_+ M \oplus \Sigma_- M$ of the complex spinor bundle of M into spinors of positive and negative chirality. The finite dimensional irreducible representations of the simply-connected Lie group $\mathrm{Spin}^0(3,1)$ are given by $\Sigma_+^{(k/2)} \otimes \Sigma_-^{(\ell/2)}$ where $k, \ell \in \mathbb{N}$. Here $\Sigma_+^{(k/2)} = \Sigma_+^{\odot k}$ is the k-th symmetric tensor product of the positive half-spinor representation Σ_+ and similarly for $\Sigma_-^{(\ell/2)}$. Let the associated vector bundles $\Sigma_\pm^{(k/2)} M$ carry the induced inner product and connection.
For $s \in \mathbb{N}$, $s \geq 1$, consider the twisted Dirac operator $D^{(s)}$ acting on sections of $\Sigma M \otimes \Sigma_+^{((s-1)/2)} M$. In the induced splitting

$$\Sigma M \otimes \Sigma_+^{((s-1)/2)} M = \Sigma_+ M \otimes \Sigma_+^{(s-1/2)} M \oplus \Sigma_- M \otimes \Sigma_+^{((s-1)/2)} M$$

the operator $D^{(s)}$ is of the form

$$\begin{pmatrix} 0 & D_-^{(s)} \\ D_+^{(s)} & 0 \end{pmatrix}$$

because Clifford multiplication by vectors exchanges the chiralities. The Clebsch-Gordan formulas [E10, Prop. II.5.5] tell us that the representation $\Sigma_+ \otimes \Sigma_+^{(\frac{s-1}{2})}$ splits as

$$\Sigma_+ \otimes \Sigma_+^{(\frac{s-1}{2})} = \Sigma_+^{(\frac{s}{2})} \oplus \Sigma_+^{(\frac{s}{2}-1)}.$$

Hence we have the corresponding parallel orthogonal projections

$$\pi_s : \Sigma_+ M \otimes \Sigma_+^{(\frac{s-1}{2})} M \to \Sigma_+^{(\frac{s}{2})} M \quad \text{and} \quad \pi_s' : \Sigma_+ M \otimes \Sigma_+^{(\frac{s-1}{2})} M \to \Sigma_+^{(\frac{s}{2}-1)} M.$$

On the other hand, the representation $\Sigma_- \otimes \Sigma_+^{(\frac{s-1}{2})}$ is irreducible. Now *Buchdahl operators* are the operators of the form

$$B^{(s)}_{\mu_1,\mu_2,\mu_3} := \begin{pmatrix} \mu_1 \cdot \pi_s + \mu_2 \cdot \pi_s' & D_-^{(s)} \\ D_+^{(s)} & \mu_3 \cdot \mathrm{id} \end{pmatrix}$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$ are constants. By definition, $B^{(s)}_{\mu_1,\mu_2,\mu_3}$ is of the form $D^{(s)} + b$, where b is of order zero. In particular, $B^{(s)}_{\mu_1,\mu_2,\mu_3}$ is a Dirac-type operator, hence it is Green-hyperbolic.
If M were Riemannian, then $D^{(s)}$ would be formally self-adjoint. Hence the operator $B^{(s)}_{\mu_1,\mu_2,\mu_3}$ would be formally self-adjoint if and only if the constants μ_1, μ_2, μ_3 are real. In Lorentzian signature, $\Sigma_+ M$ and $\Sigma_- M$ are isotropic for the natural inner product on ΣM, so that the bundles on which the Buchdahl operators act, carry no natural non-degenerate inner product.
For a definition of Buchdahl operators using indices we refer to [E12, E13, E44] and to [E28, Def. 8.1.4, p. 104].

6.2.6 The Rarita-Schwinger operator

For the Rarita-Schwinger operator on Riemannian manifolds, we refer to [E43, Sec. 2], see also [E8, Sec. 2]. In this section let the spacetime M be spin and consider the Clifford-multiplication $\gamma : T^*M \otimes \Sigma M \to \Sigma M$, $\theta \otimes \psi \mapsto \theta^\sharp \cdot \psi$, where ΣM is the complex spinor bundle of M. Then there is the representation theoretic splitting of $T^*M\otimes\Sigma M$ into the orthogonal and parallel sum

$$T^*M\otimes\Sigma M=\iota(\Sigma M)\oplus\Sigma^{3/2}M,$$

where $\Sigma^{3/2}M:=\ker(\gamma)$ and $\iota(\psi):=-\frac{1}{m}\sum_{j=1}^m e_j^*\otimes e_j\cdot\psi$. Here again $(e_j)_{1\le j\le m}$ is a local orthonormal basis of the tangent bundle. Let $\mathscr{D}$ be the twisted Dirac operator on $T^*M\otimes\Sigma M$, that is, $\mathscr{D}:=i\cdot(\mathrm{id}\otimes\gamma)\circ\nabla$, where ∇ denotes the induced covariant derivative on $T^*M\otimes\Sigma M$.

Definition 6.2.25 *The* Rarita-Schwinger operator *on the spin spacetime M is defined by $\mathscr{Q}:=(\mathrm{id}-\iota\circ\gamma)\circ\mathscr{D}:C^\infty(M,\Sigma^{3/2}M)\to C^\infty(M,\Sigma^{3/2}M)$.*

By definition, the Rarita-Schwinger operator is pointwise obtained as the orthogonal projection onto $\Sigma^{3/2}M$ of the twisted Dirac operator $\mathscr{D}$ restricted to a section of $\Sigma^{3/2}M$. Using the above formula for ι, the Rarita-Schwinger operator can be written down explicitly:

$$\mathscr{Q}\psi=i\cdot\sum_{\beta=1}^m e_\beta^*\otimes\sum_{\alpha=1}^m\varepsilon_\alpha(e_\alpha\cdot\nabla_{e_\alpha}\phi_\beta-\frac{2}{m}e_\beta\cdot\nabla_{e_\alpha}\phi_\alpha)$$

for all $\psi=\sum_{\beta=1}^m e_\beta^*\otimes\psi_\beta\in C^\infty(M,\Sigma^{3/2}M)$, where here ∇ is the standard connection on ΣM. It can be checked that $\mathscr{Q}$ is a formally self-adjoint linear differential operator of first order, with principal symbol

$$\sigma_{\mathscr{Q}}(\xi):\psi\mapsto i\Big\{(\mathrm{id}\otimes\xi^\sharp\cdot)\psi-\frac{2}{m}\sum_{\beta=1}^m e_\beta^*\otimes e_\beta\cdot(\xi^\sharp\lrcorner\psi)\Big\},$$

for all $\psi=\sum_{\beta=1}^m e_\beta^*\otimes\psi_\beta\in\Sigma^{3/2}M$. Here $X\lrcorner\psi$ denotes the insertion of the tangent vector X in the first factor, that is, $X\lrcorner\psi:=\sum_{\beta=1}^m e_\beta^*(X)\psi_\beta$.

Lemma 6.2.26 *Let M be a spin spacetime of dimension $m\ge 3$. Then the characteristic variety of the Rarita-Schwinger operator of M coincides with the set of lightlike covectors.*

Proof: By definition, the characteristic variety of $\mathscr{Q}$ is the set of nonzero covectors ξ for which $\sigma_{\mathscr{Q}}(\xi)$ is not invertible. Fix an arbitrary point $x\in M$. Let $\xi\in T_x^*M\setminus\{0\}$ be non-lightlike. Without loss of generality we may assume that ξ is normalized and that the Lorentz orthonormal basis is chosen so that $\xi^\sharp=e_1$. Hence $\varepsilon_1=1$ if ξ is spacelike and $\varepsilon_1=-1$ if ξ is timelike. Take $\psi=\sum_{\beta=1}^m e_\beta^*\otimes\psi_\beta\in\ker(\sigma_{\mathscr{Q}}(\xi))$. Then

$$\begin{aligned}0&=\sum_{\beta=1}^m e_\beta^*\otimes e_1\cdot\psi_\beta-\frac{2}{m}\sum_{\beta=1}^m e_\beta^*\otimes e_\beta\cdot\psi_1\\&=\sum_{\beta=1}^m e_\beta^*\otimes(e_1\cdot\psi_\beta-\frac{2}{m}e_\beta\cdot\psi_1),\end{aligned}$$

which implies $e_1\cdot\psi_\beta=\frac{2}{m}e_\beta\cdot\psi_1$ for all $\beta\in\{1,\dots,m\}$. Choosing $\beta=1$, we obtain $e_1\cdot\psi_1=0$ because $m\ge 3$. Hence $\psi_1=0$, from which $\psi_\beta=0$ follows for all $\beta\in\{1,\dots,m\}$. Hence $\psi=0$ and $\sigma_{\mathscr{Q}}(\xi)$ is invertible.

If $\xi \in T_x^*M \setminus \{0\}$ is lightlike, then we may assume that $\xi^\sharp = e_1 + e_2$, where $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$. Choose $\psi_1 \in \Sigma_x M \setminus \{0\}$ with $(e_1+e_2)\cdot\psi_1 = 0$. Such a ψ_1 exists because Clifford multiplication by a lightlike vector is nilpotent. Set $\psi_2 := -\psi_1$ and $\psi := e_1^* \otimes \psi_1 + e_2^* \otimes \psi_2$. Then $\psi \in \Sigma_x^{3/2} M \setminus \{0\}$ and

$$-i\sigma_{\mathscr{Q}}(\xi)(\psi) = \sum_{j=1}^{2} e_j^* \otimes \underbrace{(e_1+e_2)\cdot\psi_j}_{=0} - \frac{2}{m} e_j^* \otimes e_j \cdot (\underbrace{\psi_1+\psi_2}_{=0}) = 0.$$

This shows $\psi \in \ker(\sigma_{\mathscr{Q}}(\xi))$ and hence $\sigma_{\mathscr{Q}}(\xi)$ is not invertible. □

The same proof shows that in the Riemannian case the Rarita-Schwinger operator is elliptic.

Note 6.2.27 Since the characteristic variety of the Rarita-Schwinger operator is exactly that of the Dirac operator, Lemma 6.2.26 together with [E24, Thms. 23.2.4 & 23.2.7] imply that the Cauchy problem for $\mathscr{Q}$ is well-posed in case M is globally hyperbolic. This implies they $\mathscr{Q}$ has advanced and retarded Green's operators. Hence $\mathscr{Q}$ is not of Dirac type but it is Green-hyperbolic.

Note 6.2.28 The equations originally considered by Rarita and Schwinger in [E33] correspond to the twisted Dirac operator $\mathscr{D}$ restricted to $\Sigma^{3/2}M$ but not projected back to $\Sigma^{3/2}M$. In other words, they considered the operator

$$\mathscr{D}|_{C^\infty(M,\Sigma^{3/2}M)} : C^\infty(M,\Sigma^{3/2}M) \to C^\infty(M, T^*M \otimes \Sigma M).$$

These equations are over-determined. Therefore it is not a surprise that non-trivial solutions restrict the geometry of the underlying manifold as observed by Gibbons [E22] and that this operator has no Green's operators.

6.2.7 Combining given operators into a new one

Given two Green-hyperbolic operators we can form the direct sum and obtain a new operator in a trivial fashion. It turns out that this operator is again Green-hyperbolic. Note that the two operators need not have the same order.

Lemma 6.2.29 *Let $S_1, S_2 \to M$ be two vector bundles over the globally hyperbolic manifold M. Let P_1 and P_2 be two Green-hyperbolic operators acting on sections of S_1 and S_2 respectively. Then*

$$P_1 \oplus P_2 := \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} : C^\infty(M, S_1 \oplus S_2) \to C^\infty(M, S_1 \oplus S_2)$$

is Green-hyperbolic.

Proof: If G_1 and G_2 are advanced Green's operators for P_1 and P_2 respectively, then clearly $\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$ is an advanced Green's operator for $P_1 \oplus P_2$. The retarded case is analogous. □

It is interesting to note that P_1 and P_2 need not have the same order. Hence Green-hyperbolic operators need not be hyperbolic in the usual sense. Moreover, it is not obvious that Green-hyperbolic operators have a well-posed Cauchy problem. For instance, if P_1 is a wave operator and P_2 a Dirac-type operator, then along a Cauchy hypersurface one would have to prescribe the normal derivative for the S_1-component but not for the S_2-component.

6.3 Algebras of observables

Our next aim is to quantize the classical fields governed by Green-hyperbolic differential operators. We construct local algebras of observables and we prove that we obtain locally covariant quantum field theories in the sense of [E11].

6.3.1 Bosonic quantization

In this section we show how a quantization process based on canonical commutation relations (CCR) can be carried out for formally self-adjoint Green-hyperbolic operators. This is a functorial procedure. We define the first category involved in the quantization process.

Definition 6.3.1 *The category* GlobHypGreen *consists of the following objects and morphisms:*

- *An object in* GlobHypGreen *is a triple* (M,S,P)*, where*
  - ► M *is a globally hyperbolic spacetime,*
  - ► S *is a real vector bundle over* M *endowed with a non-degenerate inner product* $\langle\cdot,\cdot\rangle$ *and*
  - ► P *is a formally self-adjoint Green-hyperbolic operator acting on sections of* S*.*
- *A morphism between two objects* (M_1,S_1,P_1) *and* (M_2,S_2,P_2) *of* GlobHypGreen *is a pair* (f,F)*, where*
  - ► f *is a time-orientation preserving isometric embedding* $M_1 \to M_2$ *with* $f(M_1)$ *causally compatible and open in* M_2*,*
  - ► F *is a fiberwise isometric vector bundle isomorphism over* f *such that the following diagram commutes:*

$$\begin{tikzcd} C^\infty(M_2,S_2) \arrow[r, "P_2"] \arrow[d, "\mathrm{res}"'] & C^\infty(M_2,S_2) \arrow[d, "\mathrm{res}"'] \\ C^\infty(M_1,S_1) \arrow[r, "P_1"] & C^\infty(M_1,S_1), \end{tikzcd} \tag{6.4}$$

  where $\mathrm{res}(\phi) := F^{-1}\circ\phi\circ f$ *for every* $\phi\in C^\infty(M_2,S_2)$*.*

Note that morphisms exist only if the manifolds have equal dimension and the vector bundles have the same rank. Note furthermore, that the inner product $\langle\cdot,\cdot\rangle$ on S is not required to be positive or negative definite.
The causal compatibility condition, which is not automatically satisfied (see e.g. [E4, Fig. 33]), ensures the commutation of the extension and restriction maps with the Green's operators:

Lemma 6.3.2 *Let (f,F) be a morphism between two objects (M_1,S_1,P_1) and (M_2,S_2,P_2) in the category* GlobHypGreen *and let $(G_1)_\pm$ and $(G_2)_\pm$ be the respective Green's operators for P_1 and P_2. Denote by $\mathrm{ext}(\phi) \in C^\infty_c(M_2,S_2)$ the extension by 0 of $F\circ\phi\circ f^{-1} : f(M_1) \to S_2$ to M_2, for every $\phi \in C^\infty_c(M_1,S_1)$. Then*

$$\mathrm{res}\circ(G_2)_\pm\circ\mathrm{ext} = (G_1)_\pm .$$

Proof: Set $\widetilde{(G_1)}_\pm := \mathrm{res}\circ(G_2)_\pm\circ\mathrm{ext}$ and fix $\phi\in C^\infty_c(M_1,S_1)$. First observe that the causal compatibility condition on f implies that

$$\begin{aligned}
\mathrm{supp}(\widetilde{(G_1)}_\pm(\phi)) &= f^{-1}(\mathrm{supp}((G_2)_\pm\circ\mathrm{ext}(\phi)))\\
&\subset f^{-1}(J^{M_2}_\pm(\mathrm{supp}(\mathrm{ext}(\phi))))\\
&= f^{-1}(J^{M_2}_\pm(f(\mathrm{supp}(\phi))))\\
&= J^{M_1}_\pm(\mathrm{supp}(\phi)).
\end{aligned}$$

In particular, $\widetilde{(G_1)}_\pm(\phi)$ has spacelike compact support in M_1 and $\widetilde{(G_1)}_\pm$ satisfies Axiom (G_3). Moreover, it follows from (6.4) that $P_2\circ\mathrm{ext} = \mathrm{ext}\circ P_1$ on $C^\infty_c(M_1,S_1)$, which directly implies that $\widetilde{(G_1)}_\pm$ satisfies Axioms (G_1) and (G_2) as well. The uniqueness of the advanced and retarded Green's operators on M_1 yields $\widetilde{(G_1)}_\pm = (G_1)_\pm$. □

Next we show how the Green's operators for a formally self-adjoint Green-hyperbolic operator provide a symplectic vector space in a canonical way. First we see how the Green's operators of an operator and of its formally dual operator are related.

Lemma 6.3.3 *Let M be a globally hyperbolic spacetime and G_+, G_- the advanced and retarded Green's operators for a Green-hyperbolic operator P acting on sections of $S\to M$. Then the advanced and retarded Green's operators G^*_+ and G^*_- for P^* satisfy*

$$\int_M \langle G^*_\pm\phi,\psi\rangle\,\mathrm{dV} = \int_M \langle\phi, G_\mp\psi\rangle\,\mathrm{dV}$$

for all $\phi\in C^\infty_c(M,S^*)$ and $\psi\in C^\infty_c(M,S)$.

Proof: Axiom (G_1) for the Green's operators implies that

$$\begin{aligned}
\int_M \langle G^*_\pm\phi,\psi\rangle\,\mathrm{dV} &= \int_M \langle G^*_\pm\phi, P(G_\mp\psi)\rangle\,\mathrm{dV}\\
&= \int_M \langle P^*(G^*_\pm\phi), G_\mp\psi\rangle\,\mathrm{dV}\\
&= \int_M \langle\phi, G_\mp\psi\rangle\,\mathrm{dV},
\end{aligned}$$

where the integration by parts is justified since $\mathrm{supp}(G^*_\pm\phi)\cap\mathrm{supp}(G_\mp\psi) \subset J^M_\pm(\mathrm{supp}(\phi))\cap J^M_\mp(\mathrm{supp}(\psi))$ is compact. □

Proposition 6.3.4 *Let (M,S,P) be an object in the category* GlobHypGreen*. Set $G := G_+ - G_-$, where G_+, G_- are the advanced and retarded Green's operator for P, respectively.*
Then the pair $(\mathrm{SYMPL}(M,S,P),\omega)$ is a symplectic vector space, where

$$\mathrm{SYMPL}(M,S,P) := C^\infty_c(M,S)/\ker(G) \quad \text{and} \quad \omega([\phi],[\psi]) := \int_M \langle G\phi,\psi\rangle\,\mathrm{dV}.$$

Here the square brackets $[\cdot]$ denote residue classes modulo $\ker(G)$.

Proof: The bilinear form $(\phi,\psi) \mapsto \int_M \langle G\phi,\psi\rangle\,\mathrm{dV}$ on $C^\infty_\mathrm{c}(M,S)$ is skew-symmetric as a consequence of Lemma 6.3.3 because P is formally self-adjoint. Its null-space is exactly $\ker(G)$. Therefore the induced bilinear form ω on the quotient space $\mathrm{SYMPL}(M,S,P)$ is non-degenerate and hence a symplectic form. □

Put $C^\infty_\mathrm{sc}(M,S) := \{\phi \in C^\infty(M,S) \,|\, \mathrm{supp}(\phi) \text{ is spacelike compact}\}$. The next result will in particular show that we can consider $\mathrm{SYMPL}(M,S,P)$ as the space of smooth solutions of the equation $P\phi = 0$ which have spacelike compact support.

Theorem 6.3.5 *Let M be a Lorentzian manifold, let $S \to M$ be a vector bundle, and let P be a Green-hyperbolic operator acting on sections of S. Let $G_\pm$ be advanced and retarded Green's operators for P, respectively. Put*

$$G := G_+ - G_- : C^\infty_\mathrm{c}(M,S) \to C^\infty_\mathrm{sc}(M,S).$$

Then the following linear maps form a complex:

$$\{0\} \to C^\infty_\mathrm{c}(M,S) \xrightarrow{P} C^\infty_\mathrm{c}(M,S) \xrightarrow{G} C^\infty_\mathrm{sc}(M,S) \xrightarrow{P} C^\infty_\mathrm{sc}(M,S). \tag{6.5}$$

This complex is always exact at the first $C^\infty_\mathrm{c}(M,S)$. If M is globally hyperbolic, then the complex is exact everywhere.

Proof: The proof follows the lines of [E4, Thm. 3.4.7] where the result was shown for wave operators. First note that, by $(\mathrm{G}_3^\pm)$ in the definition of Green's operators, we have that $G_\pm : C^\infty_\mathrm{c}(M,S) \to C^\infty_\mathrm{sc}(M,S)$. It is clear from (G_1) and (G_2) that $PG = GP = 0$ on $C^\infty_\mathrm{c}(M,S)$, hence (6.5) is a complex.
If $\phi \in C^\infty_\mathrm{c}(M,S)$ satisfies $P\phi = 0$, then by (G_2) we have $\phi = G_+P\phi = 0$ which shows that $P|_{C^\infty_\mathrm{c}(M,S)}$ is injective. Thus the complex is exact at the first $C^\infty_\mathrm{c}(M,S)$.
From now on let M be globally hyperbolic. Let $\phi \in C^\infty_\mathrm{c}(M,S)$ with $G\phi = 0$, i.e., $G_+\phi = G_-\phi$. We put $\psi := G_+\phi = G_-\phi \in C^\infty(M,S)$ and we see that $\mathrm{supp}(\psi) = \mathrm{supp}(G_+\phi) \cap \mathrm{supp}(G_-\phi) \subset J_+(\mathrm{supp}(\phi)) \cap J_-(\mathrm{supp}(\phi))$. Since (M,g) is globally hyperbolic $J_+(\mathrm{supp}(\phi)) \cap J_-(\mathrm{supp}(\phi))$ is compact, hence $\psi \in C^\infty_\mathrm{c}(M,S)$. From $P\psi = PG_+\phi = \phi$ we see that $\phi \in P(C^\infty_\mathrm{c}(M,S))$. This shows exactness at the second $C^\infty_\mathrm{c}(M,S)$.
It remains to show that any $\phi \in C^\infty_\mathrm{sc}(M,S)$ with $P\phi = 0$ is of the form $\phi = G\psi$ with $\psi \in C^\infty_\mathrm{c}(M,S)$. Using a cut-off function decompose ϕ as $\phi = \phi_+ - \phi_-$ where $\mathrm{supp}(\phi_\pm) \subset J_\pm(K)$ where K is a suitable compact subset of M. Then $\psi := P\phi_+ = P\phi_-$ satisfies $\mathrm{supp}(\psi) \subset J_+(K) \cap J_-(K)$. Thus $\psi \in C^\infty_\mathrm{c}(M,S)$. We check that $G_+\psi = \phi_+$. Namely, for all $\chi \in C^\infty_\mathrm{c}(M,S^*)$ we have by Lemma 6.3.3

$$\int_M \langle \chi, G_+P\phi_+\rangle\,\mathrm{dV} = \int_M \langle G_-^*\chi, P\phi_+\rangle\,\mathrm{dV} = \int_M \langle P^*G_-^*\chi, \phi_+\rangle\,\mathrm{dV} = \int_M \langle \chi, \phi_+\rangle\,\mathrm{dV}.$$

The integration by parts in the second equality is justified because $\mathrm{supp}(\phi_+) \cap \mathrm{supp}(G_-^*\chi) \subset J_+(K) \cap J_-(\mathrm{supp}(\chi))$ is compact. Similarly, one shows $G_-\psi = \phi_-$. Now $G\psi = G_+\psi - G_-\psi = \phi_+ - \phi_- = \phi$ which concludes the proof. □

In particular, given an object (M,S,P) in $\mathsf{GlobHypGreen}$, the map G induces an isomorphism from

$$\mathrm{SYMPL}(M,S,P) = C^\infty_\mathrm{c}(M,S)/\ker(G) \xrightarrow{\cong} \ker(P) \cap C^\infty_\mathrm{sc}(M,S).$$

Note 6.3.6 Exactness at the first $C_c^\infty(M,S)$ in sequence (6.5) says that there are no non-trivial smooth solutions of $P\phi = 0$ with compact support. Indeed, if M is globally hyperbolic, more is true.
If $\phi \in C^\infty(M,S)$ solves $P\phi = 0$ and $\mathrm{supp}(\phi)$ is future or past-compact, then $\phi = 0$.
Here a subset $A \subset M$ is called future-compact if $A \cap J_+(x)$ is compact for any $x \in M$. Past-compactness is defined similarly.
Proof: Let $\phi \in C^\infty(M,S)$ solve $P\phi = 0$ such that $\mathrm{supp}(\phi)$ is future-compact. For any $\chi \in C_c^\infty(M,S^*)$ we have

$$\int_M \langle \chi, \phi \rangle \,\mathrm{dV} = \int_M \langle P^* G_+^* \chi, \phi \rangle \,\mathrm{dV} = \int_M \langle G_+^* \chi, P\phi \rangle \,\mathrm{dV} = 0.$$

This shows $\phi = 0$. The integration by parts is justified because $\mathrm{supp}(G_+^*\chi) \cap \mathrm{supp}(\phi) \subset J_+(\mathrm{supp}(\chi)) \cap \mathrm{supp}(\phi)$ is compact, see [E4, Lemma A.5.3]. □

Note 6.3.7 Let M be a globally hyperbolic spacetime and (M,S,P) an object in GlobHypGreen. *Then the Green's operators G_+ and G_- are unique.* Namely, if G_+ and $\tilde{G}_+$ are advanced Green's operators for P, then for any $\phi \in C_c^\infty(M,S)$ the section $\psi := G_+\phi - \tilde{G}_+\phi$ has past-compact support and satisfies $P\psi = 0$. By the previous remark, we have $\psi = 0$ which shows $G_+ = \tilde{G}_+$.

Now, let (f,F) be a morphism between two objects (M_1,S_1,P_1) and (M_2,S_2,P_2) in the category GlobHypGreen. For $\phi \in C_c^\infty(M_1,S_1)$ consider the extension by zero $\mathrm{ext}(\phi) \in C_c^\infty(M_2,S_2)$ as in Lemma 6.3.2.

Lemma 6.3.8 *Given a morphism (f,F) between two objects (M_1,S_1,P_1) and (M_2,S_2,P_2) in the category* GlobHypGreen*, extension by zero induces a symplectic linear map* $\mathrm{SYMPL}(f,F) : \mathrm{SYMPL}(M_1,S_1,P_1) \to \mathrm{SYMPL}(M_2,S_2,P_2)$.
Moreover,

$$\mathrm{SYMPL}(\mathrm{id}_M, \mathrm{id}_S) = \mathrm{id}_{\mathrm{SYMPL}(M,S,P)} \tag{6.6}$$

and for any further morphism $(f',F') : (M_2,S_2,P_2) \to (M_3,S_3,P_3)$ one has

$$\mathrm{SYMPL}((f',F') \circ (f,F)) = \mathrm{SYMPL}(f',F') \circ \mathrm{SYMPL}(f,F). \tag{6.7}$$

Proof: If $\phi = P_1\psi \in \ker(G_1) = P_1(C_c^\infty(M_1,S_1))$, then $\mathrm{ext}(\phi) = P_2(\mathrm{ext}(\psi)) \in P_2(C_c^\infty(M_2,S_2)) = \ker(G_2)$. Hence ext induces a linear map

$$\mathrm{SYMPL}(f,F) : C_c^\infty(M_1,S_1)/\ker(G_1) \to C_c^\infty(M_2,S_2)/\ker(G_2).$$

Furthermore, applying Lemma 6.3.2, we have, for any $\phi, \psi \in C_c^\infty(M_1,S_1)$

$$\int_{M_2} \langle G_2(\mathrm{ext}(\phi)), \mathrm{ext}(\psi) \rangle \,\mathrm{dV} = \int_{M_1} \langle \mathrm{res} \circ G_2 \circ \mathrm{ext}(\phi), \psi \rangle \,\mathrm{dV} = \int_{M_1} \langle G_1\phi, \psi \rangle \,\mathrm{dV},$$

hence $\mathrm{SYMPL}(f,F)$ is symplectic. Equation (6.6) is trivial and extending once or twice by 0 amounts to the same, so (6.7) holds as well. □

Note 6.3.9 Under the isomorphism $\mathrm{SYMPL}(M,S,P) \to \ker(P) \cap C_{sc}^\infty(M,S)$ induced by G, the extension by zero corresponds to an extension as a smooth solution of $P\phi = 0$ with spacelike compact support. This follows directly from Lemma 6.3.2. In other words, for any morphism (f,F) from (M_1,S_1,P_1) to (M_2,S_2,P_2) in $\mathsf{GlobHypGreen}$ we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{SYMPL}(M_1,S_1,P_1) \arrow[r, "\mathrm{SYMPL}(f,F)"] \arrow[d, "\cong"'] & \mathrm{SYMPL}(M_2,S_2,P_2) \arrow[d, "\cong"] \\
\ker(P_1)\cap C^\infty_{\mathrm{sc}}(M_1,S_1) \arrow[r, "\text{extension as}", "\text{a solution}"'] & \ker(P_2)\cap C^\infty_{\mathrm{sc}}(M_2,S_2).
\end{tikzcd}$$

Let Sympl denote the category of real symplectic vector spaces with symplectic linear maps as morphisms. Lemma 6.3.8 says that we have constructed a covariant functor

$$\mathrm{SYMPL} : \mathsf{GlobHypGreen} \longrightarrow \mathsf{Sympl}.$$

In order to obtain an algebra-valued functor, we compose SYMPL with the functor CCR which associates to any symplectic vector space its Weyl algebra. Here "CCR" stands for "canonical commutation relations". This is a general algebraic construction which is independent of the context of Green-hyperbolic operators and which is carried out in Section 6.5.2. As a result, we obtain the functor

$$\mathfrak{A}_{\mathrm{bos}} := \mathrm{CCR}\circ\mathrm{SYMPL} : \mathsf{GlobHypGreen} \longrightarrow \mathsf{C^*Alg},$$

where $\mathsf{C^*Alg}$ is the category whose objects are the unital C*-algebras and whose morphisms are the injective unit-preserving C*-morphisms.
In the remainder of this section we show that the functor $\mathrm{CCR}\circ\mathrm{SYMPL}$ is a bosonic locally covariant quantum field theory. We call two subregions M_1 and M_2 of a spacetime M *causally disjoint* if and only if $J^M(M_1)\cap M_2 = \emptyset$. In other words, there are no causal curves joining M_1 and M_2.

Theorem 6.3.10 *The functor* $\mathfrak{A}_{\mathrm{bos}} : \mathsf{GlobHypGreen} \longrightarrow \mathsf{C^*Alg}$ *is a bosonic locally covariant quantum field theory, i.e., the following axioms hold:*

(i) ***(Quantum causality)*** *Let (M_j,S_j,P_j) be objects in* $\mathsf{GlobHypGreen}$, $j=1,2,3$, *and (f_j,F_j) morphisms from (M_j,S_j,P_j) to (M_3,S_3,P_3), $j=1,2$, such that $f_1(M_1)$ and $f_2(M_2)$ are causally disjoint regions in M_3.*

Then the subalgebras $\mathfrak{A}_{\mathrm{bos}}(f_1,F_1)(\mathfrak{A}_{\mathrm{bos}}(M_1,S_1,P_1))$ and $\mathfrak{A}_{\mathrm{bos}}(f_2,F_2)(\mathfrak{A}_{\mathrm{bos}}(M_2,S_2,P_2))$ of $\mathfrak{A}_{\mathrm{bos}}(M_3,S_3,P_3)$ commute.

(ii) ***(Time slice axiom)*** *Let (M_j,S_j,P_j) be objects in* $\mathsf{GlobHypGreen}$, $j=1,2$, *and (f,F) a morphism from (M_1,S_1,P_1) to (M_2,S_2,P_2) such that there is a Cauchy hypersurface $\Sigma\subset M_1$ for which $f(\Sigma)$ is a Cauchy hypersurface of M_2. Then*

$$\mathfrak{A}_{\mathrm{bos}}(f,F) : \mathfrak{A}_{\mathrm{bos}}(M_1,S_1,P_1) \to \mathfrak{A}_{\mathrm{bos}}(M_2,S_2,P_2)$$

is an isomorphism.

Proof: We first show (i). For notational simplicity we assume without loss of generality that f_j and F_j are inclusions, $j=1,2$. Let $\phi_j\in C^\infty_{\mathrm{c}}(M_j,S_j)$. Since M_1 and M_2 are causally disjoint, the sections $G\phi_1$ and ϕ_2 have disjoint support, thus

$$\omega([\phi_1],[\phi_2]) = \int_M \langle G\phi_1,\phi_2\rangle\,\mathrm{dV} = 0.$$

Now relation (iv) in Definition 6.5.11 tells us

$$w([\phi_1]) \cdot w([\phi_2]) = w([\phi_1] + [\phi_2]) = w([\phi_2]) \cdot w([\phi_1]).$$

Since $\mathfrak{A}_{\mathrm{bos}}(f_1, F_1)(\mathfrak{A}_{\mathrm{bos}}(M_1, S_1, P_1))$ is generated by elements of the form $w([\phi_1])$ and $\mathfrak{A}_{\mathrm{bos}}(f_2, F_2)(\mathfrak{A}_{\mathrm{bos}}(M_2, S_2, P_2))$ by elements of the form $w([\phi_2])$, the assertion follows.
In order to prove (ii) we show that $\mathrm{SYMPL}(f, F)$ is an isomorphism of symplectic vector spaces provided f maps a Cauchy hypersurface of M_1 onto a Cauchy hypersurface of M_2. Since symplectic linear maps are always injective, we only need to show surjectivity of $\mathrm{SYMPL}(f, F)$. This is most easily seen by replacing $\mathrm{SYMPL}(M_j, S_j, P_j)$ by $\ker(P_j) \cap C^\infty_{\mathrm{sc}}(M_j, S_j)$ as in Remark 6.3.9. Again we assume without loss of generality that f and F are inclusions.
Let $\psi \in C^\infty_{\mathrm{sc}}(M_2, S_2)$ be a solution of $P_2\psi = 0$. Let ϕ be the restriction of ψ to M_1. Then ϕ solves $P_1\phi = 0$ and has spacelike compact support in M_1 by Lemma 6.3.11 below. We will show that there is only one solution in M_2 with spacelike compact support extending ϕ. It will then follow that ψ is the image of ϕ under the extension map corresponding to $\mathrm{SYMPL}(f, F)$ and surjectivity will be shown.
To prove uniqueness of the extension, we may, by linearity, assume that $\phi = 0$. Then ψ_+ defined by

$$\psi_+(x) := \begin{cases} \psi(x), & \text{if } x \in J_+^{M_2}(\Sigma), \\ 0, & \text{otherwise}, \end{cases}$$

is smooth since ψ vanishes in an open neighborhood of Σ. Now ψ_+ solves $P_2\psi_+ = 0$ and has past-compact support. By Remark 6.3.6, $\psi_+ \equiv 0$, i.e., ψ vanishes on $J_+^{M_2}(\Sigma)$. One shows similarly that ψ vanishes on $J_-^{M_2}(\Sigma)$, hence $\psi = 0$. $\square$

Lemma 6.3.11 *Let M be a globally hyperbolic spacetime and let $M' \subset M$ be a causally compatible open subset which contains a Cauchy hypersurface of M. Let $A \subset M$ be spacelike compact in M.*
Then $A \cap M'$ is spacelike compact in M'.

Proof: Fix a common Cauchy hypersurface Σ of M' and M. By assumption, there exists a compact subset $K \subset M$ with $A \subset J^M(K)$. Then $K' := J^M(K) \cap \Sigma$ is compact [E4, Cor. A.5.4] and contained in M'.
Moreover $A \subset J^M(K')$: let $p \in A$ and let γ be a causal curve (in M) from p to some $k \in K$. Then γ can be extended to an inextensible causal curve in M, which hence meets Σ at some point q. Because of $q \in \Sigma \cap J^M(k) \subset K'$ one has $p \in J^M(K')$.
Therefore $A \cap M' \subset J^M(K') \cap M' = J^{M'}(K')$ because of the causal compatibility of M' in M. The lemma is proved. $\square$

The quantization process described in this subsection applies in particular to formally self-adjoint wave and Dirac-type operators.

6.3.2 Fermionic quantization

Next we construct a fermionic quantization. For this we need a functorial construction of Hilbert spaces rather than symplectic vector spaces. As we shall see this seems to be possible only under much more restrictive assumptions. The underlying Lorentzian manifold M is assumed to be a globally hyperbolic spacetime as before. The vector bundle S is assumed to be complex with Hermitian inner product $\langle \cdot , \cdot \rangle$ which may be indefinite. The formally self-adjoint Green-hyperbolic operator P is assumed to be of first order.

Definition 6.3.12 *A formally self-adjoint Green-hyperbolic operator P of first order acting on sections of a complex vector bundle S over a spacetime M is of* definite type *if and only if for any $x \in M$ and any future-directed timelike tangent vector $\mathfrak{n} \in T_xM$, the bilinear map*

$$S_x \times S_x \to \mathbb{C}, \qquad (\phi, \psi) \mapsto \langle i\sigma_P(\mathfrak{n}^\flat) \cdot \phi, \psi \rangle,$$

yields a positive definite Hermitian scalar product on S_x.

Example 6.3.13 The classical Dirac operator P from Example 6.2.21 is, when defined with the correct sign, of definite type, see e.g. [E5, Sec. 1.1.5] or [E3, Sec. 2].

Example 6.3.14 If $E \to M$ is a semi-Riemannian or -Hermitian vector bundle endowed with a metric connection over a spin spacetime M, then the twisted Dirac operator from Example 6.2.22 is of definite type if and only if the metric on E is positive definite. This can be seen by evaluating the tensorized inner product on elements of the form $\sigma \otimes v$, where $v \in E_x$ is null.

Example 6.3.15 The operator $P = i(d - \delta)$ on $S = \Lambda T^*M \otimes \mathbb{C}$ is of Dirac type but not of definite type. This follows from Example 6.3.14 applied to Example 6.2.23, since the natural inner product on ΣM is not positive definite. An alternative elementary proof is the following: for any timelike tangent vector $\mathfrak{n}$ on M and the corresponding covector $\mathfrak{n}^\flat$, one has

$$\langle i\sigma_P(\mathfrak{n}^\flat)\mathfrak{n}^\flat, \mathfrak{n}^\flat \rangle = -\langle \mathfrak{n}^\flat \wedge \mathfrak{n}^\flat - \mathfrak{n} \lrcorner \mathfrak{n}^\flat, \mathfrak{n}^\flat \rangle = \langle \mathfrak{n}, \mathfrak{n} \rangle \langle 1, \mathfrak{n}^\flat \rangle = 0.$$

Example 6.3.16 The Rarita-Schwinger operator defined in Section 6.2.6 is not of definite type if the dimension of the manifolds is $m \geq 3$. This can be seen as follows. Fix a point $x \in M$ and a pointwise orthonormal basis $(e_j)_{1 \leq j \leq m}$ of T_xM with e_1 timelike. The Lorentzian metric induces inner products on ΣM and on $\Sigma^{3/2}M$ which we denote by $\langle \cdot, \cdot \rangle$. Choose $\xi := e_1^\flat \in T_x^*M$ and $\psi \in \Sigma_x^{3/2}M$. Since $\sigma_{\mathscr{Q}}(\xi)$ is pointwise obtained as the orthogonal projection of $\sigma_{\mathscr{D}}(\xi)$ onto $\Sigma_x^{3/2}M$, one has

$$\begin{aligned}
\langle -i\sigma_{\mathscr{Q}}(\xi)\psi, \psi \rangle &= \langle (\mathrm{id} \otimes \xi^\sharp \cdot)\psi, \psi \rangle - \frac{2}{m} \underbrace{\sum_{\beta=1}^{m} \langle e_\beta^* \otimes e_\beta \cdot \psi_1, \psi \rangle}_{=0} \\
&= \sum_{\beta=1}^{m} \varepsilon_\beta \langle e_1 \cdot \psi_\beta, \psi_\beta \rangle.
\end{aligned}$$

Choose, as in the proof of Lemma 6.2.26, a $\psi \in \Sigma_x^{3/2}M$ with $\psi_k = 0$ for all $3 \leq k \leq m$. For such a ψ the condition $\psi \in \Sigma_x^{3/2}M$ becomes $e_1 \cdot \psi_1 = e_2 \cdot \psi_2$. As in the proof of Lemma 6.2.26 we obtain

$$\langle -i\sigma_{\mathscr{Q}}(\xi)\psi, \psi \rangle = -\langle e_1 \cdot \psi_2, \psi_2 \rangle + \langle e_1 \cdot \psi_2, \psi_2 \rangle = 0,$$

which shows that the Rarita-Schwinger operator cannot be of definite type.

We define the category GlobHypDef, whose objects are the triples (M,S,P), where M is a globally hyperbolic spacetime, S is a complex vector bundle equipped with a complex inner product $\langle\cdot\,,\cdot\rangle$, and P is a formally self-adjoint Green-hyperbolic operator of definite type acting on sections of S. The morphisms are the same as in the category GlobHypGreen.

We construct a covariant functor from GlobHypDef to HILB, where HILB denotes the category whose objects are complex pre-Hilbert spaces and whose morphisms are isometric linear embeddings. As in Section 6.3.1, the underlying vector space is the space of classical solutions to the equation $P\phi = 0$ with spacelike compact support. We put

$$\mathrm{SOL}(M,S,P) := \ker(P) \cap C^\infty_{\mathrm{sc}}(M,S).$$

Here "SOL" stands for classical solutions of the equation $P\phi = 0$ with spacelike compact support.

Lemma 6.3.17 *Let (M,S,P) be an object in* GlobHypDef*. Let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface with its future-oriented unit normal vector field $\mathfrak{n}$ and its induced volume element* dA*. Then*

$$(\phi,\psi) := \int_\Sigma \langle i\sigma_P(\mathfrak{n}^\flat)\cdot\phi_{|\Sigma},\psi_{|\Sigma}\rangle\,\mathrm{dA}, \tag{6.8}$$

yields a positive definite Hermitian scalar product on $\mathrm{SOL}(M,S,P)$ *which does not depend on the choice of* Σ*.*

Proof: First note that $\mathrm{supp}(\phi)\cap\Sigma$ is compact since $\mathrm{supp}(\phi)$ is spacelike compact, so that the integral is well-defined. We have to show that it does not depend on the choice of Cauchy hypersurface. Let Σ' be any other smooth spacelike Cauchy hypersurface. Assume first that Σ and Σ' are disjoint and let Ω be the domain enclosed by Σ and Σ' in M. Its boundary is $\partial\Omega = \Sigma\cup\Sigma'$. Without loss of generality, one may assume that $\Sigma' \subset J^M_+(\Sigma)$. By the Green's formula [E40, p. 160, Prop. 9.1] we have for all $\phi,\psi \in C^\infty_{\mathrm{sc}}(M,S)$,

$$\int_\Omega (\langle P\phi,\psi\rangle - \langle\phi,P\psi\rangle)\,\mathrm{dV} = \int_{\Sigma'}\langle\sigma_P(\mathfrak{n}^\flat)\phi,\psi\rangle\,\mathrm{dA} - \int_\Sigma\langle\sigma_P(\mathfrak{n}^\flat)\phi,\psi\rangle\,\mathrm{dA}. \tag{6.9}$$

For $\phi,\psi\in\mathrm{SOL}(M,S,P)$ we have $P\phi = P\psi = 0$ and thus

$$0 = \int_\Sigma\langle\sigma_P(\mathfrak{n}^\flat)\phi,\psi\rangle\,\mathrm{dA} - \int_{\Sigma'}\langle\sigma_P(\mathfrak{n}^\flat)\phi,\psi\rangle\,\mathrm{dA}.$$

This shows the result in the case $\Sigma\cap\Sigma' = \emptyset$.

If $\Sigma\cap\Sigma'\neq\emptyset$ consider the subset $I^M_-(\Sigma)\cap I^M_-(\Sigma')$ of M where, as usual, $I^M_+(\Sigma)$ and $I^M_-(\Sigma)$ denote the chronological future and past of the subset Σ in M, respectively. This subset is nonempty, open, and globally hyperbolic. This follows e.g. from [E4, Lemma A.5.8]. Hence it admits a smooth spacelike Cauchy hypersurface Σ'' by Theorem 6.2.3. By construction, Σ'' meets neither Σ nor Σ' and it can be easily checked that Σ'' is also a Cauchy hypersurface of M. The result follows from the argument above being applied first to the pair (Σ,Σ'') and then to the pair (Σ'',Σ'). □

Note 6.3.18 If one drops the assumption that P be of definite type, then the above sesquilinear form $(\cdot,\cdot)$ on $\ker(P)\cap C^\infty_{\mathrm{sc}}(M,S)$ still does not depend on the choice of

Σ, however it need no longer be positive definite and can even be degenerate. Pick for instance the spin Dirac operator D_g associated to the underlying Lorentzian metric g on a spin spacetime M (see Example 6.2.21) and, keeping the spinor bundle $\Sigma_g M$ associated to g, change the metric on M so that the new metric g' has larger future and past cones at each point. Note that this implies that any globally hyperbolic subregion of (M,g') is also globally hyperbolic in (M,g). Then, denoting by D_g^* the formal adjoint of D_g with respect to the metric g', the operator $\begin{pmatrix} 0 & D_g \\ D_g^* & 0 \end{pmatrix}$ on $\Sigma_g M \oplus \Sigma_g M$ remains Green-hyperbolic but it fails to be of definite type, since there exist timelike vectors for g' which are lightlike for g. Hence the principal symbol of the operator becomes non-invertible and the bilinear form in (6.8) becomes degenerate for these g'-timelike covectors.

For any object (M,S,P) in GlobHypDef we will from now on equip $\mathrm{SOL}(M,S,P)$ with the Hermitian scalar product in (6.8) and thus turn $\mathrm{SOL}(M,S,P)$ into a pre-Hilbert space.

Given a morphism (f,F) from (M_1,S_1,P_1) to (M_2,S_2,P_2) in GlobHypDef, then this is also a morphism in GlobHypGreen and hence induces a homomorphism $\mathrm{SYMPL}(f,F) : \mathrm{SYMPL}(M_1,S_1,P_1) \to \mathrm{SYMPL}(M_2,S_2,P_2)$. As explained in Remark 6.3.9, there is a corresponding extension homomorphism $\mathrm{SOL}(f,F) : \mathrm{SOL}(M_1,S_1,P_1) \to \mathrm{SOL}(M_2,S_2,P_2)$. In other words, $\mathrm{SOL}(f,F)$ is defined such that the diagram

$$\begin{tikzcd} \mathrm{SYMPL}(M_1,S_1,P_1) \arrow[r, "\mathrm{SYMPL}(f,F)"] \arrow[d, "\cong"'] & \mathrm{SYMPL}(M_2,S_2,P_2) \arrow[d, "\cong"] \\ \mathrm{SOL}(M_1,S_1,P_1) \arrow[r, "\mathrm{SOL}(f,F)"] & \mathrm{SOL}(M_2,S_2,P_2) \end{tikzcd} \tag{6.10}$$

commutes. The vertical arrows are the vector space isomorphisms induced be the Green's propagators G_1 and G_2, respectively.

Lemma 6.3.19 *The vector space homomorphism* $\mathrm{SOL}(f,F) : \mathrm{SOL}(M_1,S_1,P_1) \to \mathrm{SOL}(M_2,S_2,P_2)$ *preserves the scalar products, i.e., it is an isometric linear embedding of pre-Hilbert spaces.*

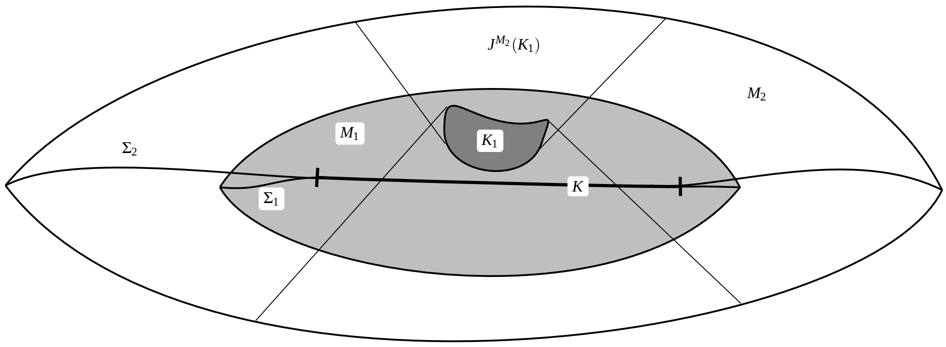
Proof: Without loss of generality we assume that f and F are inclusions. Let Σ_1 be a spacelike Cauchy hypersurface of M_1. Let $\phi_1, \psi_1 \in C^\infty_{\mathrm{sc}}(M_1,S_1)$. Denote the extension of ϕ_1 by $\phi_2 := \mathrm{SOL}(f,F)(\phi_1)$ and similarly for ψ_1.

Let $K_1 \subset M_1$ be a compact subset such that $\mathrm{supp}(\phi_2) \subset J^{M_2}(K_1)$ and $\mathrm{supp}(\psi_2) \subset J^{M_2}(K_1)$. We choose a compact submanifold $K \subset \Sigma_1$ with boundary such that $J^{M_1}(K_1) \cap \Sigma_1 \subset K$. Since Σ_1 is a Cauchy hypersurface in M_1, $J^{M_1}(K_1) \subset J^{M_1}(J^{M_1}(K_1) \cap \Sigma_1) \subset J^{M_1}(K)$.

By Theorem 6.2.5 there is a spacelike Cauchy hypersurface $\Sigma_2 \subset M_2$ containing K. Since Σ_i is a Cauchy hypersurface of M_i (where $i = 1,2$), it is met by every inextensible causal curve [E30, Lemma 14.29]. Moreover, by definition of a Cauchy hypersurface, Σ_i is achronal in M_i. Since it is also spacelike, Σ_i is even acausal [E30, Lemma 14.42]. In particular, it is met *exactly once* by every inextensible causal curve in M_i.

This implies $J^{M_2}(K_1) \subset J^{M_2}(K)$ (see Figure below): namely, pick $p \in J^{M_2}(K_1)$ and a causal curve γ in M_2 from p to some $k_1 \in K_1$. Extend γ to an inextensible causal curve $\overline{\gamma}$ in M_2. Then $\overline{\gamma}$ meets Σ_2 at some point q_2, because Σ_2 is a Cauchy hypersurface in M_2.

But $\overline{\gamma} \cap M_1$ is also an inextensible causal curve in M_1, hence it intersects Σ_1 at a point q_1, which must lie in K by definition of K. Because of $K \subset \Sigma_2$ and the uniqueness of the intersection point, one has $q_1 = q_2$. In particular, $p \in J^{M_2}(K)$.



$$J^{M_2}(K_1) \subset J^{M_2}(K)$$

We conclude $\mathrm{supp}(\phi_2) \subset J^{M_2}(K)$. Since $K \subset \Sigma_2$, we have $\mathrm{supp}(\phi_2) \cap \Sigma_2 \subset J^{M_2}(K) \cap \Sigma_2$ and $J^{M_2}(K) \cap \Sigma_2 = K$ using the acausality of Σ_2. This shows $\mathrm{supp}(\phi_2) \cap \Sigma_2 = \mathrm{supp}(\phi_1) \cap \Sigma_1$ and similarly for ψ_2. Now we get

$$(\phi_2, \psi_2) = \int_{\Sigma_2} \langle i\sigma_{P_2}(\mathfrak{n}^\flat) \cdot \phi_2, \psi_2 \rangle \, \mathrm{dA} = \int_{\Sigma_1} \langle i\sigma_{P_1}(\mathfrak{n}^\flat) \cdot \phi_1, \psi_1 \rangle \, \mathrm{dA} = (\phi_1, \psi_1)$$

and the lemma is proved. □

The functoriality of SYMPL and diagram (6.10) show that SOL is a functor from $\mathsf{GlobHypDef}$ to HILB, the category of complex pre-Hilbert spaces with isometric linear embeddings. Composing with the functor CAR (see Section 6.5.1), we obtain the covariant functor

$$\mathfrak{A}_{\mathrm{ferm}} := \mathrm{CAR} \circ \mathrm{SOL} : \mathsf{GlobHypDef} \longrightarrow \mathsf{C^*Alg}.$$

The fermionic algebras $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$ are actually $\mathbb{Z}_2$-graded algebras, see Proposition 6.5.5 (iii).

Theorem 6.3.20 *The functor $\mathfrak{A}_{\mathrm{ferm}} : \mathsf{GlobHypDef} \longrightarrow \mathsf{C^*Alg}$ is a fermionic locally covariant quantum field theory, i.e., the following axioms hold:*

(i) ***(Quantum causality)*** *Let (M_j, S_j, P_j) be objects in $\mathsf{GlobHypDef}$, $j = 1,2,3$, and (f_j, F_j) morphisms from (M_j, S_j, P_j) to (M_3, S_3, P_3), $j = 1,2$, such that $f_1(M_1)$ and $f_2(M_2)$ are causally disjoint regions in M_3.*
Then the subalgebras $\mathfrak{A}_{\mathrm{ferm}}(f_1,F_1)(\mathfrak{A}_{\mathrm{ferm}}(M_1,S_1,P_1))$ and $\mathfrak{A}_{\mathrm{ferm}}(f_2,F_2)(\mathfrak{A}_{\mathrm{ferm}}(M_2,S_2,P_2))$ of $\mathfrak{A}_{\mathrm{ferm}}(M_3,S_3,P_3)$ super-commute[1]*.*

[1] This means that the odd parts of the algebras anti-commute while the even parts commute with everything.

(ii) ***(Time slice axiom)*** *Let (M_j, S_j, P_j) be objects in* GlobHypDef, $j = 1, 2$, *and (f, F) a morphism from (M_1, S_1, P_1) to (M_2, S_2, P_2) such that there is a Cauchy hypersurface $\Sigma \subset M_1$ for which $f(\Sigma)$ is a Cauchy hypersurface of M_2. Then*

$$\mathfrak{A}_{\mathrm{ferm}}(f,F) : \mathfrak{A}_{\mathrm{ferm}}(M_1,S_1,P_1) \to \mathfrak{A}_{\mathrm{ferm}}(M_2,S_2,P_2)$$

is an isomorphism.

Proof: To show (i), we assume without loss of generality that f_j and F_j are inclusions. Let $\phi_1 \in \mathrm{SOL}(M_1,S_1,P_1)$ and $\psi_1 \in \mathrm{SOL}(M_2,S_2,P_2)$. Denote the extensions to M_3 by $\phi_2 := \mathrm{SOL}(f_1,F_1)(\phi_1)$ and $\psi_2 := \mathrm{SOL}(f_2,F_2)(\psi_1)$. Choose a compact submanifold K_1 (with boundary) in a spacelike Cauchy hypersurface Σ_1 of M_1 such that $\mathrm{supp}(\phi_1) \cap \Sigma_1 \subset K_1$ and similarly K_2 for ψ_1. Since M_1 and M_2 are causally disjoint, $K_1 \cup K_2$ is acausal. Hence, by Theorem 6.2.5, there exists a Cauchy hypersurface Σ_3 of M_3 containing K_1 and K_2. As in the proof of Lemma 6.3.19 one sees that $\mathrm{supp}(\phi_2) \cap \Sigma_3 = \mathrm{supp}(\phi_1) \cap \Sigma_1$ and similarly for ψ_2. Thus, when restricted to Σ_3, ϕ_2 and ψ_2 have disjoint support. Hence $(\phi_2, \psi_2) = 0$. This shows that the subspaces $\mathrm{SOL}(f_1,F_1)(\mathrm{SOL}(M_1,S_1,P_1))$ and $\mathrm{SOL}(f_2,F_2)(\mathrm{SOL}(M_2,S_2,P_2))$ of $\mathrm{SOL}(M_3,S_3,P_3)$ are perpendicular. Definition 6.5.1 shows that the corresponding CAR-algebras must super-commute.
To see (ii) we recall that (f,F) is also a morphism in GlobHypGreen and that we know from Theorem 6.3.10 that $\mathrm{SYMPL}(f,F)$ is an isomorphism. From diagram (6.10) we see that $\mathrm{SOL}(f,F)$ is an isomorphism. Hence $\mathfrak{A}_{\mathrm{ferm}}(f,F)$ is also an isomorphism. □

Note 6.3.21 Since causally disjoint regions should lead to commuting observables also in the fermionic case, one usually considers only the even part $\mathfrak{A}^{\mathrm{even}}_{\mathrm{ferm}}(M,S,P)$ (or a subalgebra thereof) as the observable algebra while the full algebra $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$ is called the *field algebra*.

There is a slightly different description of the functor $\mathfrak{A}_{\mathrm{ferm}}$. Let $\mathsf{HILB}_{\mathbb{R}}$ denote the category whose objects are the real pre-Hilbert spaces and whose morphisms are the isometric linear embeddings. We have the functor $\mathrm{REAL} : \mathsf{HILB} \to \mathsf{HILB}_{\mathbb{R}}$ which associates to each complex pre-Hilbert space $(V, (\cdot,\cdot))$ its underlying real pre-Hilbert space $(V, \mathfrak{Re}(\cdot,\cdot))$. By Remark 6.5.10,

$$\mathfrak{A}_{\mathrm{ferm}} = \mathrm{CAR}_{\mathrm{sd}} \circ \mathrm{REAL} \circ \mathrm{SOL}.$$

Since the self-dual CAR-algebra of a real pre-Hilbert space is the Clifford algebra of its complexification and since for any complex pre-Hilbert space V we have

$$\mathrm{REAL}(V) \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V^*,$$

$\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$ is also the Clifford algebra of $\mathrm{SOL}(M,S,P) \oplus \mathrm{SOL}(M,S,P)^* = \mathrm{SOL}(M, S \oplus S^*, P \oplus P^*)$. This is the way this functor is often described in the physics literature, see e.g. [E39, p. 115f].
Self-dual CAR-representations are more natural for real fields. Let M be globally hyperbolic and let $S \to M$ be a *real* vector bundle equipped with a real inner product $\langle\cdot,\cdot\rangle$. A formally skew-adjoint[2] differential operator P acting on sections of S is called of *definite type* if and only if for any $x \in M$ and any future-directed timelike tangent vector $\mathfrak{n} \in T_xM$, the bilinear map

$$S_x \times S_x \to \mathbb{R}, \qquad (\phi,\psi) \mapsto \langle \sigma_P(\mathfrak{n}^\flat) \cdot \phi, \psi \rangle,$$

[2] instead of self-adjoint!

yields a positive definite Euclidean scalar product on S_x. An example is given by the real Dirac operator

$$D := \sum_{j=1}^{m} \varepsilon_j e_j \cdot \nabla_{e_j}$$

acting on sections of the real spinor bundle $\Sigma^{\mathbb{R}}M$.
Given a smooth spacelike Cauchy hypersurface $\Sigma \subset M$ with future-directed timelike unit normal field $\mathfrak{n}$, we define a scalar product on $\mathrm{SOL}(M,S,P) = \ker(P) \cap C^\infty_{\mathrm{sc}}(M,S,P)$ by

$$(\phi,\psi) := \int_\Sigma \langle \sigma_P(\mathfrak{n}^\flat)\cdot \phi_{|\Sigma}, \psi_{|\Sigma}\rangle \,\mathrm{dA}.$$

With essentially the same proofs as before, one sees that this scalar product does not depend on the choice of Cauchy hypersurface Σ and that a morphism $(f,F) : (M_1,S_1,P_1) \to (M_2,S_2,P_2)$ gives rise to an extension operator $\mathrm{SOL}(f,F)$: $\mathrm{SOL}(M_1,S_1,P_1) \to \mathrm{SOL}(M_2,S_2,P_2)$ preserving the scalar product. We have constructed a functor

$$\mathrm{SOL} : \mathsf{GlobHypSkewDef} \longrightarrow \mathsf{HILB}_{\mathbb{R}}$$

where $\mathsf{GlobHypSkewDef}$ denotes the category whose objects are triples (M,S,P) with M globally hyperbolic, $S \to M$ a real vector bundle with real inner product and P a formally skew-adjoint, Green-hyperbolic differential operator of definite type acting on sections of S. The morphisms are the same as before.
Now the functor

$$\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}} := \mathrm{CAR}_{\mathrm{sd}} \circ \mathrm{SOL} : \mathsf{GlobHypSkewDef} \longrightarrow \mathsf{C^*Alg}$$

is a locally covariant quantum field theory in the sense that Theorem 6.3.20 holds with $\mathfrak{A}_{\mathrm{ferm}}$ replaced by $\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}}$.

6.4 States and quantum fields

In order to produce numbers out of our quantum field theory that can be compared to experiments, we need states, in addition to observables. We briefly recall the relation between states and representations via the GNS-construction. Then we show how the choice of a state gives rise to quantum fields and n-point functions.

6.4.1 States and representations

Recall that a *state* on a unital C*-algebra A is a linear functional $\tau : A \to \mathbb{C}$ such that

(i) τ is positive, i.e., $\tau(a^*a) \geq 0$ for all $a \in A$;

(ii) τ is normed, i.e., $\tau(1) = 1$.

One checks that for any state the sesquilinear form $A \times A \to \mathbb{C}$, $(a,b) \mapsto \tau(b^*a)$, is a positive semi-definite Hermitian product and $|\tau(a)| \leq \|a\|$ for all $a \in A$. In particular, τ is continuous.
Any state induces a representation of A. Namely, the sesquilinear form $\tau(b^*a)$ induces a scalar product $\mathfrak{so}\cdot\cdot$ on $A/\{a \in A \mid \tau(a^*a) = 0\}$. The Hilbert space completion of $A/\{a \in A \mid \tau(a^*a) = 0\}$ is denoted by $\mathscr{H}_\tau$. The action of A on $\mathscr{H}_\tau$ is induced by the multiplication in A,

$$\pi_\tau(a)[b]_\tau := [ab]_\tau,$$

where $[a]_\tau$ denotes the residue class of $a \in A$ in $A/\{a \in A \mid \tau(a^*a) = 0\}$. This representation is known as the *GNS-representation* induced by τ. The residue class $\Omega_\tau := [1]_\tau \in \mathscr{H}_\tau$ is called the *vacuum vector*. By construction, it is a cyclic vector, i.e., the orbit $\pi_\tau(A) \cdot \Omega_\tau = A/\{a \in A \mid \tau(a^*a) = 0\}$ is dense in $\mathscr{H}_\tau$.
The GNS-representation together with the vacuum vector allows to reconstruct the state since

$$\tau(a) = \tau(1^*a1) = \mathfrak{so}\pi_\tau(a)\Omega_\tau\Omega_\tau. \tag{6.11}$$

If we look at the vector state $\tilde{\tau} : \mathscr{L}(\mathscr{H}_\tau) \to \mathbb{C}$, $\tilde{\tau}(\tilde{a}) = \mathfrak{so}\tilde{a}\Omega_\tau\Omega_\tau$, on the C*-algebra $\mathscr{L}(\mathscr{H}_\tau)$ of bounded linear operators on $\mathscr{H}_\tau$, then (6.11) says that the diagram

$$\begin{tikzcd} A \arrow[rr, "\pi_\tau"] \arrow[dr, "\tau"'] & & \mathscr{L}(\mathscr{H}_\tau) \arrow[dl, "\tilde{\tau}"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes. One checks that $\|\pi_\tau\| \le 1$, see [E2, p. 20]. In particular, $\pi_\tau : A \to \mathscr{L}(\mathscr{H}_\tau)$ is continuous.
See e.g. [E2, Sec. 1.4] or [E9, Sec. 2.3] for details on states and representations of C*-algebras.

6.4.2 Bosonic quantum field

Now let (M,S,P) be an object in GlobHypGreen and τ a state on the corresponding bosonic algebra $\mathfrak{A}_{\mathrm{bos}}(M,S,P)$. Intuitively, the quantum field should be an operator-valued distribution Φ on M such that

$$e^{i\Phi(f)} = w([f])$$

for all test sections $f \in C_c^\infty(M,S)$. Here $[f]$ denotes the residue class in $\mathrm{SYMPL}(M,S,P) = C_c^\infty(M,S)/\ker G$ and $w : \mathrm{SYMPL}(M,S,P) \to \mathfrak{A}_{\mathrm{bos}}(M,S,P)$ is as in Definition 6.5.11. This suggests the definition

$$\Phi(f) := -i \left.\frac{d}{dt}\right|_{t=0} w(t[f]).$$

The problem is that w is highly discontinuous so that this derivative does not make sense. This is where states and representations come into the play. We call a state τ on $\mathfrak{A}_{\mathrm{bos}}(M,S,P)$ *regular* if for each $f \in C_c^\infty(M,S)$ and each $h \in \mathscr{H}_\tau$ the map $t \mapsto \pi_\tau(w(t[f]))h$ is continuous. Then $t \mapsto \pi_\tau(w(t[f]))$ is a strongly continuous one-parameter unitary group for any $f \in C_c^\infty(M,S)$ because

$$\pi_\tau(w((t+s)[f])) = \pi_\tau(e^{i\omega(t[f],s[f])/2}w(t[f])w(s[f])) = \pi_\tau(w(t[f]))\pi_\tau(w(s[f])).$$

Here we used Definition 6.5.11 (iv) and the fact that ω is skew-symmetric so that $\omega(t[f],s[f]) = 0$. By Stone's theorem [E34, Thm. VIII.8] this one-parameter group has a unique infinitesimal generator, i.e., a self-adjoint, generally unbounded operator $\Phi_\tau(f)$ on $\mathscr{H}_\tau$ such that

$$e^{it\Phi_\tau(f)} = \pi_\tau(w(t[f])).$$

For all h in the domain of $\Phi_\tau(f)$ we have

$$\Phi_\tau(f)h = -i \left.\frac{d}{dt}\right|_{t=0} \pi_\tau(w(t[f]))h.$$

We call the operator-valued map $f \mapsto \Phi_\tau(f)$ the *quantum field* corresponding to τ.

Definition 6.4.1 *A regular state τ on $\mathfrak{A}_{\mathrm{bos}}(M,S,P)$ is called* strongly regular *if*

(i) there is a dense subspace $\mathscr{D}_\tau \subset \mathscr{H}_\tau$ contained in the domain of $\Phi_\tau(f)$ for any $f \in C_c^\infty(M,S)$;

(ii) $\Phi_\tau(f)(\mathscr{D}_\tau) \subset \mathscr{D}_\tau$ for any $f \in C_c^\infty(M,S)$;

(iii) the map $C_c^\infty(M,S) \to \mathscr{H}_\tau$, $f \mapsto \Phi_\tau(f)h$, is continuous for every fixed $h \in \mathscr{D}_\tau$.

For a strongly regular state τ we have for all $f,g \in C_c^\infty(M,S)$, $\alpha,\beta \in \mathbb{R}$ and $h \in \mathscr{D}_\tau$:

$$\begin{aligned}
\Phi_\tau(\alpha f + \beta g)h &= -i\, \frac{d}{dt}\Big|_{t=0} \pi_\tau(w(t[\alpha f + \beta g]))h \\
&= -i\, \frac{d}{dt}\Big|_{t=0} \left\{ e^{i\alpha\beta t^2 \omega([f],[g])/2} \pi_\tau(w(\alpha t[f]))\pi_\tau(w(\beta t[g]))h \right\} \\
&= -i\, \frac{d}{dt}\Big|_{t=0} \pi_\tau(w(\alpha t[f]))h - i\, \frac{d}{dt}\Big|_{t=0} \pi_\tau(w(\beta t[g]))h \\
&= \alpha\Phi_\tau(f)h + \beta\Phi_\tau(g)h.
\end{aligned}$$

Hence $\Phi_\tau(f)$ depends linearly on f. The quantum field Φ_τ is therefore a distribution on M with values in self-adjoint operators on $\mathscr{H}_\tau$.
The *n-point functions* are defined by

$$\begin{aligned}
\tau_n(f_1,\ldots,f_n) &:= \mathfrak{so}\Phi_\tau(f_1)\cdots\Phi_\tau(f_n)\Omega_\tau\Omega_\tau \\
&= \tilde{\tau}(\Phi_\tau(f_1)\cdots\Phi_\tau(f_n)) \\
&= \tilde{\tau}\left(\left(-i\, \frac{d}{dt_1}\Big|_{t_1=0} \pi_\tau(w(t_1[f_1]))\right)\cdots\left(-i\, \frac{d}{dt_n}\Big|_{t_n=0} \pi_\tau(w(t_n[f_n]))\right)\right) \\
&= (-i)^n \, \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\cdots=t_n=0} \tilde{\tau}\left(\pi_\tau(w(t_1[f_1]))\cdots\pi_\tau(w(t_n[f_n]))\right) \\
&= (-i)^n \, \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\cdots=t_n=0} \tilde{\tau}\left(\pi_\tau(w(t_1[f_1])\cdots w(t_n[f_n]))\right) \\
&= (-i)^n \, \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\cdots=t_n=0} \tau\left(w(t_1[f_1])\cdots w(t_n[f_n])\right).
\end{aligned}$$

For a strongly regular state τ the n-point functions are continuous separately in each factor. By the Schwartz kernel theorem [E23, Thm. 5.2.1] the n-point function τ_n extends uniquely to a distribution on $M \times \cdots \times M$ (n times) in the following sense: Let $S^* \boxtimes \cdots \boxtimes S^*$ be the bundle over $M \times \cdots \times M$ whose fiber over $(x_1,\ldots,x_n)$ is given by $S^*_{x_1} \otimes \cdots \otimes S^*_{x_n}$. Then there is a unique distribution on $M \times \cdots \times M$ in the bundle $S^* \boxtimes \cdots \boxtimes S^*$, again denoted τ_n, such that for all $f_j \in C_c^\infty(M,S)$,

$$\tau_n(f_1,\ldots,f_n) = \tau_n(f_1 \otimes \cdots \otimes f_n)$$

where $(f_1 \otimes \cdots \otimes f_n)(x_1,\ldots,x_n) := f_1(x_1) \otimes \cdots \otimes f_n(x_n)$.

Theorem 6.4.2 *Let (M,S,P) be an object in* $\mathsf{GlobHypGreen}$ *and τ a strongly regular state on the corresponding bosonic algebra $\mathfrak{A}_{\mathrm{bos}}(M,S,P)$. Then* *(i)* $P\Phi_\tau = 0$ *and* $P\tau_n(f_1,\dots,f_{j-1},\cdot,f_{j+1},\dots,f_n) = 0$ *hold in the distributional sense where* $f_k \in C_c^\infty(M,S)$*,* $k \neq j$*, are fixed;*

(ii) the quantum field satisfies the canonical commutation relations, i.e.,

$$[\Phi_\tau(f),\Phi_\tau(g)]h = i\int_M \langle Gf,g\rangle\,\mathrm{dV}\cdot h$$

for all $f,g \in C_c^\infty(M,S)$ *and* $h \in \mathscr{D}_\tau$*;*

(iii) the n-point functions satisfy the canonical commutation relations, i.e.,

$$\begin{aligned}&\tau_{n+2}(f_1,\dots,f_{j-1},f_j,f_{j+1},\dots,f_{n+2})\\ &-\tau_{n+2}(f_1,\dots,f_{j-1},f_{j+1},f_j,f_{j+2},\dots,f_{n+2})\\ =\,&i\int_M \langle Gf_j,f_{j+1}\rangle\,\mathrm{dV}\cdot\tau_n(f_1,\dots,f_{j-1},f_{j+2},\dots,f_{n+2})\end{aligned}$$

for all $f_1,\dots,f_{n+2} \in C_c^\infty(M,S)$.

Proof: Since P is formally self-adjoint and $GPf = 0$ for any $f \in C_c^\infty(M,S)$, we have for any $h \in \mathscr{D}_\tau$:

$$(P\Phi_\tau)(f)h = \Phi_\tau(Pf)h = -i\left.\frac{d}{dt}\right|_{t=0}\pi_\tau(w(t\underbrace{[Pf]}_{=0}))h = -i\left.\frac{d}{dt}\right|_{t=0} h = 0.$$

This shows $P\Phi_\tau = 0$. The result for the n-point functions follows and (i) is proved. To show (ii) we observe that by Definition 6.5.11 (iv) we have on the one hand

$$w([f+g]) = e^{i\omega([f],[g])/2}w([f])w([g])$$

and on the other hand

$$w([f+g]) = e^{i\omega([g],[f])/2}w([g])w([f]),$$

hence

$$w([f])w([g]) = e^{-i\omega([f],[g])}w([g])w([f]).$$

Thus

$$\begin{aligned}\Phi_\tau(f)\Phi_\tau(g)h &= -\left.\frac{\partial^2}{\partial t\partial s}\right|_{t=s=0}\pi_\tau(w(t[f])w(s[g]))h\\ &= -\left.\frac{\partial^2}{\partial t\partial s}\right|_{t=s=0}\pi_\tau(e^{-i\omega(t[f],s[g])}w(s[g])w(t[f]))h\\ &= -\left.\frac{\partial^2}{\partial t\partial s}\right|_{t=s=0}\left\{e^{-i\omega(t[f],s[g])}\cdot\pi_\tau(w(s[g])w(t[f]))h\right\}\\ &= i\omega([f],[g])h + \Phi_\tau(g)\Phi_\tau(f)h\\ &= i\int_M\langle Gf,g\rangle\,\mathrm{dV}\cdot h + \Phi_\tau(g)\Phi_\tau(f)h.\end{aligned}$$

This shows (ii). Assertion (iii) follows from (ii). □

Note 6.4.3 As a consequence of the canonical commutation relations we get

$$[\Phi_\tau(f), \Phi_\tau(g)] = 0$$

if the supports of f and g are causally disjoint, i.e., if there is no causal curve from $\mathrm{supp}(f)$ to $\mathrm{supp}(g)$. The reason is that in this case the supports of Gf and g are disjoint. A similar remark holds for the n-point functions.

Note 6.4.4 In the physics literature one also finds the statement $\Phi(\overline{f}) = \Phi(f)^*$. This simply expresses the fact that we are dealing with a theory over the reals. We have encoded this by considering real vector bundles S, see Definition 6.3.1, and the fact that $\Phi_\tau(f)$ is always self-adjoint.

6.4.3 Fermionic quantum fields

Let (M,S,P) be an object in $\mathsf{GlobHypDef}$ and let τ be a state on the fermionic algebra $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$. For $f \in C_c^\infty(M,S)$ we put

$$\begin{aligned} \Phi_\tau(f) &:= -\pi_\tau(\mathbf{a}(Gf)^*), \\ \Phi_\tau^+(f) &:= \pi_\tau(\mathbf{a}(Gf)), \end{aligned}$$

where $\mathbf{a}$ is as in Definition 6.5.1 (compare [E18, Sec. III.B, p. 141]). Since π_τ, $\mathbf{a}$, and G are sequentially continuous (for G see [E4, Prop. 3.4.8]), so are Φ_τ and Φ_τ^+. In contrast to the bosonic case, no regularity assumption on τ is needed. Hence Φ_τ and Φ_τ^+ are distributions on M with values in the space of bounded operators on $\mathscr{H}_\tau$. Note that Φ_τ is linear while Φ_τ^+ is anti-linear.

Theorem 6.4.5 *Let (M,S,P) be an object in* $\mathsf{GlobHypDef}$ *and τ a state on the corresponding fermionic algebra $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$. Then*

(i) $P\Phi_\tau = P\Phi_\tau^+ = 0$ holds in the distributional sense;

(ii) the quantum fields satisfy the canonical anti-commutation relations, i.e.,

$$\begin{aligned} \{\Phi_\tau(f), \Phi_\tau(g)\} &= \{\Phi_\tau^+(f), \Phi_\tau^+(g)\} = 0, \\ \{\Phi_\tau(f), \Phi_\tau^+(g)\} &= i\Big(\int_M \langle Gf, g\rangle \, \mathrm{dV}\Big) \cdot \mathrm{id}_{\mathscr{H}_\tau} \end{aligned}$$

for all $f, g \in C_c^\infty(M,S)$.

Proof: Since $GP = 0$ on $C_c^\infty(M,S)$, we have $P\Phi_\tau(f) = \Phi_\tau(Pf) = -\pi_\tau(\mathbf{a}(GPf)^*) = 0$ and similarly for Φ_τ^+. This proves assertion (i).
Using Definition 6.5.1 (ii) we compute

$$\begin{aligned} \{\Phi_\tau(f), \Phi_\tau(g)\} &= \{\pi_\tau(\mathbf{a}(Gf)^*), \pi_\tau(\mathbf{a}(Gg)^*)\} \\ &= \pi_\tau(\{\mathbf{a}(Gf)^*, \mathbf{a}(Gg)^*\}) \\ &= \pi_\tau(\{\mathbf{a}(Gg), \mathbf{a}(Gf)\}^*) \\ &= 0. \end{aligned}$$

Similarly one sees $\{\Phi_\tau^+(f), \Phi_\tau^+(g)\} = 0$. Definition 6.5.1 (iii) also yields

$$\{\Phi_\tau(f), \Phi_\tau^+(g)\} = -\pi_\tau(\{\mathbf{a}(Gf)^*, \mathbf{a}(Gg)\}) = -(Gf, Gg) \cdot \mathrm{id}_{\mathscr{H}_\tau}.$$

To prove assertion (ii) we have to verify

$$(Gf, Gg) = -i \int_M \langle Gf, g\rangle \, \mathrm{dV} \tag{6.12}$$

Let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface. Since $\mathrm{supp}(G_+g)$ is past-compact, we can find a Cauchy hypersurface $\Sigma' \subset M$ in the past of Σ which does not intersect $\mathrm{supp}(G_+g) \subset J_+^M(\mathrm{supp}(g))$. Denote the region between Σ and Σ' by Ω'. The Green's formula (6.9) yields

$$\begin{aligned}
(Gf, G_+g) &= \int_\Sigma \langle i\sigma_P(\mathfrak{n}^\flat)\cdot Gf, G_+g\rangle \, \mathrm{dA} \\
&= \int_{\Sigma'} \langle i\sigma_P(\mathfrak{n}^\flat)\cdot Gf, G_+g\rangle \, \mathrm{dA} + i\int_{\Omega'} (\langle PGf, G_+g\rangle - \langle Gf, PG_+g\rangle)\, \mathrm{dV} \\
&= -i \int_{\Omega'} \langle Gf, g\rangle \, \mathrm{dV}
\end{aligned}$$

because $PG_+g = g$ and $PGf = 0$. Since Σ' can be chosen arbitrarily to the past, this shows

$$(Gf, G_+g) = -i \int_{J_-(\Sigma)} \langle Gf, g\rangle \, \mathrm{dV}. \tag{6.13}$$

A similar computation yields

$$(Gf, G_-g) = i \int_{J_+(\Sigma)} \langle Gf, g\rangle \, \mathrm{dV}. \tag{6.14}$$

Subtracting (6.14) from (6.13) yields (6.12) and concludes the proof of assertion (ii). □

Note 6.4.6 Similarly to the bosonic case, we find

$$\{\Phi_\tau(f), \Phi_\tau^+(g)\} = 0$$

if the supports of f and g are causally disjoint.

Note 6.4.7 Using the anti-commutation relations in Theorem 6.4.5 (ii), the computation of n-point functions can be reduced to those of the form

$$\tau_{n,n'}(f_1,\ldots,f_n,g_1,\ldots,g_{n'}) = \langle \Omega_\tau, \Phi_\tau(f_1)\cdots\Phi_\tau(f_n)\Phi_\tau^+(g_1)\cdots\Phi_\tau^+(g_{n'})\Omega_\tau\rangle_\tau.$$

As in the bosonic case, the n-point functions satisfy the field equation in the distributional sense in each argument and extend to distributions on $M\times\cdots\times M$.

If one uses the self-dual fermionic algebra $\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}}(M,S,P)$ instead of $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$, then one gets the quantum field

$$\Psi_\tau(f) := \pi_\tau(\mathbf{b}(Gf))$$

where **b** is as in Definition 6.5.6. Then the analogue to Theorem 6.4.5 is

Theorem 6.4.8 *Let (M,S,P) be an object in* GlobHypSkewDef *and τ a state on the corresponding self-dual fermionic algebra $\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}}(M,S,P)$. Then*

(i) $P\Psi_\tau = 0$ holds in the distributional sense;

(ii) *the quantum field takes values in self-adjoint operators,* $\Psi_\tau(f) = \Psi_\tau(f)^*$ *for all* $f \in C_c^\infty(M,S)$*;*

(iii) *the quantum fields satisfy the canonical anti-commutation relations, i.e.,*

$$\{\Psi_\tau(f), \Psi_\tau(g)\} = \int_M \langle Gf, g\rangle \, \mathrm{dV} \cdot \mathrm{id}_{\mathscr{H}_\tau}$$

for all $f, g \in C_c^\infty(M,S)$.

Note 6.4.9 It is interesting to compare the concept of locally covariant quantum field theories as proposed in [E11] to the axiomatic approach to quantum field theory on Minkowski space based on the Gårding-Wightman axioms as exposed in [E35, Sec. IX.8]. Property 1 (relativistic invariance of states) and Property 6 (Poincaré invariance of the field) in [E35] are replaced by functoriality (covariance). Property 4 (invariant domain for fields) and Property 5 (regularity of the field) have been encoded in strong regularity of the state used to define the quantum field in the bosonic case and are automatic in the fermionic case. Property 7 (local commutativity or microscopic causality) is contained in Theorems 6.4.2 and 6.4.5. Property 3 (existence and uniqueness of the vacuum) has no analogue and is replaced by the *choice* of a state. Property 8 (cyclicity of the vacuum) is then automatic by the general properties of the GNS-construction.
There remains one axiom, Property 2 (spectral condition), which we have not discussed at all. It gets replaced by the Hadamard condition on the state chosen. It was observed by Radzikowski [E32] that earlier formulations of this condition are equivalent to a condition on the wave front set of the 2-point function. Much work has been put into constructing and investigating Hadamard states for various examples of fields, see e.g. [E15, E16, E19, E25, E36, E37, E38, E42] and the references therein.

6.5 Algebras of canonical (anti-) commutation relations

We collect the necessary algebraic facts about CAR and CCR-algebras.

6.5.1 CAR algebras

The symbol "CAR" stands for "canonical anti-commutation relations". These algebras are related to pre-Hilbert spaces. We always assume the Hermitian inner product $(\cdot\,,\cdot)$ to be linear in the first argument and anti-linear in the second.

Definition 6.5.1 *A* CAR-representation *of a complex pre-Hilbert space* $(V, (\cdot\,,\cdot))$ *is a pair* $(\mathbf{a}, A)$*, where* A *is a unital* C^**-algebra and* $\mathbf{a} : V \to A$ *is an anti-linear map satisfying:*

(i) $A = C^*(\mathbf{a}(V))$,

(ii) $\{\mathbf{a}(v_1), \mathbf{a}(v_2)\} = 0$ *and*

(iii) $\{\mathbf{a}(v_1)^*, \mathbf{a}(v_2)\} = (v_1, v_2) \cdot 1$,

for all $v_1, v_2 \in V$.

We want to discuss CAR-representations in terms of C*-Clifford algebras, whose definition we recall. Given a complex pre-Hilbert vector space $(V,(\cdot\,,\cdot))$, we denote by $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of V considered as a real vector space and by $q_{\mathbb{C}}$ the complex-bilinear extension of $\mathfrak{Re}(\cdot\,,\cdot)$ to $V_{\mathbb{C}}$. Let $\mathrm{Cl}_{\mathrm{alg}}(V_{\mathbb{C}}, q_{\mathbb{C}})$ be the algebraic Clifford algebra of $(V_{\mathbb{C}}, q_{\mathbb{C}})$. It is an associative complex algebra with unit and contains $V_{\mathbb{C}}$ as a vector subspace. Its multiplication is called Clifford multiplication and denoted by "$\cdot$". It satisfies the Clifford relations

$$v \cdot w + w \cdot v = -2q_{\mathbb{C}}(v,w)1 \tag{6.15}$$

for all $v,w \in V_{\mathbb{C}}$. Define the $*$-operator on $\mathrm{Cl}_{\mathrm{alg}}(V_{\mathbb{C}}, q_{\mathbb{C}})$ to be the unique anti-multiplicative and anti-linear extension of the anti-linear map $V_{\mathbb{C}} \to V_{\mathbb{C}}$, $v_1 + iv_2 \mapsto -(\overline{v_1 + iv_2}) = -(v_1 - iv_2)$ for all $v_1, v_2 \in V$. In other words,

$$*\Big(\sum_{i_1<\ldots<i_k} \alpha_{i_1,\ldots,i_k} z_{i_1} \cdot \ldots \cdot z_{i_k}\Big) = (-1)^k \sum_{i_1<\ldots<i_k} \overline{\alpha_{i_1,\ldots,i_k}} \cdot \overline{z_{i_k}} \cdot \ldots \cdot \overline{z_{i_1}}$$

for all $k \in \mathbb{N}$ and $z_{i_1}, \ldots, z_{i_k} \in V_{\mathbb{C}}$. Let $\|\cdot\|_\infty$ be defined by

$$\|a\|_\infty := \sup_{\pi \in \mathrm{Rep}(V)} (\|\pi(a)\|)$$

for every $a \in \mathrm{Cl}_{\mathrm{alg}}(V_{\mathbb{C}}, q_{\mathbb{C}})$, where $\mathrm{Rep}(V)$ denotes the set of all (isomorphism classes of) $*$-homomorphisms from $\mathrm{Cl}_{\mathrm{alg}}(V_{\mathbb{C}}, q_{\mathbb{C}})$ to C*-algebras. Then $\|\cdot\|_\infty$ can be shown to be a well-defined C*-norm on $\mathrm{Cl}_{\mathrm{alg}}(V_{\mathbb{C}}, q_{\mathbb{C}})$, see e.g. [E31, Sec. 1.2].

Definition 6.5.2 *The C*-Clifford algebra of a pre-Hilbert space $(V,(\cdot\,,\cdot))$ is the C*-completion of $\mathrm{Cl}_{\mathrm{alg}}(V_{\mathbb{C}}, q_{\mathbb{C}})$ with respect to the C*-norm $\|\cdot\|_\infty$ and the star operator defined above.*

Theorem 6.5.3 *For every complex pre-Hilbert space $(V,(\cdot\,,\cdot))$, the C*-Clifford algebra $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ provides a* CAR*-representation of $(V,(\cdot\,,\cdot))$ via $\mathbf{a}(v) = \frac{1}{2}(v + iJv)$, where J is the complex structure of V.*
Moreover, CAR*-representations have the following universal property: Let $\widehat{A}$ be any unital C*-algebra and $\widehat{\mathbf{a}} : V \to \widehat{A}$ be any anti-linear map satisfying Axioms* (ii) *and* (iii) *of Definition 6.5.1. Then there exists a unique C*-morphism $\widetilde{\alpha} : \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \to \widehat{A}$ such that*

$$\begin{tikzcd} V \arrow[r, "\widehat{\mathbf{a}}"] \arrow[d, "\mathbf{a}"'] & \widehat{A} \\ \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \arrow[ur, dotted, "\widetilde{\alpha}"'] & \end{tikzcd}$$

commutes. Furthermore, $\widetilde{\alpha}$ is injective.

Proof: Define $p_\mp : V \to \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ by $p_-(v) := \frac{1}{2}(v + iJv)$ and $p_+(v) := \frac{1}{2}(v - iJv)$. Since $p_-(Jv) = -ip_-(v)$, the map $\mathbf{a} = p_-$ is anti-linear. Because of $\mathbf{a}(v) - \mathbf{a}(v)^* = p_-(v) + p_+(v) = v$, the C*-subalgebra of $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ generated by the image of $\mathbf{a}$ contains V. Hence $\mathbf{a}(V)$ generates $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ as a C*-algebra. Axiom (i) in Definition 6.5.1 is proved.
Let $v_1, v_2 \in V$, then

$$\begin{aligned} \{\mathbf{a}(v_1), \mathbf{a}(v_2)\} &= p_-(v_1) \cdot p_-(v_2) + p_-(v_2) \cdot p_-(v_1) \\ &= -2q_{\mathbb{C}}(p_-(v_1), p_-(v_2)) \cdot 1 \\ &= 0, \end{aligned}$$

which is Axiom (ii) in Definition 6.5.1. Furthermore,

$$\begin{aligned}
\{\mathbf{a}(v_1)^*, \mathbf{a}(v_2)\} &= -p_+(v_1)\cdot p_-(v_2) - p_-(v_2)\cdot p_+(v_1) \\
&= 2q_{\mathbb{C}}(p_+(v_1), p_-(v_2))\cdot 1 \\
&= \mathfrak{Re}(v_1, v_2)\cdot 1 + i\mathfrak{Re}(v_1, Jv_2)\cdot 1 \\
&= (v_1, v_2)\cdot 1,
\end{aligned}$$

which shows Axiom (iii) in Definition 6.5.1. Therefore $(\mathbf{a}, \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}))$ is a CAR-representation of $(V, (\cdot,\cdot))$.
The second part of the theorem follows from $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ being simple, i.e., from the non-existence of non-trivial closed two-sided $*$-invariant ideals, see [E31, Thm. 1.2.2]. Let $\widehat{\mathbf{a}} : V \to \widehat{A}$ be any other anti-linear map satisfying (ii) and (iii) in Definition 6.5.1. Since $\mathbf{a}$ and $\widehat{\mathbf{a}}$ are injective (which is clear by Axiom (iii)) one may set $\alpha(\mathbf{a}(v)) := \widehat{\mathbf{a}}(v)$ for all $v \in V$. Axioms (ii) and (iii) allow us to extend α to a C*-morphism $\widetilde{\alpha} : C^*(\mathbf{a}(V)) = \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \to \widehat{A}$. The injectivity of $\widehat{\mathbf{a}}$ implies the non-triviality of $\widetilde{\alpha}$ which, together with the simplicity of $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$, provides the injectivity of $\widetilde{\alpha}$. Therefore we found an injective C*-morphism $\widetilde{\alpha} : \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \to \widehat{A}$ with $\widetilde{\alpha}\circ\mathbf{a} = \widehat{\mathbf{a}}$. It is unique since it is determined by $\mathbf{a}$ and $\widehat{\mathbf{a}}$ on a subset of generators. This concludes the proof of Theorem 6.5.3. □

For an alternative description of the CAR-representation in terms of creation and annihilation operators on the fermionic Fock space we refer to [E9, Prop. 5.2.2].

Corollary 6.5.4 *For every complex pre-Hilbert space $(V, (\cdot,\cdot))$ there exists a* CAR-*representation of $(V, (\cdot,\cdot))$, unique up to C*-isomorphism.*

Proof: The existence has already been proved in Theorem 6.5.3. Let $(\widehat{\mathbf{a}}, \widehat{A})$ be any CAR-representation of $(V, (\cdot,\cdot))$. Theorem 6.5.3 states the existence of a unique injective C*-morphism $\widetilde{\alpha} : \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \to \widehat{A}$ such that $\widetilde{\alpha}\circ\mathbf{a} = \widehat{\mathbf{a}}$. Now $\widetilde{\alpha}$ has to be surjective since Axiom (i) holds for $(\widehat{\mathbf{a}}, \widehat{A})$. □

From now on, given a complex pre-Hilbert space $(V, (\cdot,\cdot))$, we denote the C*-algebra $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ associated with the CAR-representation $(\mathbf{a}, \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}))$ of $(V, (\cdot,\cdot))$ by $\mathrm{CAR}(V, (\cdot,\cdot))$. We list the properties of CAR-representations which are relevant for quantization, see also [E9, Vol. II, Thm. 5.2.5, p. 15].

Proposition 6.5.5 *Let $(V, (\cdot,\cdot))$ be a complex pre-Hilbert space and $(\mathbf{a}, \mathrm{CAR}(V, (\cdot,\cdot)))$ its* CAR-*representation.*

(i) For every $v \in V$ one has $\|\mathbf{a}(v)\| = |v| = (v,v)^{\frac{1}{2}}$, where $\|\cdot\|$ denotes the C*-norm on $\mathrm{CAR}(V, (\cdot,\cdot))$.

(ii) The C*-algebra $\mathrm{CAR}(V, (\cdot,\cdot))$ is simple, i.e., it has no closed two-sided $*$-ideals other than $\{0\}$ and the algebra itself.

(iii) The algebra $\mathrm{CAR}(V, (\cdot,\cdot))$ is $\mathbb{Z}_2$-graded,

$$\mathrm{CAR}(V, (\cdot,\cdot)) = \mathrm{CAR}^{\mathrm{even}}(V, (\cdot,\cdot)) \oplus \mathrm{CAR}^{\mathrm{odd}}(V, (\cdot,\cdot)),$$

and $\mathbf{a}(V) \subset \mathrm{CAR}^{\mathrm{odd}}(V, (\cdot,\cdot))$.

(iv) Let $f : V \to V'$ be an isometric linear embedding, where $(V', (\cdot\,,\cdot)')$ is another complex pre-Hilbert space. Then there exists a unique injective C^*-morphism $\mathrm{CAR}(f) : \mathrm{CAR}(V, (\cdot\,,\cdot)) \to \mathrm{CAR}(V', (\cdot\,,\cdot)')$ such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\mathbf{a}"] & V' \arrow[d, "\mathbf{a}'"] \\ \mathrm{CAR}(V,(\cdot\,,\cdot)) \arrow[r, "\mathrm{CAR}(f)"] & \mathrm{CAR}(V',(\cdot\,,\cdot)') \end{tikzcd}$$

commutes.

Proof: We show assertion (i) . On the one hand, the C*-property of the norm $\|\cdot\|$ implies

$$\begin{aligned} \|\mathbf{a}(v)\|^4 &= \|\mathbf{a}(v)\mathbf{a}(v)^*\|^2 \\ &= \|(\mathbf{a}(v)\mathbf{a}(v)^*)^2\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\mathbf{a}(v)\mathbf{a}(v)^*)^2 &= \mathbf{a}(v)\{\mathbf{a}(v)^*, \mathbf{a}(v)\}\mathbf{a}(v)^* \\ &= |v|^2\mathbf{a}(v)\mathbf{a}(v)^*, \end{aligned}$$

where we used $\mathbf{a}(v)^2 = 0$ which follows from the second axiom. We deduce that

$$\begin{aligned} \|\mathbf{a}(v)\|^4 &= |v|^2 \cdot \|\mathbf{a}(v)\mathbf{a}(v)^*\| \\ &= |v|^2 \cdot \|\mathbf{a}(v)\|^2. \end{aligned}$$

Since $\mathbf{a}$ is injective, we obtain the result.
Assertion (ii) follows from $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ being simple, see [E31, Thm. 1.2.2]. Alternatively, it can be deduced from the universal property formulated in Theorem 6.5.3.
To see (iii) we recall that the Clifford algebra $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ has a $\mathbb{Z}_2$-grading where the even part is generated by products of an even number of vectors in $V_{\mathbb{C}}$ and, similarly, the odd part is the vector space span of products of an odd number of vectors in $V_{\mathbb{C}}$, see [E31, p. 27]. This is compatible with the Clifford relations (6.15). Clearly, $\mathbf{a}(V) \subset \mathrm{CAR}^{\mathrm{odd}}(V, (\cdot\,,\cdot))$.
It remains to show (iv). It is straightforward to check that $\mathbf{a}' \circ f$ satisfies Axioms (ii) and (iii) in Definition 6.5.1. The result follows from Theorem 6.5.3. □

One easily sees that $\mathrm{CAR}(\mathrm{id}) = \mathrm{id}$ and that $\mathrm{CAR}(f' \circ f) = \mathrm{CAR}(f') \circ \mathrm{CAR}(f)$ for all isometric linear embeddings $V \xrightarrow{f} V' \xrightarrow{f'} V''$. Therefore we have constructed a covariant functor

$$\mathrm{CAR} : \mathsf{HILB} \longrightarrow \mathsf{C^*Alg},$$

where HILB denotes the category whose objects are the complex pre-Hilbert spaces and whose morphisms are the isometric linear embeddings.
For *real* pre-Hilbert spaces there is the concept of *self-dual* CAR-representations.

Definition 6.5.6 *A* self-dual CAR-representation *of a real pre-Hilbert space* $(V, (\cdot\,,\cdot))$ *is a pair* $(\mathbf{b}, A)$*, where A is a unital C^*-algebra and $\mathbf{b} : V \to A$ is an $\mathbb{R}$-linear map satisfying:*

(i) $A = C^*(\mathbf{b}(V))$,

(ii) $\mathbf{b}(v) = \mathbf{b}(v)^*$ *and*

(iii) $\{\mathbf{b}(v_1), \mathbf{b}(v_2)\} = (v_1, v_2) \cdot 1$,

for all $v, v_1, v_2 \in V$.

Given a self-dual CAR-representation, one can extend $\mathbf{b}$ to a $\mathbb{C}$-linear map from the complexification $V_{\mathbb{C}}$ to A. This extension $\mathbf{b} : V_{\mathbb{C}} \to A$ then satisfies $\mathbf{b}(\overline{v}) = \mathbf{b}(v)^*$ and $\{\mathbf{b}(v_1), \mathbf{b}(v_2)\} = (v_1, \overline{v_2}) \cdot 1$ for all $v, v_1, v_2 \in V_{\mathbb{C}}$. These are the axioms of a self-dual CAR-representation as in [E1, p. 386].

Theorem 6.5.7 *For every real pre-Hilbert space* $(V, (\cdot\,,\cdot))$, *the* C^**-Clifford algebra* $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ *provides a self-dual* CAR*-representation of* $(V, (\cdot\,,\cdot))$ *via* $\mathbf{b}(v) = \frac{i}{\sqrt{2}} v$.
Moreover, self-dual CAR*-representations have the following universal property: Let* $\widehat{A}$ *be any unital* C^**-algebra and* $\widehat{\mathbf{b}} : V \to \widehat{A}$ *be any* $\mathbb{R}$*-linear map satisfying Axioms* (ii) *and* (iii) *of Definition 6.5.6. Then there exists a unique* C^**-morphism* $\widetilde{\beta} : \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \to \widehat{A}$ *such that*

$$\begin{tikzcd} V \arrow[r, "\widehat{\mathbf{b}}"] \arrow[d, "\mathbf{b}"'] & \widehat{A} \\ \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}) \arrow[ur, dotted, "\widetilde{\beta}"] & \end{tikzcd}$$

commutes. Furthermore, $\widetilde{\beta}$ *is injective.*

Corollary 6.5.8 *For every real pre-Hilbert space* $(V, (\cdot\,,\cdot))$ *there exists a* CAR*-representation of* $(V, (\cdot\,,\cdot))$, *unique up to* C^**-isomorphism.*

From now on, given a real pre-Hilbert space $(V, (\cdot\,,\cdot))$, we denote the C*-algebra $\mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}})$ associated with the self-dual CAR-representation $(\mathbf{b}, \mathrm{Cl}(V_{\mathbb{C}}, q_{\mathbb{C}}))$ of $(V, (\cdot\,,\cdot))$ by $\mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot))$.

Proposition 6.5.9 *Let* $(V, (\cdot\,,\cdot))$ *be a real pre-Hilbert space and* $(\mathbf{b}, \mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot)))$ *its self-dual* CAR*-representation.*

(i) *For every* $v \in V$ *one has* $\|\mathbf{b}(v)\| = \frac{1}{\sqrt{2}}|v|$, *where* $\|\cdot\|$ *denotes the* C^**-norm on* $\mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot))$.

(ii) *The* C^**-algebra* $\mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot))$ *is simple.*

(iii) *The algebra* $\mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot))$ *is* $\mathbb{Z}_2$*-graded,*

$$\mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot)) = \mathrm{CAR}_{\mathrm{sd}}^{\mathrm{even}}(V, (\cdot\,,\cdot)) \oplus \mathrm{CAR}_{\mathrm{sd}}^{\mathrm{odd}}(V, (\cdot\,,\cdot)),$$

and $\mathbf{b}(V) \subset \mathrm{CAR}_{\mathrm{sd}}^{\mathrm{odd}}(V, (\cdot\,,\cdot))$.

(iv) *Let* $f : V \to V'$ *be an isometric linear embedding, where* $(V', (\cdot\,,\cdot)')$ *is another real pre-Hilbert space. Then there exists a unique injective* C^**-morphism* $\mathrm{CAR}_{\mathrm{sd}}(f) : \mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot)) \to \mathrm{CAR}_{\mathrm{sd}}(V', (\cdot\,,\cdot)')$ *such that*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\mathbf{b}"'] & V' \arrow[d, "\mathbf{b}'"] \\ \mathrm{CAR}_{\mathrm{sd}}(V, (\cdot\,,\cdot)) \arrow[r, "\mathrm{CAR}_{\mathrm{sd}}(f)"] & \mathrm{CAR}_{\mathrm{sd}}(V', (\cdot\,,\cdot)') \end{tikzcd}$$

commutes.

The proofs are similar to the ones for CAR-representations of complex pre-Hilbert spaces. We have constructed a functor

$$\mathsf{CAR}_{\mathrm{sd}} : \mathsf{HILB}_{\mathbb{R}} \longrightarrow \mathsf{C^*Alg},$$

where $\mathsf{HILB}_{\mathbb{R}}$ denotes the category whose objects are the real pre-Hilbert spaces and whose morphisms are the isometric linear embeddings.

Note 6.5.10 Let $(V,(\cdot\,,\cdot))$ be a complex pre-Hilbert space. If we consider V as a real vector space, then we have the real pre-Hilbert space $(V,\mathfrak{Re}(\cdot\,,\cdot))$. For the corresponding CAR-representations we have

$$\mathrm{CAR}(V,(\cdot\,,\cdot)) = \mathrm{CAR}_{\mathrm{sd}}(V,\mathfrak{Re}(\cdot\,,\cdot)) = \mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$$

and

$$\mathbf{b}(v) = \frac{i}{\sqrt{2}}(\mathbf{a}(v) - \mathbf{a}(v)^*).$$

6.5.2 CCR algebras

In this section, we recall the construction of the representation of any (real) symplectic vector space by the so-called canonical commutation relations (CCR). Proofs can be found in [E4, Sec. 4.2].

Definition 6.5.11 *A* CCR-representation *of a symplectic vector space* (V,ω) *is a pair* (w,A)*, where* A *is a unital* C^**-algebra and* w *is a map* $V \to A$ *satisfying:*

(i) $A = C^*(w(V))$,

(ii) $w(0) = 1$,

(iii) $w(-\phi) = w(\phi)^*$,

(iv) $w(\phi+\psi) = e^{i\omega(\phi,\psi)/2}w(\phi)\cdot w(\psi)$,

for all $\phi,\psi \in V$.

The map w is in general neither linear, nor any kind of group homomorphism, nor continuous [E4, Prop. 4.2.3].

Example 6.5.12 Given any symplectic vector space (V,ω), consider the Hilbert space $H := L^2(V,\mathbb{C})$, where V is endowed with the counting measure. Define the map w from V into the space $\mathscr{L}(H)$ of bounded endomorphisms of H by

$$(w(\phi)F)(\psi) := e^{i\omega(\phi,\psi)/2}F(\phi+\psi),$$

for all $\phi,\psi \in V$ and $F \in H$. It is well-known that $\mathscr{L}(H)$ is a C*-algebra with the operator norm as C*-norm, and that the map w satisfies the Axioms (ii)-(iv) from Definition 6.5.11, see e.g. [E4, Ex. 4.2.2]. Hence setting $A := C^*(w(V))$, the pair (w,A) provides a CCR-representation of (V,ω).

This is essentially the only example of CCR-representation:

Theorem 6.5.13 *Let (V,ω) be a symplectic vector space and $(\hat{w},\widehat{A})$ be a pair satisfying the Axioms* (ii)-(iv) *of Definition 6.5.11. Then there exists a unique C^*-morphism $\Phi : A \to \widehat{A}$ such that $\Phi \circ w = \hat{w}$, where (w,A) is the* CCR*-representation from Example 6.5.12. Moreover, Φ is injective.*
In particular, (V,ω) has a CCR*-representation, unique up to C^*-isomorphism.*

We denote the C*-algebra associated to the CCR-representation of (V,ω) from Example 6.5.12 by $\mathrm{CCR}(V,\omega)$. As a consequence of Theorem 6.5.13, we obtain the following important corollary.

Corollary 6.5.14 *Let (V,ω) be a symplectic vector space and $(w,\mathrm{CCR}(V,\omega))$ its* CCR*-representation.*

(i) The C^*-algebra $\mathrm{CCR}(V,\omega)$ is simple, i.e., it has no closed two-sided $*$-ideals other than $\{0\}$ and the algebra itself.

(ii) Let (V',ω') be another symplectic vector space and $f : V \to V'$ a symplectic linear map. Then there exists a unique injective C^*-morphism $\mathrm{CCR}(f) : \mathrm{CCR}(V,\omega) \to \mathrm{CCR}(V',\omega')$ such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "w"] & V' \arrow[d, "w'"] \\ \mathrm{CCR}(V,\omega) \arrow[r, "\mathrm{CCR}(f)"] & \mathrm{CCR}(V',\omega') \end{tikzcd}$$

commutes.

Obviously $\mathrm{CCR}(\mathrm{id}) = \mathrm{id}$ and $\mathrm{CCR}(f' \circ f) = \mathrm{CCR}(f') \circ \mathrm{CCR}(f)$ for all symplectic linear maps $V \xrightarrow{f} V' \xrightarrow{f'} V''$, so that we have constructed a covariant functor

$$\mathrm{CCR} : \mathsf{Sympl} \longrightarrow \mathsf{C^*Alg}.$$

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