Geometrisation

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Abstract. Short overview of Thurston's geometrisation conjecture, mainly based on [2, Sec. 1 & 2] and [12, Sec. 1 & 2] (details about the topological part - Sections 1 and 2 - can be found in [7, Ch. 1]).

In the whole talk and unless otherwise stated the symbol M will denote a compact connected *orientable* 3-dimensional manifold (shortly: 3-manifold). The (possibly empty) boundary of M will be denoted by ∂M and $M \setminus \partial M$ will be called the interior of M. We can (and shall) assume that M is smooth since every 3-manifold has a unique smooth structure up to diffeomorphism, see e.g. [11, Sec. 3] for a short survey on that question.

1 Canonical decomposition of compact 3-manifolds

In this section we describe how every 3-manifold can be split into finitely many topologically beautiful pieces, where "beautiful" means "prime":

Definition 1.1 *A 3-manifold M is called*

- prime *if and only if* ($M = P \sharp Q \implies P = \mathbb{S}^3$ *or* $Q = \mathbb{S}^3$)*, i.e., if M cannot be written as a connected sum in a non-trivial way.*
- irreducible *if and only if every embedding* $\mathbb{S}^2 \hookrightarrow M$ *extends to an embedding* $\overline{B}^3 \hookrightarrow M$*, i.e., if every embedded 2-sphere in M bounds an embedded 3-ball.*

Notes 1.2

1. Every irreducible 3-manifold M is prime. Let namely P, Q be 3-manifolds with $M = P \sharp Q$. Then the embedded $\mathbb{S}^2$ along which both P and Q are glued together bounds a ball. This implies that $P \setminus \overline{B}^3$ or $Q \setminus \overline{B}^3$ must coincide with that ball: the image $\overline{B}$ of $\overline{B}^3$ under the embedding must lie in exactly one of the two pieces since otherwise the embedded $\mathbb{S}^2$ would be two-sided; as a compact set $\overline{B}$ has to be closed and since it has the same boundary as its containing piece, it is open in that piece. Therefore $P = \mathbb{S}^3$ or $Q = \mathbb{S}^3$.
2. As a consequence of the (already involved) *sphere theorem* (see e.g. [7, Sec. 3.1]), every irreducible 3-manifold has trivial second homotopy group $\pi_2(M)$.

The 3-sphere $\mathbb{S}^3$ is irreducible in virtue of

Alexander's theorem: *Every embedded 2-sphere in* $\mathbb{R}^3$ [*hence in* $\mathbb{S}^3$] *bounds an embedded 3-ball.*

For a proof based on Morse theory and the 2-dimensional Schönflies' theorem (*if γ is a simple closed curve in $\mathbb{R}^2$, then there exists an homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(\gamma) = \mathbb{S}^1$*), see [7, Sec. 1.1].

All but one prime manifolds are irreducible:

*Of course none of the results is from me; all the mistakes are. Thanks to Bernd Ammann for very helpful and enlightening discussions around the subject.

Lemma 1.3 *Let M be a prime non-irreducible 3-manifold, then M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$.*

Proof: First we prove that $\mathbb{S}^1 \times \mathbb{S}^2$ is prime. We have to show that, if $S \subset \mathbb{S}^1 \times \mathbb{S}^2$ is an embedded 2-sphere such that $(\mathbb{S}^1 \times \mathbb{S}^2) \setminus S$ has 2 connected components (shortly: a *separating* 2-sphere), then one of those components must be an embedded 3-ball. Let S be such a 2-sphere and denote by T and U the closures of the two connected components of $(\mathbb{S}^1 \times \mathbb{S}^2) \setminus S$. By van Kampen's theorem one has $\mathbb{Z} = \pi_1(\mathbb{S}^1 \times \mathbb{S}^2) = \pi_1(T) * \pi_1(U)$, in particular $\pi_1(T) = 1$ or $\pi_1(U) = 1$, say $\pi_1(T) = 1$. Identify the universal cover of $\mathbb{S}^1 \times \mathbb{S}^2$ with $\mathbb{R}^3 \setminus \{0\}$ and consider a lift of the inclusion map $T \subset \mathbb{S}^1 \times \mathbb{S}^2$ to $\mathbb{R}^3 \setminus \{0\}$. Denote by $\widetilde{T}$ the (diffeomorphic) image of T under that map. Then Alexander's theorem implies that the embedded 2-sphere $\partial\widetilde{T}$ bounds an embedded 3-ball in $\mathbb{R}^3$. But since $\partial\widetilde{T}$ already bounds the compact manifold $\widetilde{T}$, we deduce with the same argument as in Note 1.2 that $\widetilde{T}$ is an embedded 3-ball. In particular, T is an embedded 3-ball in $\mathbb{S}^1 \times \mathbb{S}^2$. This proves that $\mathbb{S}^1 \times \mathbb{S}^2$ is prime. Note that $\mathbb{S}^1 \times \mathbb{S}^2$ is not irreducible, since e.g. $\{1\} \times \mathbb{S}^2 \subset \mathbb{S}^1 \times \mathbb{S}^2$ is an embedded 2-sphere whose complement is connected.
Conversely let M be a prime non-irreducible 3-manifold. Then there exists an embedded 2-sphere S such that $M \setminus S$ is connected (shortly: a *non-separating* 2-sphere). If S were a connected component of the boundary, then the prime manifold M would be diffeomorphic to a closed 3-ball $\overline{B}^3$ (take a neighbourhood N of that connected component of the boundary with N diffeomorphic to $\mathbb{S}^2 \times [0, 1[$ and consider the separating 2-sphere $\mathbb{S}^2 \times \{1\}$), which is already irreducible by Alexander's theorem, contradiction. Therefore $S \not\subseteq \partial M$. Up to deforming S by an isotopy[1], we may assume that $S \cap \partial M = \varnothing$ and hence $S \subset M \setminus \partial M$.

Claim: *If the interior of a 3-manifold M contains a non-separating 2-sphere, then M can be written as $M = P \sharp (\mathbb{S}^1 \times \mathbb{S}^2)$ for some 3-manifold P.*

Proof of claim: Let S be a non-separating 2-sphere in $M \setminus \partial M$ and consider an open neighbourhood N of S in M. Since both S and M are orientable, the normal bundle of S in M is trivial and, up to making N smaller, one can assume that N is diffeomorphic to $S\times]-1, 1[$ and that $\overline{N}$ is diffeomorphic to $S \times [-1, 1]$ (here we use the fact that $S \subset M \setminus \partial M$). In particular $M \setminus \overline{N}$ is a deformation retract of the path-connected space $M \setminus S$, therefore $M \setminus \overline{N}$ (and hence $M \setminus N = \overline{M \setminus \overline{N}}$) is path-connected as well. Take a path joining in $M \setminus \overline{N}$ a point of $S \times \{-1\}$ to a point of $S \times \{1\}$ and denote by N' an open (sufficiently small) tubular neighbourhood of that path in M. Then $\overline{N} \cup \overline{N'}$ is a 3-manifold which is diffeomorphic to $(\mathbb{S}^1 \times \mathbb{S}^2) \setminus ((\mathbb{S}^2 \setminus \overline{D}) \times I)$ for an open disk $D \subset \mathbb{S}^2$ and an open interval I. Since $(\mathbb{S}^2 \setminus \overline{D}) \times I$ is diffeomorphic to $D \times I \cong B^3$, the manifold $\overline{N} \cup \overline{N'}$ is diffeomorphic to the complement of an open embedded 3-ball in $\mathbb{S}^1 \times \mathbb{S}^2$. Define P to be the 3-manifold obtained by gluing a 3-ball along the boundary S of $(N \cup N')^c$ (note that S is an embedded 2-sphere from the preceding argument), then $M = P \sharp (\mathbb{S}^1 \times \mathbb{S}^2)$. $\surd$

Since by assumption M is prime, the manifold P in the connected sum $M = P \sharp (\mathbb{S}^1 \times \mathbb{S}^2)$ must be $\mathbb{S}^3$, therefore M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$. $\square$

The main theorem of this section, which is due to H. Kneser [10], is the following:

Theorem 1.4 (canonical decomposition of 3-manifolds) *Every 3-manifold M can be written as a connected sum of the form*

$$M = P_1 \sharp \ldots \sharp P_p \sharp \underbrace{\mathbb{S}^1 \times \mathbb{S}^2 \sharp \ldots \sharp \mathbb{S}^1 \times \mathbb{S}^2}_{q}$$

for finitely many irreducible 3-manifolds $P_1, \ldots, P_p$ and $q \in \mathbb{N}$. This decomposition is unique up to permutation of the P_i's and insertion or deletion of $\mathbb{S}^3$'s.

Idea of proof: Existence and uniqueness are proved separately. For the existence, assume M to contain an embedded non-separating 2-sphere S. As in the proof of Lemma 1.3, one may assume that $S \subset M \setminus \partial M$. The claim in the proof of Lemma 1.3 implies the existence of a 3-manifold P with $M = P \sharp (\mathbb{S}^1 \times \mathbb{S}^2)$. If P also contains an embedded non-separating 2-sphere, then another $\mathbb{S}^1 \times \mathbb{S}^2$ can be split off M. The process can be inductively repeated, splitting off $\mathbb{S}^1 \times \mathbb{S}^2$'s as along as embedded non-separating 2-spheres exist.

[1] The complement of a point in $\mathbb{S}^2$ is contractible, so we can assume $S \cap \partial M = \varnothing$ or $S \cap \partial M = \{pt\}$; in the latter case, pushing S away from ∂M in a small neighbourhood of the point of $S \cap \partial M$ does not change the number of connected components of $M \setminus S$.

Since each $\mathbb{S}^1 \times \mathbb{S}^2$ produces an $H_1(\mathbb{S}^1 \times \mathbb{S}^2) \cong \mathbb{Z}$-component in the first homology group $H_1(M)$ of M and $H_1(M)$ is finitely generated (remember that M is compact), this process must stop after finitely many steps. Thus it remains to cover the case where every embedded 2-sphere in M separates. To proceed, one shows the existence of an integer n with the following property: for any family of disjoint embedded (separating) 2-spheres in M such that the complement of their union does *not* contain any punctured 3-sphere (the complement of finitely many points in $\mathbb{S}^3$), the cardinality of the family is at most n. If such an n exists, then the existence of the decomposition is achieved. If namely M is not irreducible, then there exists a separating embedded 2-sphere in M which does not bound an embedded 3-ball, hence M can be written as a non-trivial connected sum. The same argument may be carried out on both factors in the connected sum, and this can be repeated inductively on each factor. But the existence of n precisely implies that this algorithm must stop after finitely many steps; this in turn implies that each factor we obtain at the end is irreducible (choosing a further $\mathbb{S}^2$ only splits off an $\mathbb{S}^3$ - in any factor). The hard and technical proof of the existence of n, which essentially reduces to the case where the 2-sphere is "transversal" in some sense to a fixed triangulation of M, can be found in [7, pp. 5-7].
As for the uniqueness of the decomposition, one first has to show that - after deleting all $\mathbb{S}^3$'s in the connected sum - the irreducible factors coincide up to permutation (see [7, pp. 7-8]). The number of $\mathbb{S}^1 \times \mathbb{S}^2$'s is then fixed by the rank of the "second factor" in $H_1(M)$. $\square$

For non-orientable manifolds, the following criterion is available for testing irreducibility (see [7, Prop. 1.6 p.8]):

Lemma 1.5 *If the total space of a covering is irreducible, then so is its base.*

2 Splitting irreducible compact 3-manifolds along incompressible 2-tori

In this section we decompose irreducible 3-manifolds further on. Before stating the result, we need to introduce new notions.

Definition 2.1 *An embedded surface S in a 3-manifold M is called*

- 2-sided *(resp. 1-sided) in M if and only if its normal bundle in M is trivial (resp. non-trivial).*
- properly embedded *in M if and only if $\partial S \subset \partial M$ and $S \pitchfork \partial M$ (the surface S intersects ∂M transversally).*
- ∂-parallel *in M if and only if it is isotopic with fixed boundary to a surface embedded in ∂M.*

Definition 2.2 *A properly embedded surface S in a 3-manifold M is called* incompressible *in M if and only if it has no $\mathbb{S}^2$-component, no ∂-parallel D^2-component and, for every embedded disk D in M with $D \cap S = \partial D$, there exists an embedded disk D' in S such that $\partial D' = \partial D$.*

By definition a properly embedded surface S in M is incompressible if and only if every embedded disk in M having its boundary on S bounds an embedded disk in S.

Lemma 2.3 *Let S be an embedded surface in a 3-manifold M.*

1. *The surface S is incompressible if and only if every of its connected components is.*
2. *If S is not an $\mathbb{S}^2$ or a D^2, then it is incompressible as soon as the induced group homomorphism $\pi_1(S) \to \pi_1(M)$ is injective (in this case the embedded S is called* π_1-injective *in M). The converse is true if S is 2-sided in M.*
3. *Neither $\mathbb{S}^3$ nor $\mathbb{R}^3$ contains any incompressible surface.*
4. *In case S is a 2-sided torus in an irreducible M, it is compressible if and only if either it bounds an embedded solid torus $\mathbb{S}^1 \times D^2$ or it lies in an embedded 3-ball (in M).*
5. *If S is incompressible in M, then M is irreducible if and only if $M \setminus S$ is.*

In particular, a properly embedded orientable 2-torus T in M is incompressible if and only if it is π_1-injective.

Definition 2.4 *An irreducible* 3*-manifold* M *is called* atoroidal *if and only if every incompressible* 2*-torus in* M *is* ∂*-parallel.*

Any irreducible 3-manifold can be decomposed along incompressible 2-tori such that eventually all incompressible tori in the remaining pieces lie on the boundary of those (up to isotopy):

Proposition 2.5 *For any irreducible* 3*-manifold* M*, there exists a finite family of disjoint incompressible* 2*-tori in* M *such that each connected component of the complement of their union in* M *is atoroidal.*

Note that each connected component of the complement is again irreducible by Lemma 2.3. However, one should beware that - in contrast to the prime decomposition - the decomposition in Proposition 2.5 is in general *not* unique, see [7, Ex. p.13] for an example.

Next we introduce the concept of Seifert fibered 3-manifold (Definition 2.6 below), which itself requires a good understanding of the "baby" Seifert fibering. The latter is defined as follows: for relatively prime integers p, q let $X_{p,q}$ be the (topological) quotient ${}^{[0,1]\times D^2}/_{\sim}$ with $(0,x) \sim (1, r_{\frac{2\pi p}{q}}(x))$ and where r_θ denotes the rotation of angle θ around the origin in $\mathbb{R}^2$. The second projection $[0,1]\times D^2 \longrightarrow D^2$ induces a continuous and surjective map $X_{p,q} \xrightarrow{\pi} D^2_{p,q}$, where $D^2_{p,q} := {}^{D^2}/_{x \sim r_{\frac{2\pi p}{q}}(x)}$ (note that $D^2_{p,q}$ is a 2-dimensional *orbifold* in a natural way, with singularity group ${}^{\mathbb{Z}}/_{q\mathbb{Z}}$ at $[0]$). The preimage of each point in $D^2_{p,q}$ (which is called "fibre") is diffeomorphic to a circle which is made out of q different segments glued together "one after the other"[2] except when the base point is $[0]$, where the circle consists of only one such segment with both endpoints identified. Each sufficiently small disk which is transversal to the fibre intersects each other fibre exactly once in the former case and q times in the latter one.

Definition 2.6 *Let* M *be a* 3*-manifold.*

- *A* Seifert fibering *of* M *is a decomposition of* M *into disjoint circles such that each of those circles has an open neighbourhood which is diffeomorphic (preserving fibres) to an open neighbourhood of some fibre in* $X_{p,q} \xrightarrow{\pi} D^2_{p,q}$ *for some relatively prime integers* p, q*.*
- *The* 3*-manifold* M *is called* Seifert fibered *if and only if it admits a Seifert fibering.*

If M is Seifert fibered, then the quotient space ${}^{M}/_{\mathbb{S}^1}$ is a 2-dimensional orbifold with finitely many singularities as soon as M (hence ${}^{M}/_{\mathbb{S}^1}$) is compact. The statement which should be considered as analogous to Theorem 1.4 for irreducible manifolds is the following one, which is due to W. Jaco and P. Shalen [8] and independently to K. Johannson [9]:

Theorem 2.7 (Jaco-Shalen-Johannson decomposition of irreducible 3-manifolds) *For any irreducible* 3*-manifold* M*, there exists a finite family of disjoint incompressible* 2*-tori in* M *such that each connected component of the complement of their union in* M *is either atoroidal or a Seifert fibered manifold. Such a family with the least number of such* 2*-tori is unique up to isotopy.*

3 Locally homogeneous metrics in dimension 3

Here we aim at briefly describing the 8 "model geometries" in dimension 3. From now on the symbol g will denote a Riemannian metric on a smooth manifold M.

Definition 3.1 *A Riemannian manifold* (M, g) *is called*

- homogeneous *if and only if its isometry group acts transitively on* M*.*
- modelled *on a Riemannian manifold* $(\overline{M}, \overline{g})$ *if and only if it is locally isometric to* $(\overline{M}, \overline{g})$*.*

[2] The preimage $\pi^{-1}(\{[x]\})$ is the union of $\{[(t, r_{\frac{2\pi k p}{q}}(x))], t \in [0,1]\}$ over $k \in \{0, \dots, q-1\}$.

- locally homogeneous *if and only if it is complete and modelled on a homogeneous Riemannian manifold.*

In the following we are only interested in those locally homogeneous 3-dimensional Riemannian manifolds of *finite volume.*

Theorem 3.2 (W.P. Thurston, see [19]) *Let (M,g) be a locally homogeneous 3-dimensional orientable Riemannian manifold of* finite volume. *Denote by $(\overline{M},\overline{g})$ the simply-connected complete homogeneous Riemannian manifold on which (M,g) is modelled. Then there are only the following possibilities:*

- *The manifold $(\overline{M},\overline{g})$ has constant sectional curvature, in which case, up to rescaling the metric:*

    1. *The manifold $(\overline{M},\overline{g})$ is isometric to $(\mathbb{S}^3,\mathrm{can})$ and (M,g) is isometric to $({}^{\mathbb{S}^3}\!/_{\Gamma},\mathrm{can})$ for a finite subgroup $\Gamma\subset\mathrm{SO}_4$ acting freely on $\mathbb{S}^3$; in particular, M is compact.*
    2. *The manifold $(\overline{M},\overline{g})$ is isometric to $(\mathbb{R}^3,\mathrm{can})$ and (M,g) is isometric to some compact quotient $({}^{\mathbb{R}^3}\!/_{\Gamma},\mathrm{can})$ for a subgroup $\Gamma\subset\mathrm{SO}_3\rtimes\mathbb{R}^3$ acting freely on $\mathbb{R}^3$.*
    3. *The manifold $(\overline{M},\overline{g})$ is isometric to $(\mathbb{H}^3,\mathrm{can})$ and (M^3,g) is isometric to some quotient of finite volume $({}^{\mathbb{H}^3}\!/_{\Gamma},\mathrm{can})$ for a subgroup $\Gamma\subset\mathrm{SO}^{\uparrow}(3,1)$ acting freely on $\mathbb{H}^3$.*

- *The manifold $(\overline{M},\overline{g})$ is isometric to a (non-trivial) Riemannian product, in which case, up to rescaling the metric:*

    4. *The manifold $(\overline{M},\overline{g})$ is isometric to $(\mathbb{S}^2\times\mathbb{R},\mathrm{can}\oplus dt^2)$ and (M,g) is isometric to either $(\mathbb{S}^2\times\mathbb{S}^1,\mathrm{can}\oplus dt^2)$ or to a connected sum $\mathbb{R}\mathrm{P}^3\sharp\,\mathbb{R}\mathrm{P}^3$; in particular, M is compact.*
    5. *The manifold $(\overline{M},\overline{g})$ is isometric to $(\mathbb{H}^2\times\mathbb{R},\mathrm{can}\oplus dt^2)$ and (M,g) is isometric to a quotient $({}^{\Sigma\times\mathbb{S}^1}\!/_{\Gamma},\mathrm{can}\oplus dt^2)$, where Γ is a finite subgroup of the isometry group and Σ is a hyperbolic surface of finite area.*

- *The manifold $(\overline{M},\overline{g})$ is isometric to a simply-connected Lie group with left invariant metric, in which case:*

    6. *The manifold $(\overline{M},\overline{g})$ is isometric to the Heisenberg group $\left\{\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix},\ a,b,c\in\mathbb{R}\right\}$ with some left invariant metric and (M,g) is isometric to a finite quotient of an $\mathbb{S}^1$-bundle over the 2-torus, in particular it is compact (then (M,g) is called a* nilmanifold*).*
    7. *The manifold $(\overline{M},\overline{g})$ is isometric to the solvable group $\mathbb{R}^2\rtimes\mathbb{R}$ with some left invariant metric, where $\mathbb{R}$ acts on $\mathbb{R}^2$ via $t\mapsto\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}$ and (M,g) is isometric to a finite quotient of a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$, in particular it is compact (then (M,g) is called a* solvmanifold*).*
    8. *The manifold $(\overline{M},\overline{g})$ is isometric to the universal cover $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ of the special linear group $\mathrm{SL}_2(\mathbb{R})$ with some left-invariant metric and (M,g) is isometric to a Seifert fibred manifold whose orbit space is a hyperbolic 2-dimensional orbifold of finite area.*

Each non-compact hyperbolic 3-manifold with finite volume consists of a compact part to which finitely many warped products of the form $(\mathbb{T}^2\times]0,\infty[,e^{-2t}g_{\mathrm{flat}}\oplus dt^2)$ (for some flat metric g_{flat} on the 2-torus) are attached. In particular, such a manifold is diffeomorphic to the interior of a compact 3-manifold with π_1-injective tori as boundary components. A closer look to 5. and 8. also shows that, if in those cases M is non-compact, then it is also diffeomorphic to the interior of a compact 3-manifold with π_1-injective tori as boundary components. Therefore one obtains the following

Corollary 3.3 *Let (M,g) be a locally homogeneous 3-dimensional orientable Riemannian manifold of finite volume. Then either M is closed or M is non-compact and in that case it is diffeomorphic to the interior of a compact 3-manifold with π_1-injective tori as boundary components.*

The pros in topology also know that the families appearing in 1., 2., 4., 5., 6. and 8. of Theorem 3.2 can all be described as Seifert fibred manifolds, sometimes in different ways.

4 Thurston's geometrisation and the Poincaré conjecture

Roughly speaking, Thurston's geometrisation conjecture states that any closed 3-manifold can be decomposed into finitely many pieces, each of those carrying a "nice" metric:

Theorem 4.1 (Thurston's geometrisation conjecture [18]) *Given any closed orientable prime 3-manifold M, there exist finitely many disjoint embedded 2-tori in M such that each connected component in the complement of their union admits a locally homogeneous Riemannian metric of finite volume.*

Thurston's geometrisation conjecture is no more a conjecture but a theorem, as a consequence G. Perelman's work on the Ricci-flow with surgery [13, 14, 15], which itself builds on R. Hamilton's earlier achievements (e.g. [3, 4, 5, 6]) and on Thurston's proof in particular cases, see Section 5 below for a very short account and e.g. [1] or [17] for a very long one.

The solution to the geometrisation conjecture provides one to the Poincaré conjecture:

Corollary 4.2 (Poincaré conjecture [16]) *Any simply-connected closed (connected) 3-manifold is diffeomorphic to $\mathbb{S}^3$.*

Proof: Let M be any such 3-manifold. Note that M is automatically orientable as soon as it is simply-connected. Decompose M into prime factors (Theorem 1.4). Since M is simply-connected, so is each prime factor in the decomposition, in particular each of those is actually irreducible (Lemma 1.3) and therefore we may assume that M is irreducible (if this case is covered, then we are done because of $\mathbb{S}^3\sharp\,\mathbb{S}^3\sharp\ldots\sharp\,\mathbb{S}^3 \stackrel{\text{diff.}}{\cong} \mathbb{S}^3$). Now decompose M as in Theorem 4.1. Then Corollary 3.3 (combined with Lemma 2.3.2) implies that this decomposition must be trivial since each 2-torus along which the manifold is decomposed in Theorem 4.1 must be incompressible, not only in the pieces it bounds but also in M by van Kampen's theorem.[3] Hence M has a locally homogeneous Riemannian metric of finite volume. Because of $\pi_1(M) = 1$, the simply-connected homogeneous manifold on which M is modelled is M itself, in particular this model manifold must be compact. Looking back at the classification of homogeneous models (Theorem 3.2), we observe that the only remaining possibility for M is $\mathbb{S}^3$. $\square$

5 Ricci-flow and geometrisation

In this section (mainly based on [12, Sec. 3-4]) we briefly describe how Ricci flows provides the decomposition in Theorem 4.1.

The basic idea - originally due to R. Hamilton - is that, for any given metric g_0 on the closed 3-manifold M, if one lets g_0 evolve under the Ricci-flow ($\frac{\partial g}{\partial t} = -2\mathrm{ric}_g$ with $g(0) = g_0$), then the metric $g(t)$ obtained should converge to a locally homogeneous metric as t grows, at least on some "large part" of the manifold; in particular, this part carries such a metric.

Stated this way the idea is naive, as Hamilton himself noticed, since in general the Ricci-flow develops singularities: as we have already seen in former talks, elementary computations show that the scalar curvature of $g(t)$ explodes in finite time as soon as g_0 has positive scalar curvature.

Nevertheless, on any closed manifold, the Ricci-flow always exists on a small interval and is unique, as Hamilton showed in his earliest papers on the topic. Moreover, he characterized the first time where singularities appear: if $g(t)$ exists for all $t \in [0, T[$ but not after $T < \infty$, then there is at least one point in the manifold where the Riemann curvature tensor must explode as t approaches T. Specializing to dimension 3, Hamilton [3] proved that, if one starts with a metric with *non-negative* Ricci curvature on some closed M, then the Ricci-flow produces either a flat metric for all times, or a metric $g(t)$ for which $(M, g(t))$ is covered by $\mathbb{R} \times \mathbb{S}^2(r(t))$ where $r(t)$ goes to 0 in finite time, or a metric which, after rescaling it such as to keep the diameter constant, converges to a round metric (i.e., with positive sectional curvature) in finite time. In all three cases the geometrisation conjecture is obviously fulfilled, where only geometries of type

[3] Actually, using $\pi_1(M) = 1$, Lemma 2.3.2 forbides the existence of incompressible tori in M, so that the decomposition of M into atoroidal pieces (Theorem 2.7) just consists of M itself.

1, 2 and 4 appear.

As a next step, Hamilton [6] looked at the situation where the Ricci-flow exists for *all time* and the Riemann curvature tensor *behaves like* $O(\frac{1}{t})$ as t goes to ∞. Then only two possibilities can occur: either M has a flat metric or M can be written as the union of finitely many manifolds carrying a hyperbolic metric with finite volume, connected sums of Seifert fibred manifolds and of manifolds admitting a solv-metric. The hyperbolic metrics appear as limits of $\frac{g(t)}{t}$ as t goes to ∞, where $g(t)$ is the Ricci-flow on the original manifold. On the complement of the hyperbolic part, the metric $\frac{g(t)}{t}$ collapses[4] with bounded curvature and Cheeger-Gromov's compactness results combined with 3-dimensional topological techniques imply that only connected sums of Seifert fibred manifolds and solvmanifolds can appear. Furthermore, all 2-tori T in the cusps $T \times \mathbb{R}_+$ of the hyperbolic pieces are incompressible tori and so are all boundary tori of the Seifert fibred summands. Though not as straightforward as before, the geometrisation conjecture can be deduced in that case, see [12, Cor. 3.3.3].

Still the Ricci-flow generally develops singularities and therefore a new idea has to be sown for Hamilton's Ricci-flow ansatz to be extended. The breakthrough came with the concept of *Ricci-flow with surgery*. The intuition behind it is that, after removing the possible singularities occurring during the Ricci-flow, one obtains new pieces on which the Ricci-flow evolves to the local homogeneous metric we want. Of course, there are numerous difficulties to implement this process, for instance: which parts exactly should be removed? After removing those parts, should other manifolds be glued in, in that case which ones and how? Assuming this process to work, do singularities appear only finitely many times? Or, if not, can they accumulate in time?

The first question when wanting to perform Ricci-flow beyond singularities is: how do those look like? The good news is that only two kinds of singularities can occur. In the first type, the metric *shrinks*, i.e., it goes down to 0 but still in a way where the underlying geometry can be controlled: rescaling the metric so as to keep the diameter fixed provides a locally homogeneous metric as limit when getting near to the singular time. In the second type (characterized by G. Perelman), the metric develops either a long thin tube diffeomorphic to $\mathbb{S}^2 \times I$ for some interval I or the union of such a thin tube with a positively-curved "cap" at one end. As a matter of fact, regions where those singularities occur also satisfy the geometrisation conjecture.
It is in the proof of the characterization of the singularity-type that the $\mathcal{W}$-entropy enters, both to show that the "limit" in some sense is a smooth manifold (this works because of the control of the injectivity radius provided by the entropy) and to determine the behaviour of the metric itself: the monotonicity formula for the $\mathcal{W}$-entropy implies that, if the entropy does not increase on some interval, then the Ricci-flow has to be a Ricci-soliton on that interval.

Now Ricci-flow with surgery consists of the following algorithm: let the Ricci-flow start on the manifold M, go to the next singular time, throw away those regions of the manifolds where singularities of the first type occur, cut singular regions of the second type off and insert by surgery an "almost-flat" 3-ball at the ends of the thin tubes; then flow the new metrics again and proceed as above. Since all intermediate operations consist of surgeries along 2-spheres, the geometrisation conjecture holds on the original manifold as soon as it holds on the pieces after finitely many steps: the decomposition in Theorem 4.1 is given by the connected sum of the pieces which have been split apart in the process. Note that the surgery process generated by the Ricci-flow may - actually will after finitely many surgeries - produce trivial pieces, corresponding to splitting off $\mathbb{S}^3$'s.

The last and decisive step, an authentic *tour de force* due to G. Perelman [13, 14, 15], consists in generalizing Hamilton's latter result on long-time-limits of Ricci-flow to Ricci-flow with surgery, even *without* the assumption on the behaviour of the curvature tensor on long time. Combined to the impossibility for the "singular times" to accumulate (a fact which is also due to Perelman), this provides the existence of a finite time where the decomposition of the original manifold fits with that of the geometrisation conjecture; every further surgery will split off an $\mathbb{S}^3$.

[4] in which sense? To be cleared.

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