Penrose's singularity theorem

Nicolas Ginoux

Seminar on general relativity theory - University of Regensburg

November 5, 2013

Abstract: We discuss Penrose's singularity theorem as stated and proved in [1, Sec. 2.9], which itself closely follows [4, Ch. 14].

1 Main statement

Theorem 1.1 (Penrose's singularity theorem [4, Thm. 14.61])
Let (M^n, g) *be a spacetime with*

- noncompact *Cauchy hypersurface,*
- $\mathrm{ric}_g(X, X) \geq 0$ *for all* $X \in TM$ lightlike*,*
- *a nonempty* compact achronal spacelike *(embedded) submanifold* N^{n-2} *with* past-oriented timelike *mean curvature vector field.*[1]

Then (M^n, g) *is* not *future lightlike geodesically complete, i.e., there exists a noncomplete future-directed lightlike geodesic in* (M^n, g)*.*

Proof: We argue by contradiction and assume that (M^n, g) were future lightlike geodesically complete. Up to restricting ourselves to a connected component of M^n containing a connected component of N, we may assume that M itself is connected.
Claim 1: *The subset* $\partial I_+(N)$ *is a nonempty compact achronal topological hypersurface of* (M^n, g)*.*
Proof of Claim 1: Since M^n has a Cauchy hypersurface, it is globally hyperbolic, see e.g. [2, Sec. 3.1]. As a consequence, all subsets of the form $J_\pm(x)$, $x \in M$, are closed in M and so is $J_+(N)$ since N is compact (see also [2,

[1] Such a submanifold is called *strictly trapped* by M. Kriele, see [3, Def. 9.2.1].

Sec. 3.1]). Therefore, because of $I_+(N) \subset J_+(N) \subset \overline{I_+(N)}$ (true in general) we obtain $J_+(N) = \overline{I_+(N)}$ and thus $\partial I_+(N) = J_+(N) \setminus I_+(N)$ by the fact that the chronological future $I_+(N)$ is open in M. Proposition 1.2 applied to the (nonempty) future set $I_+(N)$ implies that $\partial I_+(N)$ is a closed (as a subset) achronal topological hypersurface of M; note that $I_+(N) = M$ cannot occur since otherwise it would contradict the achronality of N. By Proposition 1.3 below, we also deduce that $\partial I_+(N)$ is compact. It remains to show that $\partial I_+(N) \neq \varnothing$: but if $\partial I_+(N) = \varnothing$, then $I_+(N) = J_+(N)$ would hold, in particular the nonempty open and closed subset $I_+(N)$ must coincide with M since M is connected; again, this would contradict the achronality of N. $\surd$

Claim 2: *The hypersurface $\partial I_+(N)$ is homeomorphic to a (hence any) Cauchy hypersurface of (M^n, g).*

Proof of Claim 2: Let S be any topological Cauchy hypersurface of (M^n, g). Note that, M being assumed to be connected, so is S. Let X be a complete smooth (future-oriented) timelike vector field whose integral curves are inextendible[2]; for instance, pick any *complete* Riemannian metric h on M (there is always one), any smooth timelike vector field $\widetilde{X}$ on M and set $X := \frac{\widetilde{X}}{\sqrt{h(\widetilde{X},\widetilde{X})}}$.
We show that the flow of X provides the desired homeomorphism. Namely denote by $\phi^X : \mathbb{R} \times M \to M$ the flow of X and consider the map

$$\begin{array}{rcl} \rho : \partial I_+(N) & \longrightarrow & S \\ x & \longmapsto & y \text{ where } \phi^X(\mathbb{R}, x) \cap S = \{y\}. \end{array}$$

Note that the map ρ is well-defined since the timelike curve $t \mapsto \phi^X(t, x)$ is inextendible and S is a Cauchy hypersurface, hence is met exactly once by that curve. By definition, ρ can be written as the composition $p_2 \circ \psi^{-1}$, where $p_2 : \mathbb{R} \times S \to S$ is the projection onto the second factor and $\psi : \mathbb{R} \times S \to M$, $(t, x) \mapsto \phi^X(t, x)$ is a homeomorphism, see e.g. [1, Satz 2.5.13]; in particular, ρ is continuous. Since $\partial I_+(N)$ is achronal by Claim 1, ρ is injective. By compactness of $\partial I_+(N)$, $\rho : \partial I_+(N) \to \rho(\partial I_+(N))$ is a homeomorphism, hence $\rho(\partial I_+(N))$ is an $n-1$-dimensional topological manifold. Brouwer's theorem about the invariance of the domain (see e.g. [5, Thm 1.31]) implies that $\rho(\partial I_+(N))$ must be open in S[3]. It follows that $\rho(\partial I_+(N))$, being closed, open in S and nonempty (Claim 1), must coincide with S. Therefore, $\rho : \partial I_+(N) \to S$ is a homeomorphism. $\surd$

We obtain a contradiction since $\partial I_+(N)$ is compact whereas no Cauchy hypersurface of (M^n, g) is by assumption. Therefore, (M^n, g) is not future lightlike geodesically complete. $\square$

[2]as curves, which is not guaranteed by the fact that X is complete!

[3]This is a *nontrivial* statement!

Proposition 1.2 *Let (M^n, g) be a connected spacetime and $\varnothing \neq A \subsetneq M$ be any future set (i.e., $I_+(A) \subset A$). Then ∂A is an achronal closed (as a subset) topological hypersurface of M^n.*

Proof: See e.g. [1, Kor. 2.5.8]. $\square$

As emphasized in [4, Ch. 14], the main idea in the proof of Theorem 1.1 consists in turning the *differential geometric* condition for N to be (strictly) trapped into a *causal* condition:

Proposition 1.3 *Let (M^n, g) be a future lightlike geodesically complete spacetime satisfying $\mathrm{ric}_g(X, X) \geq 0$ for all $X \in TM$ lightlike. Let N^{n-2} be a compact achronal spacelike submanifold with past-oriented timelike mean curvature vector field. Then N is* future-trapped*, i.e., $J_+(N) \setminus I_+(N)$ is compact.*

Proof: Fix an arbitrary Riemannian metric h on M and consider the submanifold

$$\widehat{N} := \left\{X \in T^\perp N,\ X \text{ future-oriented lightlike with } h(X, X) = 1\right\}$$

of $T^\perp N$. Since N has codimension 2 in M, there are at each point $x \in N$ exactly two future-oriented lightlike vectors with unit h-length in $T_x^\perp N$, hence the restriction of the projection map $\pi : T^\perp N \to N$ to $\widehat{N}$ is a two-fold covering map $\widehat{N} \to N$. In particular, $\widehat{N}$ is compact since N is. Now the map $T^\perp N \to \mathbb{R}$, $X \mapsto g(H(\pi(X)), X)$, is continuous and positive on $\widehat{N}$ (for any $v \in \mathbb{R}^m$ past-oriented timelike and $w \in J_+(0)$, one has $\langle\langle v, w\rangle\rangle > 0$), therefore there exists a $b > 0$ with

$$g(H(\pi(X)), X) \geq \frac{1}{b} \qquad \forall\, X \in \widehat{N}.$$

Because M is assumed to be future lightlike geodesically complete, the geodesic $t \mapsto \exp(tX)$ is defined on $[0, \infty[$ for any $X \in TM$ future-oriented lightlike; in particular, for any $X \in \widehat{N}$, the geodesic $c_X : t \mapsto \exp(tX)$ is defined on $[0, b]$. By Lemma 1.4 below, c_X must have a focal point in $]0, b]$. Pick any point $q \in J_+(N) \setminus I_+(N)$. By Theorem 1.5 below, there exists a (future-oriented) lightlike geodesic c in M with $c(0) \in N$, $\dot{c}(0) \in T_{c(0)}^\perp N$, (w.l.o.g.) $h(\dot{c}(0), \dot{c}(0)) = 1$ and *without any focal point* between $c(0)$ and q. By uniqueness of geodesics, we have $c = c_X$ for $X = \dot{c}(0) \in \widehat{N}$; by the preceding argument, if $t_0 \in [0, \infty[$ is such that $c(t_0) = q$, then $t_0 \leq b$. It follows that

$$J_+(N) \setminus I_+(N) \subset \exp(K),$$

where $K := \left\{tX,\, X \in \widehat{N} \text{ and } t \in [0,b]\right\} \subset T^\perp N$. Since K is compact, so is $\exp(K)$. It remains to prove that $J_+(N) \setminus I_+(N)$ is closed in $\exp(K)$. But this follows from $\exp(K) \subset J_+(N)$ by definition of K: if $(q_m)_m$ is any sequence of $J_+(N) \setminus I_+(N)$ converging to some $q \in \exp(K)$, then $q \in J_+(N)$ by $\exp(K) \subset J_+(N)$, and on the other hand $q \notin I_+(N)$ since $I_+(N)$ is open. On the whole, $J_+(N) \setminus I_+(N)$ is compact, QED. □

The central argument in the proof of Theorem 1.1 is hence reduced to the following lemma.

Lemma 1.4 *Let (M^n, g) be any $n(\geq 3)$-dimensional Lorentzian manifold, $N^{n-2} \subset M$ be any spacelike submanifold and $c : [0,b] \to M$ be any lightlike geodesic with $c(0) = x \in N$ and $\dot{c}(0) \in T_x^\perp N$, for some $b > 0$. Assume furthermore that*

i) $\mathrm{ric}_g(\dot{c}(t), \dot{c}(t)) \geq 0$ *for all* $t \in [0,b]$ *and*

ii) $g(H(x), \dot{c}(0)) \geq \frac{1}{b}$*, where* $H := \frac{1}{n-2}\mathrm{tr}_g(\mathrm{I\!I})$ *is the mean curvature vector of* N *in* M.

Then c has a focal point in $]0,b]$.

Proof: First recall that a focal point is a point along the geodesic c for which nontrivial *infinitesimal*[4] geodesic deformations of c exist fixing that point while keeping the deformed geodesic normal to the submanifold N. In other words, call $c(t_0)$ ($t_0 \in]0,b]$) a focal point iff the space

$$\left\{J \text{ Jacobi v.f. along } c,\ J(0) \in T_{c(0)}N, \left(\frac{\nabla^M J}{dt}(0)\right)^T = -\mathrm{I\!I}^*(J(0), \dot{c}(0)) \text{ and } J(t_0) = 0\right\}$$

is nonzero, where $\mathrm{I\!I}^*(X, \nu) \in TN$ is defined by $g(\mathrm{I\!I}^*(X,\nu), Y) = g(\mathrm{I\!I}(X,Y), \nu)$ for all $X, Y \in T_xN$ and $\nu \in T_x^\perp N$.

We argue by contradiction and assume c had no focal point in $]0,b]$. Fix an orthonormal basis $\{e_1, \ldots, e_{n-2}\}$ of T_xN and consider the Jacobi vector fields $J_1, \ldots, J_{n-2}$ along c with $J_j(0) = e_j$ as well as $\frac{\nabla J_j}{dt}(0)^T = -\mathrm{I\!I}^*(e_j, \dot{c}(0))$. Let $J_0(t) := t\dot{c}(t)$ be the Jacobi vector field along c with $J_0(0) = 0$ and $\frac{\nabla J_0}{dt}(0) = \dot{c}(0)$.

Claim A: *For all $t \in]0,b]$, the vectors $J_0(t), J_1(t), \ldots, J_{n-2}(t)$ form a basis of $\dot{c}(t)^\perp \subset T_{c(t)}M$.*

Proof of Claim A: Since c is lightlike, we have $g(J_0(t), \dot{c}(t)) = 0$ for all $t \in$

[4]Thank you Bernd.

$[0,b]$. Moreover, because $J_1,\ldots,J_{n-2}$ are Jacobi vector fields along c, we have $\frac{d^2}{dt^2}g(J_j,\dot{c}) = -g(R_{J_j,\dot{c}}\dot{c},\dot{c}) = 0$ on $[0,b]$; but by $g(J_j(0),\dot{c}(0)) = g(e_j,\dot{c}(0)) = 0$ and $\frac{d}{dt}(g(J_j,\dot{c}))(0) = -g(\mathrm{I\!I}^*(e_j,\dot{c}(0)),\dot{c}(0)) = 0$ (since $\dot{c}(0) \in T_{c(0)}^{\perp}N$), we deduce that $g(J_j,\dot{c}) = 0$ on $[0,b]$, for all $1 \leq j \leq n-2$. This shows that $J_j(t) \in \dot{c}(t)^{\perp}$ for all $0 \leq j \leq n-2$.
If $J_0(t), J_1(t),\ldots,J_{n-2}(t)$ were linearly dependent for some $t \in]0,b]$, then there would exist $(\lambda_0,\lambda_1,\ldots,\lambda_{n-2}) \in \mathbb{R}^{n-1}\setminus\{0\}$ with $\sum_{j=0}^{n-2}\lambda_j J_j(t) = 0$. The Jacobi vector field $J := \sum_{j=0}^{n-2}\lambda_j J_j$ would satisfy $J(t) = 0$, $J(0) = \sum_{j=1}^{n-2}\lambda_j e_j \in T_{c(0)}N \setminus \{0\}$ as well as

$$\frac{\nabla J}{dt}(0)^T = \lambda_0 \underbrace{\dot{c}(0)^T}_{0} + \sum_{j=1}^{n-2}\lambda_j \frac{\nabla J_j}{dt}(0)^T = -\mathrm{I\!I}^*(\sum_{j=1}^{n-2}\lambda_j e_j, \dot{c}(0)) = -\mathrm{I\!I}^*(J(0),\dot{c}(0)).$$

This would imply that t is a focal point for c in $]0,b]$, contradiction to the assumption. Therefore, $\{J_0(t),J_1(t),\ldots,J_{n-2}(t)\}$ is a basis of $\dot{c}(t)^{\perp}$. $\surd$
Claim B: *For all* $0 \leq i,j \leq n-2$ *and* $t \in [0,b]$, *we have* $g(\frac{\nabla J_i}{dt},J_j) = g(J_i,\frac{\nabla J_j}{dt})$.
Proof of Claim B: We already know that, since J_i and J_j are Jacobi vector fields, there exists a constant $\lambda_{ij} \in \mathbb{R}$ with $g(\frac{\nabla J_i}{dt},J_j) - g(J_i,\frac{\nabla J_j}{dt}) = \lambda_{ij}$ on $[0,b]$ (the derivative of the l.h.s. vanishes because of $g(R_{X,Y}Z,T) = g(R_{Z,T}X,Y)$). For $1 \leq i,j \leq n-2$, we have

$$\begin{aligned} g(\frac{\nabla J_i}{dt}(0),J_j(0)) - g(J_i(0),\frac{\nabla J_j}{dt}(0)) &= -g(\mathrm{I\!I}^*(e_i,\dot{c}(0)),e_j) + g(\mathrm{I\!I}^*(e_j,\dot{c}(0)),e_i) \\ &= -g(\mathrm{I\!I}(e_i,e_j),\dot{c}(0)) + g(\mathrm{I\!I}(e_j,e_i),\dot{c}(0)) \\ &= 0 \end{aligned}$$

and for $i = 0$, $1 \leq j \leq n-2$, we have

$$g(\frac{\nabla J_0}{dt}(0),J_j(0)) - g(J_0(0),\frac{\nabla J_j}{dt}(0)) = g(\dot{c}(0),e_j) - 0 = 0.$$

On the whole, the constant λ_{ij} vanishes, QED. $\surd$
Claim C: *Let* $V \in \Gamma(c^*TM)$ *be any vector field along* c *with* $V(0) \in T_{c(0)}N$, $V(b) = 0$ *as well as* $V(t) \in \dot{c}(t)^{\perp}$ *for all* $t \in [0,b]$. *Then*

$$\int_0^b \left\{ g(\frac{\nabla V}{dt},\frac{\nabla V}{dt}) - g(R_{\dot{c},V}V,\dot{c}) \right\} dt \geq g(\mathrm{I\!I}(V(0),V(0)),\dot{c}(0)),$$

with equality iff $V(t)$ *is proportional to* $\dot{c}(t)$ *for all* $t \in [0,b]$.
Proof of Claim C: By Claim A, there exist smooth functions $f_0,\ldots,f_{n-2}$

on $]0,b]$ with $V = \sum_{j=0}^{n-2} f_j J_j$ on $]0,b]$. The functions $tf_0, f_1, \ldots, f_{n-2}$ can be even smoothly extended onto $[0,b]$ with $\lim\limits_{t\to 0^+} tf_0(t) = 0$ since $V(0) \in T_{c(0)}N = \mathrm{Span}(e_1, \ldots, e_{n-2})$. Let $X := \sum_{j=0}^{n-2} f_j' J_j$ and $Y := \sum_{j=0}^{n-2} f_j \frac{\nabla J_j}{dt}$ on $]0,b]$. Then $\frac{\nabla V}{dt} = X + Y$ with $g(X,X) \geq 0$ on $]0,b]$ (for $X \perp \dot{c}$ and $\dot{c}$ is lightlike). Now $g(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}) - g(R_{\dot{c},V}V, \dot{c}) = \frac{d}{dt}\left(g(V, \frac{\nabla V}{dt})\right) - g(\frac{\nabla^2 V}{dt^2} - R_{\dot{c},V}\dot{c}, V)$, with

$$\begin{aligned}
\frac{\nabla^2 V}{dt^2} - R_{\dot{c},V}\dot{c} &= \sum_{j=0}^{n-2} f_j'' J_j + 2f_j' \frac{\nabla J_j}{dt} + f_j \frac{\nabla^2 J_j}{dt^2} - f_j R_{\dot{c},J_j}\dot{c} \\
&= \sum_{j=0}^{n-2} f_j'' J_j + 2f_j' \frac{\nabla J_j}{dt} \qquad (J_j = \text{ Jacobi vector field}),
\end{aligned}$$

so that

$$\begin{aligned}
g(\frac{\nabla^2 V}{dt^2} - R_{\dot{c},V}\dot{c}, V) &= \sum_{i,j=0}^{n-2} f_i'' f_j g(J_i, J_j) + 2f_i' f_j g(\frac{\nabla J_i}{dt}, J_j) \\
&\overset{\text{(ClaimB)}}{=} \sum_{i,j=0}^{n-2} (f_i' f_j g(J_i, J_j))' - f_i' f_j' g(J_i, J_j) - 2f_i' f_j g(\frac{\nabla J_i}{dt}, J_j) \\
&\qquad + 2f_i' f_j g(\frac{\nabla J_i}{dt}, J_j) \\
&= \sum_{i,j=0}^{n-2} (f_i' f_j g(J_i, J_j))' - g(X,X).
\end{aligned}$$

Fixing $\varepsilon \in]0,b[$ and integrating, we obtain

$$\begin{aligned}
\int_\varepsilon^b \left\{ g(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}) - g(R_{\dot{c},V}V, \dot{c}) \right\} dt &= \left[g(V, \frac{\nabla V}{dt}) - \sum_{i,j=0}^{n-2} f_i' f_j g(J_i, J_j) \right]_\varepsilon^b \\
&\quad + \int_\varepsilon^b g(X,X) dt,
\end{aligned}$$

with

$$\begin{aligned}
g(V, \frac{\nabla V}{dt}) - \sum_{i,j=0}^{n-2} f_i' f_j g(J_i, J_j) &= \sum_{i,j=0}^{n-2} f_i f_j' g(J_i, J_j) + f_i f_j g(J_i, \frac{\nabla J_j}{dt}) - f_i' f_j g(J_i, J_j) \\
&= g(V, Y),
\end{aligned}$$

so that

$$\begin{aligned}\int_\varepsilon^b \left\{ g(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}) - g(R_{\dot c, V}V, \dot c)\right\} dt &= [g(V,Y)]_\varepsilon^b + \int_\varepsilon^b g(X,X)dt \\ &= -g(V(\varepsilon), Y(\varepsilon)) + \int_\varepsilon^b g(X,X)dt.\end{aligned}$$

Note that, since $g(X,X) \geq 0$ on $]0,b]$, the integral $\int_0^b g(X,X)dt$ is well-defined, nonnegative and $\int_0^b g(X,X)dt = \lim_{\varepsilon\to 0} \int_\varepsilon^b g(X,X)dt$. Moreover, since $g(J_0(t), \frac{\nabla J_0}{dt}(t)) = \frac{1}{2}\frac{d}{dt}\left(g(J_0,J_0)\right)(t) = 0$ for all $t \in [0,b]$, we have by Claim A

$$g(V,Y) = \sum_{j=1}^{n-2} f_0 f_j g(J_0, \frac{\nabla J_j}{dt}) + \sum_{i,j=1}^{n-2} f_i f_j g(J_i, \frac{\nabla J_j}{dt}))$$

on $]0,b]$, so that, using $f_0 J_0 = (tf_0)\dot c$ and $\lim_{t\to 0^+} tf_0(t) = 0$, we obtain

$$\begin{aligned}\lim_{t\to 0^+} g(V(t), Y(t)) &= \sum_{j=1}^{n-2} \lim_{t\to 0^+} (tf_0(t)) f_j(t) g(\dot c, \frac{\nabla J_j}{dt})(t) \\ &\quad + \sum_{i,j=1}^{n-2} f_i(0) f_j(0) g(J_i(0), \frac{\nabla J_j}{dt})(0)) \\ &= -\sum_{i,j=1}^{n-2} f_i(0) f_j(0) g(e_i, \mathrm{II}^*(e_j, \dot c(0))) \\ &= -g(\mathrm{II}(V(0), V(0)), \dot c(0)).\end{aligned}$$

We deduce that

$$\begin{aligned}\int_0^b \left\{ g(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}) - g(R_{\dot c, V}V, \dot c)\right\} dt &= \lim_{\varepsilon\to 0^+} -g(V(\varepsilon), Y(\varepsilon)) + \int_\varepsilon^b g(X,X)dt \\ &= g(\mathrm{II}(V(0), V(0)), \dot c(0)) + \int_0^b g(X,X)dt.\end{aligned}$$

The desired inequality follows from $\int_0^b g(X,X)dt \geq 0$. Moreover, the inequality is an equality iff $g(X,X) = 0$ on $]0,b]$, which is equivalent to X being proportional to $\dot c$. But the latter implies $f_j' = 0$ for all $j \geq 1$ and, because $f_j(b) = 0$, also $f_j = 0$ for all $j \geq 1$; that is, that V is proportional to $\dot c$ (in particular $V(0) = 0$). Conversely, if V is proportional to $\dot c$, then so is X and the equality holds. $\surd$

Let E be any parallel vector field along c with $E(0) \in T_{c(0)}N$ and let

$V(t) := (1 - \frac{t}{b})E(t)$ for all $t \in [0, b]$. Then V is a vector field along c, nonproportional to $\dot{c}$ and with $g(V, \dot{c}) = 0$. Claim C implies

$$\begin{aligned} 0 &< \int_0^b \left\{ g(\frac{\nabla V}{dt}, \frac{\nabla V}{dt}) - g(R_{\dot{c},V}V, \dot{c}) \right\} dt - g(\mathrm{II}(V(0), V(0)), \dot{c}(0)) \\ &= \int_0^b \frac{1}{b^2} g(E(0), E(0)) - (1 - \frac{t}{b})^2 g(R_{\dot{c},E}E, \dot{c}) dt - g(\mathrm{II}(E(0), E(0)), \dot{c}(0)). \end{aligned}$$

At this point we need another remark.
Claim D: *Let $X \in T_xM$ be any lightlike vector on a Lorentzian manifold (M^n, g). Let $\{e_1, \ldots, e_{n-2}\}$ be a spacelike orthonormal family of $X^\perp$. Then*

$$\mathrm{ric}_g(X, X) = \sum_{j=1}^{n-2} g(R_{X,e_j}e_j, X).$$

Proof of Claim D: Since $\{e_1, \ldots, e_{n-2}\}^\perp$ is 2-dimensional and has signature $(1, 1)$, there exists a Lorentzian orthonormal basis $\{e_{n-1}, e_n\}$ of $\{e_1, \ldots, e_{n-2}\}^\perp$ such that $g(e_{n-1}, e_{n-1}) = 1$, $g(e_n, e_n) = -1$ and $e_{n-1} + e_n = \lambda X$ for some $\lambda \in \mathbb{R}^\times$. By definition of Ricci curvature,

$$\mathrm{ric}_g(X, X) = \sum_{j=1}^{n-1} g(R_{X,e_j}e_j, X) - g(R_{X,e_n}e_n, X),$$

where $g(R_{X,e_{n-1}+e_n}e_{n-1}, X) = \lambda g(R_{X,X}e_{n-1}, X) = 0 = g(R_{X,e_{n-1}+e_n}e_n, X)$. As a consequence,

$$g(R_{X,e_{n-1}}e_{n-1}, X) - g(R_{X,e_n}e_n, X) = -g(R_{X,e_n}e_{n-1}, X) + g(R_{X,e_{n-1}}e_n, X) = 0,$$

which gives the claim. $\surd$
If one lets $E := e_j$, $1 \leq j \leq n-2$, and sums the last inequality over j, one obtains by Claim D:

$$0 < \frac{n-2}{b} - \int_0^b (1 - \frac{t}{b})^2 \underbrace{\mathrm{ric}_g(\dot{c}, \dot{c})}_{\geq 0} dt - (n-2) g(H(c(0)), \dot{c}(0)) \leq 0,$$

contradiction. Therefore, c must have a focal point in $]0, b]$. $\square$

Theorem 1.5 *Let (M^n, g) be a spacetime with spacelike (embedded) submanifold N. Let $c : [0, b] \to M$ be any future-oriented causal curve with $c(0) \in N$. Then, unless c is up to reparametrization a lightlike geodesic with*

$\dot{c}(0) \in T^{\perp}N$ *and without any focal point in* $]0, b[$*, there exists arbitrarily closed to* c *a future-oriented* timelike *curve from* N *to* $c(b)$*.*[5]

Proof: See [1, Satz 2.2.12]. □

2 Example

We consider the Schwarzschild spacetime: for some $m \in \mathbb{R}_+^\times$ and $h : \mathbb{R}_+^\times \to]-\infty, 1[$, $r \mapsto 1 - \frac{2m}{r}$, let

$$(M^4, g) := (\mathbb{R} \times (]0, 2m[\cup]2m, \infty[) \times \mathbb{S}^2, -h(r)dt^2 \oplus \frac{1}{h(r)}dr^2 \oplus r^2\langle\cdot\,,\cdot\rangle),$$

where $(\mathbb{S}^2, \langle\cdot\,,\cdot\rangle)$ denotes the standard 2-dimensional sphere of Gauß-curvature 1. Then (M^4, g) is Ricci-flat, globally hyperbolic with $(\mathbb{R} \times \{m\} \times \mathbb{S}^2) \cup (\{0\}\times]2m, \infty[\times\mathbb{S}^2)$ as spacelike Cauchy hypersurface and the mean curvature vector field of the 2-dimensional compact achronal spacelike submanifold $N^2 := \{0\} \times \{r_0\} \times \mathbb{S}^2$ (where $r_0 \in]0, 2m[\cup]2m, \infty[$) of (M^4, g) is given by $H = -\frac{h(r_0)}{r_0}\frac{\partial}{\partial r}$, see [1, Sec. 2.9] for details. Moreover, the lightlike geodesics of the Schwarzschild half-plane $(\mathbb{R} \times (]0, 2m[\cup]2m, \infty[), -h(r)dt^2 \oplus \frac{1}{h(r)}dr^2)$ are the parametrized curves of the form

$$s \mapsto (\varepsilon(as + 2m\ln(|as + b - 2m|)), as + b)$$

for some $\varepsilon \in \{\pm 1\}$ and $(a, b) \in \mathbb{R}^\times \times \mathbb{R}$.

1. *Case* $r_0 > 2m$: Then the vector field $\frac{\partial}{\partial r}$ (and so H) is spacelike, so that Theorem 1.1 cannot be applied. And indeed all future-oriented lightlike geodesics in $(\mathbb{R}\times]2m, \infty[\times\mathbb{S}^2, -h(r)dt^2 \oplus \frac{1}{h(r)}dr^2 \oplus r^2\langle\cdot\,,\cdot\rangle)$ are future-complete.

2. *Case* $0 < r_0 < 2m$: Then H is future-oriented timelike. If the standard time-orientation of $(\mathbb{R}\times]0, 2m[\times\mathbb{S}^2, -h(r)dt^2 \oplus \frac{1}{h(r)}dr^2 \oplus r^2\langle\cdot\,,\cdot\rangle)$ is reversed (i.e., defined by $-\frac{\partial}{\partial r}$), then Theorem 1.1 applies. And indeed there exist past-oriented lightlike geodesics that are not complete: fixing $x_0 \in \mathbb{S}^2$, the curve $s \mapsto (-s + 2m\ln(s + 2m), -s, x_0)$ is a lightlike geodesic which is both past- and future-incomplete[6].

[5] This also applies to the case when $N = \varnothing$: in that case, there is no condition on $c(0)$ and the conclusion obviously holds true!

[6] That curve is defined on $]-2m, 0[$ and cannot be extended beyond $s = 0$, because it runs into the black-hole singularity of the Schwarzschild spacetime.

References

[1] C. Bär, *Lorentzgeometrie*, lecture notes, 2006.

[2] N. Ginoux, *Teilskript zur Vorlesung Lorentzgeometrie*, lecture notes, 2010.

[3] M. Kriele, *Spacetime. Foundations of general relativity and differential geometry*, Lecture Notes in Physics **59**, Springer, 1999.

[4] B. O'Neill, *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics **103**, Academic Press, 1983.

[5] J.W. Vick, *Homology theory. An introduction to algebraic topology*, Pure and Applied Mathematics **53**, Academic Press, 1973.