A splitting theorem for Riemannian manifolds of generalised Ricci-Hessian type

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Abstract. In this paper, we study and partially classify those Riemannian manifolds carrying a non-identically vanishing function f whose Hessian is minus f times the Ricci-tensor of the manifold.

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1 Introduction

In this paper, we are interested in those Riemannian manifolds (M^n, g) supporting a non-identically-vanishing function f satisfying

$$\nabla^2 f = -f \cdot \mathrm{Ric} \tag{1}$$

on M, where $\nabla^2 f := \nabla\nabla f$ denotes the Hessian of f and Ric the Ricci-tensor of (M^n, g), both seen as $(1,1)$-tensor fields. This equation originates in the search for nontrivial solutions to the so-called *skew-Killing-spinor-equation* [3].

Equation (1) looks much like that considered by C. He, P. Petersen and W. Wylie in their search for warped product Einstein metrics where functions f are considered whose Hessian is a *positive* scalar multiple of $f \cdot (\mathrm{Ric} - \lambda \mathrm{Id})$ for some real constant λ, see e.g. [6, Eq. (1.4)]. However, no attempt has been made to deal with the *negative* case since. In another direction, J. Corvino proved [1, Prop. 2.7] that a positive function f satisfies $\nabla^2 f = f \cdot \mathrm{Ric} - (\Delta f) \cdot \mathrm{Id}$ on M if and only if the (Lorentzian) metric $\overline{g} := -f^2 dt^2 \oplus g$ is Einstein on $\mathbb{R} \times M$. In [5] the more general situation was considered where the r.h.s. of (1) is replaced by $f \cdot q$ for some *a priori* arbitrary symmetric tensor field q on TM; but the statements formulated in [5] are only valid when, for a given fixed q, the space of functions f satisfying $\nabla^2 f = f \cdot q$ has dimension at least 2, see e.g. [5, Theorem A]. Thus we are left with an open problem in case we know about only one such function f.

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Along the same line, S. Güler and S.A. Demirbağ define a Riemannian manifold (M^n, g) to be quasi Einstein if and only if there exist smooth functions u, α, λ on M such that

$$\mathrm{Ric} + \nabla^2 u - \alpha du \otimes \nabla u = \lambda \cdot \mathrm{Id},$$

see [4, Eq. (1.1)]. It is easy to see that, for $\alpha = -1$ and $\lambda = 0$, a function u solves that equation if and only if $f := e^u$ solves (1). However the results of [4] cannot be compared with ours since the quasi Einstein condition seems to be interesting only in the case where $\alpha > 0$ and $u > 0$.

Independently from [5, 6], F.E.S. Feitosa, A.A. Freitas Filho, J.N.V. Gomes and R.S. Pina define gradient almost Ricci soliton warped products by means of functions $f > 0, \varphi, \lambda$ satisfying in particular

$$\frac{m}{f}\nabla^2 f + \lambda \cdot \mathrm{Id} = \mathrm{Ric} + \nabla^2 \varphi$$

for some nonzero real constant m, see [2, Eq. (1.4)]. Our equation (1) is the special case of that equation where $m = -1$ and $\lambda = \varphi = 0$. But again [2] only deals with the case where $m > 0$; besides, only positive f are considered. Therefore, no result of [2] can be used in our setting.

Ricci-flat manifolds carry obvious solutions to (1), just pick constant functions. Constant functions are actually all solutions to (1) in case M is Ricci-flat and closed. If M is Ricci-flat, complete but noncompact, then there exists a nonconstant solution to (1) if and only if M is the Riemannian product of $\mathbb{R}$ with a (complete) Ricci-flat manifold N; in that case, f is an affine-linear function of the $t \in \mathbb{R}$-coordinate. In the search for further nonconstant functions satisfying (1), a natural setting that comes immediately to mind is the case where (M^n, g) is Einstein, because then (1) gets close to the Obata resp. Tashiro equation. But surprisingly enough only certain two-dimensional spaceforms carry such functions among complete Einstein manifolds, see Lemma 2.1 below. Further examples can be constructed taking Riemannian products of manifolds carrying a function solving (1) – e.g. some 2-dimensional spaceforms – with any Ricci-flat Riemannian manifold. It is however *a priori* unclear whether other examples exist besides those.

We show here that, under further geometric assumptions, only products of two-dimensional spaceforms with a Ricci-flat Riemannian manifold can appear, see Theorem 2.4. In particular, we obtain a classification result covering to some extent the missing case in [6].

2 Main result and proof

2.1 Preliminary remarks

We start with preliminary results, most of which are elementary or already proved in the literature. From now on, we shall denote by S the scalar curvature of M and, for any function h on M, by ∇h the gradient vector field of h w.r.t. g on M. First observe that the equation $\nabla^2 f = -f \cdot \mathrm{Ric}$ is of course linear in f but is also invariant under metric rescaling: if $\overline{g} = \lambda^2 g$ for some nonzero real number λ, then $\overline{\nabla}^2 f = \lambda^{-2}\overline{\nabla}^2 f$ (this comes from the rescaling of the gradient) and $\overline{\mathrm{Ric}} = \lambda^{-2}\mathrm{Ric}$.

Lemma 2.1 *Let (M^n, g) be any connected Riemannian manifold carrying a smooth real-valued function f satisfying* (1) *on M.*

1. *The gradient vector field ∇f of f w.r.t. g satisfies*

$$\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S. \tag{2}$$

2. *There exists a real constant μ such that*

$$f\Delta f + 2|\nabla f|^2 = \mu. \qquad (3)$$

3. *If $n > 2$ and f is everywhere positive or negative, then f solves* (1) *if and only if, setting $u := \frac{1}{2-n}\ln|f|$, the metric $\overline{g} := e^{2u}g$ satisfies $\overline{\mathrm{ric}} = (\overline{\Delta}u)\overline{g} - (n-2)(n-3)du \otimes du$ on M and in that case $\overline{\Delta}u = -\frac{\mu}{n-2}e^{2(n-3)u}$. In particular, if $n = 3$, the existence of a positive solution f to* (1) *is equivalent to $(M, f^{-2}g)$ being Einstein with scalar curvature $-3\overline{\Delta}\ln|f|$.*

4. *If M is closed and f is everywhere positive or negative, then f is constant on M.*

5. *If nonempty, the vanishing set $N_0 := f^{-1}(\{0\})$ of f is a scalar-flat totally geodesic hypersurface of M. Moreover, N_0 is flat as soon as it is 3-dimensional and carries a nonzero parallel vector field.*

6. *If furthermore M is non-Ricci-flat, Einstein or 2-dimensional, then $n = 2$ and M has constant curvature. In particular, when (M^2, g) is complete, there exists a nonconstant function f satisfying* (1) *if and only if, up to rescaling the metric, the manifold (M^2, g) is isometric to either the round sphere $\mathbb{S}^2$ and f is a nonzero eigenfunction associated to the first positive Laplace eigenvalue; or to flat $\mathbb{R}^2$ or cylinder $\mathbb{S}^1 \times \mathbb{R}$ and f is an affine-linear function; or to the hyperbolic plane $\mathbb{H}^2$ and f is a solution to the Tashiro equation $\nabla^2 f = f \cdot \mathrm{Id}$.*

7. *If S is constant, then outside the set of critical points of f, the vector field $\nu := \frac{\nabla f}{|\nabla f|}$ is geodesic. Moreover, assuming (M^n, g) to be also complete,*
   - *(a) if $S > 0$, then up to rescaling the metric as well as f, we may assume that $S = 2$ and that $\mu = f\Delta f + 2|\nabla f|^2 = 2$ on M, in which case the function f has 1 as maximum and -1 as minimum value and those are the only critical values of f;*
   - *(b) if $S = 0$, then up to rescaling f, we may assume that $\mu = 2$ on M, in which case f has no critical value and $f(M) = \mathbb{R}$, in particular M is noncompact;*
   - *(c) if $S < 0$, then up to rescaling the metric, we may assume that $S = -2$ on M, in which case one of the following holds:*
     - *i. if $\mu > 0$, then up to rescaling f we may assume that $\mu = 2$, in which case f has no critical value and $f(M) = \mathbb{R}$, in particular M is noncompact;*
     - *ii. if $\mu = 0$, then f has no critical value and empty vanishing set and, up to changing f into $-f$, we have $f(M) = (0, \infty)$, in particular M is noncompact;*
     - *iii. if $\mu < 0$, then up to rescaling f we may assume that $\mu = -2$, in which case f has a unique critical value, which, up to changing f into $-f$, can be assumed to be a minimum; moreover, $f(M) = [1, \infty)$, in particular M is noncompact.*

Proof: The proof of the first statement follows that of [7, Lemma 4]. On the one hand, we take the codifferential of $\nabla^2 f$ and obtain, choosing a local orthonormal basis $(e_j)_{1\leq j\leq n}$ of TM and using Bochner's formula for 1-forms:

$$\begin{aligned}
\delta\nabla^2 f &= -\sum_{j=1}^{n} \left(\nabla_{e_j}\nabla^2 f\right)(e_j) \\
&= -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j}\nabla f - \nabla_{\nabla_{e_j}e_j}\nabla f \\
&= \nabla^*\nabla(\nabla f) \\
&= \Delta(\nabla f) - \mathrm{Ric}(\nabla f).
\end{aligned}$$

On the other hand, by (1) and the formula $\delta\mathrm{Ric} = -\frac{1}{2}\nabla S$,

$$\begin{aligned}\delta\nabla^2 f &= \delta\left(-f\cdot\mathrm{Ric}\right)\\ &= \mathrm{Ric}(\nabla f) - f\cdot\delta\mathrm{Ric}\\ &= \mathrm{Ric}(\nabla f) + \frac{f}{2}\nabla S.\end{aligned}$$

Comparing both identities, we deduce that $\Delta(\nabla f) = 2\mathrm{Ric}(\nabla f) + \frac{f}{2}\nabla S$. But identity (1) also gives

$$\Delta f = -\mathrm{tr}\left(\nabla^2 f\right) = fS, \tag{4}$$

so that $\Delta(\nabla f) = \nabla(\Delta f) = \nabla(fS) = S\nabla f + f\nabla S$ and therefore $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S$, which is (2).
By (1) and (2), we have

$$\begin{aligned}2\nabla(|\nabla f|^2) &= 4\nabla^2_{\nabla f} f\\ &= -4f\cdot\mathrm{Ric}(\nabla f)\\ &= -4f\cdot\left(\frac{S}{2}\nabla f + \frac{f}{4}\nabla S\right)\\ &= -2Sf\nabla f - f^2\nabla S\\ &= -\nabla(Sf^2)\\ &\overset{(4)}{=} -\nabla(f\Delta f),\end{aligned}$$

from which (3) follows.
If f vanishes nowhere, then up to changing f into $-f$, we may assume that $f > 0$ on M. Writing f as $e^{(2-n)u}$ for some real-valued function u (that is, $u = \frac{1}{2-n}\ln f$), the Ricci-curvatures (as $(0,2)$-tensor fields) ric and $\overline{\mathrm{ric}}$ of (M,g) and $(M, \overline{g} = e^{2u}g)$ respectively are related as follows:

$$\overline{\mathrm{ric}} = \mathrm{ric} + (2-n)(\nabla du - du\otimes du) + (\Delta u - (n-2)|du|_g^2)g. \tag{5}$$

But $\nabla df = (n-2)^2 f\cdot du\otimes du + (2-n)f\cdot\nabla du$ and the Laplace operators Δ of (M,g) and $\overline{\Delta}$ of $(M,\overline{g})$ are related via $\overline{\Delta}v = e^{-2u}\cdot(\Delta v - (n-2)g(du,dv))$ for any function v, so that

$$\begin{aligned}\overline{\mathrm{ric}} &= \mathrm{ric} + \frac{1}{f}\nabla df - (n-2)^2 du\otimes du + (n-2)du\otimes du + (\overline{\Delta}u)\overline{g}\\ &= \mathrm{ric} + \frac{1}{f}\nabla df - (n-2)(n-3)du\otimes du + (\overline{\Delta}u)\overline{g}.\end{aligned}$$

As a consequence, f satisfies (1) if and only if $\overline{\mathrm{ric}} = (\overline{\Delta}u)\overline{g} - (n-2)(n-3)du\otimes du$ holds on M. Moreover,

$$\begin{aligned}f\Delta f + 2|df|_g^2 &= f\cdot\left(-(n-2)^2 f|du|_g^2 - (n-2)f\Delta u\right) + 2(n-2)^2 f^2|du|_g^2\\ &= -(n-2)f^2\cdot\left(\Delta u - (n-2)|du|_g^2\right)\\ &= -(n-2)f^2\cdot e^{2u}\cdot\overline{\Delta}u\\ &= -(n-2)e^{2(2-n)u}\cdot e^{2u}\cdot\overline{\Delta}u\\ &= -(n-2)e^{2(3-n)u}\cdot\overline{\Delta}u,\end{aligned}$$

in particular (3) yields $\overline{\Delta}u = -\frac{\mu}{n-2}e^{2(n-3)u}$. In dimension 3, we notice that $\overline{\Delta}u = \frac{S}{3}$. This shows the third statement.

If f vanishes nowhere, then again we may assume that $f > 0$ on M. Since M is closed, f has a minimum and a maximum. At a point x where f attains its maximum, we have $\mu = f(x)(\Delta f)(x) + 2|\nabla_x f|^2 = f(x)(\Delta f)(x) \geq 0$. In the same way, $\mu = f(y)(\Delta f)(y) \leq 0$ at any point y where f attains its minimum. We deduce that $\mu = 0$ which, by integrating the identity $f\Delta f + 2|\nabla f|^2 = \mu$ on M, yields $df = 0$. This shows the fourth statement.
The first part of the fifth statement is the consequence of the following very general fact [5, Prop. 1.2], that we state and reprove here for the sake of completeness: if some smooth real-valued function f satisfies $\nabla^2 f = fq$ for some quadratic form q on M, then the subset $N_0 = f^{-1}(\{0\})$ is – if nonempty – a totally geodesic smooth hypersurface of M. First, it is a smooth hypersurface because of $d_x f \neq 0$ for all $x \in N_0$: namely if $c\colon \mathbb{R} \to M$ is any geodesic with $c(0) = x$, then the function $y := f \circ c$ satisfies the second order linear ODE $y'' = \langle \nabla^2_{\dot c} f, \dot c\rangle = q(\dot c, \dot c) \cdot y$ on $\mathbb{R}$ with the initial condition $y(0) = 0$; if $d_x f = 0$, then $y'(0) = 0$ and hence $y = 0$ on $\mathbb{R}$, which would imply that $f = 0$ on M by geodesic connectedness, contradiction. To compute the shape operator W of N_0 in M, we define $\nu := \frac{\nabla f}{|\nabla f|}$ to be a unit normal to N_0. Then for all $x \in N_0$ and $X \in T_x M$,

$$\begin{aligned}
\nabla^M_X \nu &= X\left(\frac{1}{|\nabla f|}\right) \cdot \nabla f + \frac{1}{|\nabla f|} \cdot \nabla^M_X \nabla f \\
&= -\frac{X\left(|\nabla f|^2\right)}{2|\nabla f|^3} \cdot \nabla f + \frac{1}{|\nabla f|} \cdot \nabla^M_X \nabla f \\
&= \frac{1}{|\nabla f|} \cdot \left(\nabla^2_X f - \langle \nabla^2_X f, \nu\rangle \cdot \nu\right), \qquad (6)
\end{aligned}$$

in particular $W_x = -(\nabla\nu)_x = 0$ because of $\left(\nabla^2 f\right)_x = f(x) q_x = 0$. This shows that N_0 lies totally geodesically in M.
Now recall Gauß equations for Ricci curvature: for every $X \in TN_0$,

$$\mathrm{Ric}_{N_0}(X) = \mathrm{Ric}(X)^T - R^M_{X,\nu}\nu + \mathrm{tr}_g(W) \cdot WX - W^2 X,$$

where $\mathrm{Ric}(X)^T = \mathrm{Ric}(X) - \mathrm{ric}(X,\nu)\nu$ is the component of the Ricci curvature that is tangential to the hypersurface N_0. As a straightforward consequence, if S_{N_0} denotes the scalar curvature of N_0,

$$S_{N_0} = S - 2\mathrm{ric}(\nu,\nu) + \left(\mathrm{tr}_g(W)\right)^2 - |W|^2.$$

Here, $W = 0$ and $\mathrm{Ric}(\nu) = \frac{S}{2}\nu$ along N_0 because N_0 lies totally geodesically in M, so that

$$S_{N_0} = S - 2\mathrm{ric}(\nu,\nu) = S - S = 0.$$

This proves N_0 to be scalar-flat. If moreover N_0 is 3-dimensional and carries a parallel vector field, then it is locally the Riemannian product of a scalar-flat – and hence flat – surface with a line, therefore N_0 is flat. This shows the fifth statement.
In dimension 2, we can write $\mathrm{Ric} = \frac{S}{2}\mathrm{Id} = K\mathrm{Id}$, where K is the Gauß curvature. But we also know that $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S = K\nabla f + \frac{f}{2}\nabla K$. Comparing both identities and using the fact that $\{f \neq 0\}$ is dense in M leads to $\nabla K = 0$, that is, M has constant Gauß curvature. Up to rescaling the metric as well as f, we may assume that $S, \mu \in \{-2, 0, 2\}$. If M^2 is complete with constant $S > 0$ (hence $K = 1$) and f is nonconstant, then $\mu > 0$ so that, by Obata's solution to the equation $\nabla^2 f + f \cdot \mathrm{Id}_{TM} = 0$, the manifold M must be isometric to the round sphere of radius 1 and the function f must be a nonzero eigenfunction associated to the first positive eigenvalue of the Laplace operator on $\mathbb{S}^2$, see [9, Theorem A]. If M^2 is complete and has vanishing curvature, then its universal cover is the flat $\mathbb{R}^2$ and the lift $\tilde f$ of f to $\mathbb{R}^2$ must be an affine-linear function of the form $\tilde f(x) = \langle a, x\rangle + b$ for some nonzero $a \in \mathbb{R}^2$ and some $b \in \mathbb{R}$; since the only possible

nontrivial quotients of $\mathbb{R}^2$ on which $\tilde{f}$ may descend are of the form ${}^{\mathbb{R}}\!/\!_{\mathbb{Z}\cdot\check{a}} \times \mathbb{R}$ for some nonzero $\check{a} \in a^{\perp}$, the manifold M itself must be either flat $\mathbb{R}^2$ or such a flat cylinder. If M^2 is complete with constant $S < 0$, then f satisfies the Tashiro equation $\nabla^2 f = f \cdot \mathrm{Id}_{TM}$. But then Y. Tashiro proved that (M^2, g) must be isometric to the hyperbolic plane of constant sectional curvature -1, see e.g. [11, Theorem 2 p.252]. Note that the functions f listed above on $\mathbb{S}^2$, $\mathbb{R}^2$, $\mathbb{S}^1 \times \mathbb{R}$ or $\mathbb{H}^2$ obviously satisfy (1).

If (M^n, g) is Einstein with $n \geq 3$, then it has constant scalar curvature and $\mathrm{Ric} = \frac{S}{n} \cdot \mathrm{Id}$. But again the identity $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f + \frac{f}{4}\nabla S = \frac{S}{2}\nabla f$ yields $n = 2$ unless $S = 0$ and thus M is Ricci-flat. Therefore, $n = 2$ is the only possibility for non-Ricci-flat Einstein M. This shows the sixth statement.

If S is constant, then $\mathrm{Ric}(\nabla f) = \frac{S}{2}\nabla f$. As a consequence, $\nabla^2_{\nabla f} f = -f\mathrm{Ric}(\nabla f) = -\frac{Sf}{2}\nabla f$. But, as already observed in e.g. [10, Prop. 1], away from its vanishing set, the gradient of f is a pointwise eigenvector of its Hessian if and only if the vector field $\nu = \frac{\nabla f}{|\nabla f|}$ is geodesic, see (6) above.

Assuming furthermore (M^n, g) to be complete, we can rescale as before f and g such that $S, \mu \in \{-2, 0, 2\}$. In case $S > 0$ and hence $S = 2$, necessarily $\mu > 0$ holds and thus $\mu = 2$. But then $f^2 + |\nabla f|^2 = 1$, so that the only critical points of f are those where $f^2 = 1$, which by $f^2 \leq 1$ shows that the only critical points of f are those where $f = \pm 1$ and hence where f takes a maximum or minimum value. Outside critical points of f, we may consider the function $y := f \circ \gamma \colon \mathbb{R} \to \mathbb{R}$, where $\gamma \colon \mathbb{R} \to M$ is a maximal integral curve of the geodesic vector field ν. Then y satisfies $y' = |\nabla f| \circ \gamma > 0$ and $y(t)^2 + y'(t)^2 = 1$, so that $y' = \sqrt{1 - y^2}$ and therefore there exists some $\phi \in \mathbb{R}$ such that

$$y(t) = \cos(t + \phi) \qquad \forall\, t \in \mathbb{R}.$$

Since that function obviously changes sign and 0 is not a critical value of f, we can already deduce that f changes sign, in particular $N_0 = f^{-1}(\{0\})$ is nonempty. Moreover, the explicit formula for y shows that f must have critical points, which are precisely those where cos reaches its minimum or maximum value. This shows statement 7.(a).

In case $S = 0$, we have $\Delta f = 0$ and therefore (3) becomes $|\nabla f|^2 = \mu$ on M, in particular $\mu > 0$ and f has no critical point on M. But because of $|\nabla f| = 1$, the function $y = f \circ \gamma$ is in fact equal to $t \mapsto t + \phi$ for some constant $\phi \in \mathbb{R}$. This shows that $f(M) = \mathbb{R}$ and in particular that M cannot be compact. This proves statement 7.(b).

In case $S < 0$ and thus $S = -2$, there are still three possibilities for μ:

- If $\mu > 0$, then $\mu = 2$ and (3) becomes $-f^2 + |\nabla f|^2 = 1$, hence f has no critical point. If γ is any integral curve of the normalized gradient vector field $\nu = \frac{\nabla f}{|\nabla f|}$, then the function $y := f \circ \gamma$ satisfies the ODEs $y' = \sqrt{1 + y^2}$, therefore $y(t) = \sinh(t + \phi)$ for some real constant ϕ. In particular, $f(M) = \mathbb{R}$ and M cannot be compact.

- If $\mu = 0$, then (3) becomes $f^2 = |\nabla f|^2$. But since no point where f vanishes can be a critical point by the fifth statement, f has no critical point and therefore must be of constant sign. Up to turning f into $-f$, we may assume that $f > 0$ and thus $f = |\nabla f|$. Along any integral curve γ of $\nu = \frac{\nabla f}{|\nabla f|}$, the function $y := f \circ \gamma$ satisfies $y' = y$ and hence $y(t) = C \cdot e^t$ for some positive constant C. This shows $f(M) = (0, \infty)$, in particular M cannot be compact.

- If $\mu < 0$, then $\mu = -2$ and (3) becomes $-f^2 + |\nabla f|^2 = -1$. As a consequence, because of $f^2 = 1 + |\nabla f|^2 \geq 1$, the function f has constant sign and hence we may assume that $f \geq 1$ up to changing f into $-f$. In particular, the only possible critical value of f is 1, which is an absolute minimum of f. If γ is any integral curve of the normalized gradient vector field $\nu = \frac{\nabla f}{|\nabla f|}$, which is defined at least on the set of regular points of f, then the function $y := f \circ \gamma$ satisfies the ODEs $y' = \sqrt{y^2 - 1}$, therefore $y(t) = \cosh(t + \phi)$ for some real constant ϕ. Since that function has an absolute minimum, it must have a critical point. It remains to notice that $f(M) = [1, \infty)$ and thus that M cannot be compact. This shows statement 7.(c) and concludes the proof of Lemma 2.1.

$\square$

Example 2.2 In dimension 3, Lemma 2.1 implies that, starting with any Einstein – or, equivalently, constant-sectional-curvature- – manifold (M^3, g) and any real function u such that $\Delta u = \frac{S}{3}$, the function $f := e^{-u}$ satisfies (1) on the manifold $(M, \overline{g} = e^{-2u} g)$. In particular, since there is an infinite-dimensional space of harmonic functions on any nonempty open subset M of $\mathbb{R}^3$, there are many nonhomothetic conformal metrics on such M for which nonconstant solutions of (1) exist. On any nonempty open subset of the 3-dimensional hyperbolic space $\mathbb{H}^3$ with constant sectional curvature -1, there is also an infinite-dimensional affine space of solutions to the Poisson equation $\Delta u = -2$: in geodesic polar coordinates about any fixed point $p \in \mathbb{H}^3$, assuming u to depend only on the geodesic distance r to p, that Poisson equation is a second-order linear ODE in $u(r)$ and therefore has infinitely many affinely independent solutions. In particular, there are also lots of conformal metrics on $\mathbb{H}^3$ for which nonconstant solutions of (1) exist.
Note however that, although $\mathbb{H}^3$ is conformally equivalent to the unit open ball $\mathbb{B}^3$ in $\mathbb{R}^3$, we do not obtain the same solutions to the equation depending on the metric we start from. Namely, we can construct solutions of (1) starting from the Euclidean metric g and from the hyperbolic metric $e^{2w} g$ on $\mathbb{B}^3$, where $e^{2w(x)} = \frac{4}{(1-|x|^2)^2}$ at any $x \in \mathbb{B}^3$. In both cases we obtain solutions of (1) by conformal change of the metric. Since g and $e^{2w} g$ lie in the same conformal class, the question arises whether solutions coming from $e^{2w} g$ can coincide with solutions coming from g on $\mathbb{B}^3$. Assume f were a solution of (1) arising by conformal change of g (by e^{-2u} for some $u \in C^\infty(\mathbb{B}^3)$) and by conformal change of $e^{2w} g$ (by e^{-2v} for some $v \in C^\infty(\mathbb{B}^3)$). Then $f = e^{-u} = e^{-v}$ and thus $v = u$ would hold, therefore u would satisfy $\Delta_g u = 0$ as well as $\Delta_{e^{2w} g} u = -2$, in particular

$$\begin{aligned} 0 &= \Delta_g u \\ &= e^{2w} \Delta_{e^{2w} g} u + \langle dw, du \rangle_g \\ &= -2e^{2w} + \langle dw, du \rangle_g . \end{aligned}$$

But the r.h.s. of the last identity has no reason to vanish in general. Note also that the conformal metrics themselves have no reason to coincide, since otherwise $e^{-2v} e^{2w} g = e^{-2u} g$ would hold hence $u = v - w$ as well and the same kind of argument would lead to an equation that is generally not fulfilled.

Note 2.3 If S is a nonzero constant and M is closed, then the function f is an eigenfunction for the scalar Laplace operator associated to the eigenvalue S on (M, g) and it has at least two nodal domains. Mind however that S is not necessarily the first positive Laplace eigenvalue on (M, g). E.g. consider the Riemannian manifold $M = \mathbb{S}^2 \times \Sigma^{n-2}$ which is the product of standard $\mathbb{S}^2$ with a closed Ricci-flat manifold Σ^{n-2}, then the first positive Laplace eigenvalue of Σ can be made arbitrarily small by rescaling its metric; since the Laplace spectrum of M is the sum of the Laplace spectra of $\mathbb{S}^2$ and Σ, the first Laplace eigenvalue on M can be made as close to 0 as desired by rescaling the metric on Σ.

2.2 Classification in presence of a particular Killing vector field

Next we aim at describing the structure of M using the flow $(F_t^\nu)_t$ of ν. Namely, outside the possible critical points of f, the manifold M is locally diffeomorphic via $(F_t^\nu)_t$ to the product $I \times N_c$ of an open interval with a regular level hypersurface N_c of f. Moreover, the induced metric has the form

$dt^2 \oplus g_t$ for some one-parameter-family of Riemannian metrics on N_c. To determine g_t, one would need to know the Lie derivative of g w.r.t. ν; but for all $X, Y \in TN_c$,

$$(\mathcal{L}_\nu g)(X,Y) = \langle \nabla_X \nu, Y\rangle + \langle \nabla_Y \nu, X\rangle \stackrel{(6)}{=} \frac{2}{|\nabla f|}\left(\langle \nabla^2_X f, Y\rangle - \langle \nabla^2_X f, \nu\rangle\langle \nu, Y\rangle\right) = -\frac{2f}{|\nabla f|}\mathrm{ric}(X,Y)$$

and we do not know *a priori* more about the Ricci curvature of M. Besides, we have *a priori* no information either on the critical subsets $\{\nabla f = 0\}$, we do not even know whether they are totally geodesic submanifolds or not.

Therefore, we introduce more assumptions. We actually introduce some that fit to the particular geometric setting induced by so-called skew Killing spinors, see [3].

Theorem 2.4 *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 3$ and constant scalar curvature S carrying a nonconstant real-valued smooth function f satisfying (1). Up to rescaling the metric as well as f we can assume that $S = 2\epsilon$ with $\epsilon \in \{-1, 0, 1\}$ and $\max(f) = 1 = -\min(f)$ if $S = 2$, $|\nabla f| = 1$ if $S = 0$ and $-f^2 + |\nabla f|^2 \in \{-1, 0, 1\}$ if $S = -2$. Assume also the existence of a non-identically vanishing Killing vector field η on M such that*

- *the vector fields η and ∇f are pointwise orthogonal,*
- *the vector field $\eta_{|_{N_c}}$ is parallel along $N_c := f^{-1}(\{c\})$ for every regular value c of f,*
- *if $S \neq 0$, the vector field η satisfies $\inf_M(|\eta|) = \inf_M(|\nabla f|)$ and vanishes where ∇f does,*
- *also $\nabla_\eta \eta = \epsilon f \nabla f$ holds in case $n > 4$ or f has no critical point.*

Then M is isometric to either the Riemannian product $S(\epsilon) \times \Sigma^{n-2}$ of the simply-connected complete surface with curvature ϵ with a complete Ricci-flat manifold Σ in case $\epsilon \neq 0$, or to the Riemannian product of $\mathbb{R}$ with some complete Ricci-flat manifold carrying a nonzero parallel vector field in case $\epsilon = 0$.

Proof: The assumption that $\eta \perp \nabla f$ not only means that the flow of η preserves the level hypersurfaces of f, but also implies that $[\eta, \nabla f] = [\eta, \nu] = 0$: for

$$0 = X(\langle \eta, \nabla f\rangle) = \langle \nabla_X \eta, \nabla f\rangle + \langle \eta, \nabla^2_X f\rangle = \langle \nabla^2_\eta f - \nabla_{\nabla f}\eta, X\rangle \qquad \forall X \in TM,$$

so that $\nabla^2_\eta f = \nabla_{\nabla f}\eta$, that is, $\nabla_{\nabla f}\eta = -f\mathrm{Ric}(\eta)$ and it also follows that

$$[\eta, \nabla f] = \nabla_\eta \nabla f - \nabla_{\nabla f}\eta = \nabla^2_\eta f - \nabla^2_\eta f = 0.$$

As a further consequence of $[\eta, \nabla f] = 0$, using again that η is Killing,

$$[\eta, \nu] = \eta(\frac{1}{|\nabla f|})\nabla f = -\frac{\langle \nabla_\eta \nabla f, \nabla f\rangle}{|\nabla f|^3}\nabla f = -\frac{\langle \nabla_{\nabla f}\eta, \nabla f\rangle}{|\nabla f|^3}\nabla f = 0.$$

In particular, the flow of ν preserves η and conversely the flow of η preserves both ∇f and ν. Next we examine the assumption that $\eta_{|_{N_c}}$ is parallel on $N_c = f^{-1}(\{c\})$, which is a smooth hypersurface for all but finitely many values of c by Lemma 2.1. By Gauß-Weingarten formula, $\nabla^{N_c}\eta = 0$ is equivalent to

$$\nabla_X \eta = \nabla^{N_c}_X \eta + \langle W\eta, X\rangle\nu = \langle W\eta, X\rangle\nu \stackrel{(6)}{=} -\frac{1}{|\nabla f|}\langle \nabla^2_\eta f, X\rangle\nu = \frac{f}{|\nabla f|}\langle \mathrm{Ric}(\eta), X\rangle\nu$$

for all $X \in TN_c$, where $W = -\nabla\nu$ is the Weingarten-endomorphism-field of N_c in M. With $\nabla_\nu \eta = -\frac{f}{|\nabla f|}\mathrm{Ric}(\eta)$, the above identity is equivalent to

$$\nabla\eta = \frac{f}{|\nabla f|} \cdot (\mathrm{Ric}(\eta) \otimes \nu - \nu \otimes \mathrm{Ric}(\eta)) = \frac{f}{|\nabla f|} \cdot \mathrm{Ric}(\eta) \wedge \nu. \tag{7}$$

In particular $\mathrm{rk}(\nabla\eta) \leq 2$ on the subset of regular points of f. Moreover, η cannot vanish anywhere on the subset of regular points of f: for if η vanished at some regular point x, then η would vanish along the level hypersurface containing x and, being preserved by the flow of ν, it would have to vanish identically on a nonempty open subset of M and therefore on M, which would be a contradiction. Thus $\eta^{-1}(\{0\}) \subset (\nabla f)^{-1}(\{0\})$. On the other hand, the assumption $(\nabla f)^{-1}(\{0\}) \subset \eta^{-1}(\{0\})$ yields $\eta^{-1}(\{0\}) = (\nabla f)^{-1}(\{0\})$. In particular, when nonempty, $(\nabla f)^{-1}(\{0\})$ is a totally geodesic submanifold of M (vanishing set of a Killing vector field) and has even codimension, which is positive otherwise η would vanish identically.
By e.g. [8, Sec. 2.5], the tangent bundle of $\eta^{-1}(\{0\})$ is given by $\ker(\nabla\eta)$ and therefore it has pointwise dimension at most $n-2$. But since $\mathrm{rk}(\nabla\eta) \leq 2$ on $M \setminus \eta^{-1}(\{0\})$, which is a dense open subset of M, the inequality $\mathrm{rk}(\nabla\eta) \leq 2$ must hold along $\eta^{-1}(\{0\})$ by continuity, in particular $\dim(\ker(\nabla\eta)) \geq n-2$ and thus $\dim(\ker(\nabla\eta)) = n-2$ along $\eta^{-1}(\{0\})$. On the whole, when nonempty, the set of critical points of f is a possibly disconnected $(n-2)$-dimensional totally geodesic submanifold of M.
As a further step, we translate Gauß equations for Ricci curvature along each N_c in our context. Denoting $W = -\nabla\nu = \frac{f}{|\nabla f|}\mathrm{Ric}^T$ the Weingarten-endomorphism-field of N_c in M, where Ric^T is the pointwise othogonal projection of Ric onto TM, we have $\mathrm{tr}(W) = \frac{f}{|\nabla f|} \cdot \frac{S}{2}$ by $\mathrm{Ric}(\nu) = \frac{S}{2}\nu$. As a consequence, we have, for all $X \in TN_c$:

$$\begin{aligned}
\mathrm{Ric}(X) &= \mathrm{Ric}(X)^T \\
&= \mathrm{Ric}_{N_c}(X) + W^2X - \mathrm{tr}(W)WX + R_{X,\nu}\nu \\
&= \mathrm{Ric}_{N_c}(X) + \frac{f^2}{|\nabla f|^2} \cdot \left(\mathrm{Ric}^2(X) - \frac{S}{2}\mathrm{Ric}(X)\right) + R_{X,\nu}\nu.
\end{aligned}$$

We can compute the curvature term $R_{X,\nu}\nu$ explicitely: choosing a vector field X that is pointwise tangent to the regular level hypersurfaces of f, we can always assume w.l.o.g. that $[X,\nu] = 0$ at the point where we compute, so that, with $\nabla_\nu\nu = 0$,

$$\begin{aligned}
R_{X,\nu}\nu &= -\nabla_\nu\nabla_X\nu \\
&= \nabla_\nu\left(\frac{f}{|\nabla f|}\mathrm{Ric}(X)\right) \\
&= \nu\left(\frac{f}{|\nabla f|}\right) \cdot \mathrm{Ric}(X) + \frac{f}{|\nabla f|} \cdot (\nabla_\nu \mathrm{Ric})(X) + \frac{f}{|\nabla f|} \cdot \mathrm{Ric}(\nabla_\nu X) \\
&= \frac{\nu(f)|\nabla f| - f\nu(|\nabla f|)}{|\nabla f|^2} \cdot \mathrm{Ric}(X) + \frac{f}{|\nabla f|} \cdot (\nabla_\nu \mathrm{Ric})(X) + \frac{f}{|\nabla f|} \cdot \mathrm{Ric}(\nabla_X \nu) \\
&= \frac{|\nabla f|^2 - f\frac{\langle \nabla_\nu \nabla f, \nabla f\rangle}{|\nabla f|}}{|\nabla f|^2} \cdot \mathrm{Ric}(X) + \frac{f}{|\nabla f|} \cdot (\nabla_\nu \mathrm{Ric})(X) - \frac{f^2}{|\nabla f|^2} \cdot \mathrm{Ric}^2(X) \\
&= \left(1 + \frac{Sf^2}{2|\nabla f|^2}\right) \cdot \mathrm{Ric}(X) + \frac{f}{|\nabla f|} \cdot (\nabla_\nu \mathrm{Ric})(X) - \frac{f^2}{|\nabla f|^2} \cdot \mathrm{Ric}^2(X).
\end{aligned}$$

We can deduce that, for every $X \in TN_c$,

$$\mathrm{Ric}_{N_c}(X) = \mathrm{Ric}(X) - \frac{f^2}{|\nabla f|^2} \cdot \left(\mathrm{Ric}^2(X) - \frac{S}{2}\mathrm{Ric}(X)\right) - R_{X,\nu}\nu$$

$$
\begin{aligned}
&= \operatorname{Ric}(X) - \frac{f^2}{|\nabla f|^2}\cdot\left(\operatorname{Ric}^2(X) - \frac{S}{2}\operatorname{Ric}(X)\right)\\
&\quad -\left(1+\frac{Sf^2}{2|\nabla f|^2}\right)\cdot\operatorname{Ric}(X) - \frac{f}{|\nabla f|}\cdot(\nabla_\nu \operatorname{Ric})(X) + \frac{f^2}{|\nabla f|^2}\cdot\operatorname{Ric}^2(X)\\
&= -\frac{f}{|\nabla f|}\cdot(\nabla_\nu\operatorname{Ric})(X). \qquad (8)
\end{aligned}
$$

That identity has important consequences. First, choosing a local o.n.b. $(e_j)_{1\leq j\leq n-1}$ of TN_c,

$$
\begin{aligned}
S_{N_c} &= \sum_{j=1}^{n}\langle \operatorname{Ric}_{N_c}(e_j), e_j\rangle\\
&= -\frac{f}{|\nabla f|}\cdot\sum_{j=1}^{n}\langle(\nabla_\nu\operatorname{Ric})(e_j), e_j\rangle\\
&= -\frac{f}{|\nabla f|}\cdot\left(\sum_{j=1}^{n}\langle(\nabla_\nu\operatorname{Ric})(e_j), e_j\rangle + \langle(\nabla_\nu\operatorname{Ric})(\nu),\nu\rangle\right) + \frac{f}{|\nabla f|}\cdot\langle(\nabla_\nu\operatorname{Ric})(\nu),\nu\rangle,
\end{aligned}
$$

with $(\nabla_\nu\operatorname{Ric})(\nu) = \nabla_\nu(\operatorname{Ric}(\nu)) - \operatorname{Ric}(\nabla_\nu\nu) = \nabla_\nu(\frac{S}{2}\nu) = 0$, so that

$$
S_{N_c} = -\frac{f}{|\nabla f|}\cdot\operatorname{tr}(\nabla_\nu\operatorname{Ric}) = -\frac{f}{|\nabla f|}\cdot\nu(\operatorname{tr}(\operatorname{Ric})) = -\frac{f}{|\nabla f|}\cdot\nu(S) = 0.
$$

Therefore, each level hypersurface N_c is scalar-flat.
In the case where $n = 3$ or 4, the manifold N_c is locally the Riemannian product of a flat manifold with an interval and is hence also flat, in particular $\operatorname{Ric}_{N_c} = 0$, which in turn implies that

$$
\nabla_\nu\operatorname{Ric} = 0. \qquad (9)
$$

This equation, which holds on the dense open subset $\{\nabla f \neq 0\}$, means that all eigenspaces and eigenvalues of the Ricci-tensor of M are preserved under parallel transport along integral curves of ν. Assume first that f has critical points, in particular $S \neq 0$. Along the critical submanifold $N_{\text{crit}} := (\nabla f)^{-1}(\{0\})$, one has $\ker(\operatorname{Ric}) \supset TN_{\text{crit}}$: if $c\colon I \to N_{\text{crit}}$ is any smooth curve, then $f\circ c$ is constant and therefore $0 = (f\circ c)'' = \langle\nabla^2_{\dot c} f, \dot c\rangle$ (the gradient of f vanishes along N_{crit}), so that $\operatorname{ric}(\dot c,\dot c) = 0$. But $\nabla^2 f$ and thus Ric is either nonpositive or nonnegative along N_{crit} because N_{crit} is a set of minima or maxima of f as we have seen in Lemma 2.1, therefore $\operatorname{Ric}(\dot c) = 0$. In particular, 0 is an eigenvalue of multiplicity at least $n-2$ of the Ricci-tensor; since the Ricci-eigenvalues are constant along the integral curves of ν, it can be deduced that 0 is an eigenvalue of multiplicity at least $n-2$ everywhere in M. But the multiplicity cannot be greater that $n-2$, otherwise Ric would have only one nonzero eigenvalue (namely $\frac{S}{2}\in\{\pm1\}$) and hence its trace would be $\frac{S}{2}$, contradiction. Therefore 0 is an eigenvalue of multiplicity exactly $n-2$ of Ric at every point in M. It remains to notice that at regular points, one has $\operatorname{Ric}(\nu) = \frac{S}{2}\nu$ and $\operatorname{Ric}(\eta)\perp\nu$, so that, using $\operatorname{Ric}(\eta)\perp\ker(\operatorname{Ric})$, we deduce that $\operatorname{Ric}^2(\eta)$ is proportional to $\operatorname{Ric}(\eta)$, the eigenvalue being necessarily equal to $\frac{S}{2}$, that is,

$$
\operatorname{Ric}^2(\eta) = \frac{S}{2}\operatorname{Ric}(\eta). \qquad (10)
$$

This allows for η to be normalized as we explain next. Namely we would like $\nabla_\eta\eta = \epsilon f\nabla f$ to hold on M. Let $\gamma\colon(-\varepsilon,\varepsilon)\to M$ be any integral curve of ν with starting point $\gamma(0)$ in some regular level hypersurface of f; we have already seen in the proof of Lemma 2.1 that $y := f\circ\gamma$ does not depend on the starting point $\gamma(0)$ of γ in a fixed level hypersurface of f. Since, as explained above, the vector field ν is geodesic on M and η is parallel along each N_c, the function $h := |\eta|^2 \circ \gamma$ only depends on t and not on the starting point $\gamma(0)$. In other words, $\nabla(|\eta|^2) = \nu(|\eta|^2)\nu$. But

$$\nu(|\eta|^2) = 2\langle \nabla_\nu \eta, \eta \rangle = -2\langle \nabla_\eta \eta, \nu \rangle = -2\frac{f}{|\nabla f|}\mathrm{ric}(\eta, \eta), \tag{11}$$

so that $\mathrm{ric}(\eta, \eta)$ also only depends on t. By (8) and $\mathrm{Ric}_{N_c}(\eta) = 0$ because of $\eta_{|N_c}$ being parallel, we have, outside $N_0 = f^{-1}(\{0\})$,

$$\begin{aligned} \nu(\mathrm{ric}(\eta, \eta)) &= (\nabla_\nu \mathrm{ric})(\eta, \eta) + 2\mathrm{ric}(\nabla_\nu \eta, \eta) \\ &\overset{(8)}{=} -\frac{|\nabla f|}{f}\underbrace{\langle \mathrm{Ric}_{N_c}(\eta), \eta\rangle}_{0} -2\frac{f}{|\nabla f|}\mathrm{ric}(\mathrm{Ric}(\eta), \eta) \\ &= -2\frac{f}{|\nabla f|}|\mathrm{Ric}(\eta)|^2. \end{aligned} \tag{12}$$

Note here that both (11) and (12) are valid in any dimension and without the condition (10). Combining (12) with (10) and using $\nu(|\nabla f|^2) = -2\epsilon f|\nabla f|$, we deduce that

$$\nu(\mathrm{ric}(\eta, \eta)) = -2\epsilon\frac{f}{|\nabla f|}\mathrm{ric}(\eta, \eta) = \frac{\nu(|\nabla f|^2)}{|\nabla f|^2}\mathrm{ric}(\eta, \eta).$$

As a consequence, there exists a real constant C, that has the sign of ϵ, such that $\mathrm{ric}(\eta, \eta) = C\cdot|\nabla f|^2$ and thus $\nabla_\eta \eta = Cf\nabla f$ on M. Therefore, up to replacing η by $\frac{1}{\sqrt{C\epsilon}} \cdot \eta$, we may assume that $\nabla_\eta \eta = \epsilon f \nabla f$ on M. Note that this concerns only the case where $n \in \{3, 4\}$ and f has critical points, otherwise we *assume* $\nabla_\eta \eta = \epsilon f\nabla f$ to hold on M.
Assuming from now on $\nabla_\eta \eta = \epsilon f \nabla f$ and $n \geq 3$ to hold, it can be deduced that $|\eta|^2 = -\epsilon f^2 + \mathrm{cst}$ for some $\mathrm{cst} \in \mathbb{R}$: Namely $\nabla(|\eta|^2) = -2\nabla_\eta \eta = -2\epsilon f\nabla f = -\epsilon\nabla(f^2)$ and the set of regular points of f is connected. Moreover, using e.g. (11), we have $\mathrm{ric}(\eta, \eta) = \epsilon|\nabla f|^2$; differentiating that identity w.r.t. ν and using (12) yields $|\mathrm{Ric}(\eta)| = |\epsilon| \cdot |\nabla f|$. In case $S = 2$, we have $\eta = 0$ on $(\nabla f)^{-1}(\{0\}) = f^{-1}(\{\pm 1\})$, so that $\mathrm{cst} = 1$ and hence $|\eta| = \sqrt{1 - f^2} = |\nabla f|$. In case $S = 0$, we have $|\eta|^2 = \mathrm{cst}$, from which $\mathrm{ric}(\eta, \eta) = 0$ and even $\mathrm{Ric}(\eta) = 0$ follow. In case $S = -2$, we have $-f^2 + |\nabla f|^2 = \frac{\mu}{2} \in \{-1, 0, 1\}$, so that $|\eta|^2 = f^2 + \mathrm{cst} = |\nabla f|^2 - \frac{\mu}{2} + \mathrm{cst}$. Now the assumption $\inf_M (|\eta|) = \inf_M (|\nabla f|)$ yields $\mathrm{cst} - \frac{\mu}{2} = 0$, in particular $|\eta| = |\nabla f|$.
To sum up, in all cases we obtain $\mathrm{ric}(\eta, \eta) = \epsilon|\mathrm{Ric}(\eta)| \cdot |\eta|$, which is exactly the equality case in Cauchy-Schwarz inequality. We can thus deduce that $\mathrm{Ric}(\eta)$ is proportional to η and hence $\mathrm{Ric}(\eta) = \frac{S}{2}\eta$.

If $n > 4$ or if f has no critical point, it remains to show that $\mathrm{Ric}(\eta) = \frac{S}{2}\eta$ implies $\ker(\mathrm{Ric}) = \{\eta, \nu\}^\perp$: for we already know from $\mathrm{Ric}_{N_c} = -\frac{f}{|\nabla f|} \cdot (\nabla_\nu \mathrm{Ric})$ that $S_{N_c} = 0$. But by the Gauß formula, $S_{N_c} = S - 2\mathrm{ric}(\nu, \nu) + \mathrm{tr}(W)^2 - |W|^2$, so that, with $S - 2\mathrm{ric}(\nu, \nu) = 0$, we deduce that

$$\begin{aligned} 0 &= \frac{f^2}{|\nabla f|^2} \cdot \left(\mathrm{tr}(\mathrm{Ric}^T)^2 - |\mathrm{Ric}^T|^2\right) \\ &= \frac{f^2}{|\nabla f|^2} \cdot \left(\frac{S^2}{4} - \frac{|\mathrm{Ric}(\eta)|^2}{|\eta|^2} - |\mathrm{Ric}_{\{\eta,\nu\}^\perp}|^2\right) \\ &= \frac{f^2}{|\nabla f|^2} \cdot \left(\frac{S^2}{4} - \frac{S^2}{4} - |\mathrm{Ric}_{\{\eta,\nu\}^\perp}|^2\right) \\ &= -\frac{f^2}{|\nabla f|^2} \cdot |\mathrm{Ric}_{\{\eta,\nu\}^\perp}|^2, \end{aligned}$$

from which $\mathrm{Ric}_{\{\eta,\nu\}^\perp}=0$ follows.
We have now all we need to conclude that both distributions $\mathrm{Span}(\eta,\nu)$ and its orthogonal complement are integrable and totally geodesic, the first one being of constant curvature ϵ and the second one being Ricci-flat (hence flat if $n=3$ or 4). Namely we already know that $\mathrm{Span}(\eta,\nu)$ is integrable since $[\eta,\nu]=0$. Moreover,

$$\begin{aligned}
\nabla_\eta\eta &= \epsilon f\nabla f=\epsilon f|\nabla f|\nu\\
\nabla_\eta\nu &= -\frac{f}{|\nabla f|}\mathrm{Ric}(\eta)=-\frac{Sf}{2|\nabla f|}\eta=-\frac{\epsilon f}{|\nabla f|}\eta\\
\nabla_\nu\eta &= \nabla_\eta\nu=-\frac{\epsilon f}{|\nabla f|}\eta\\
\nabla_\nu\nu &= 0,
\end{aligned}$$

so that all above expressions lie in $\mathrm{Span}(\eta,\nu)$, in particular $\mathrm{Span}(\eta,\nu)$ is totally geodesic. As for $\mathrm{Span}(\eta,\nu)^\perp$, we compute, for all $X,Y\in\Gamma\left(\mathrm{Span}(\eta,\nu)^\perp\right)$,

$$\begin{aligned}
\langle\nabla_XY,\eta\rangle &= -\langle Y,\nabla_X\eta\rangle\\
&\overset{(7)}{=} -\frac{f}{|\nabla f|}\cdot(\mathrm{ric}(\eta,X)\langle\nu,Y\rangle-\langle\nu,X\rangle\mathrm{ric}(\eta,Y))\\
&= 0
\end{aligned}$$

and, using $\mathrm{Span}(\eta,\nu)^\perp=\ker(\mathrm{Ric})$,

$$\begin{aligned}
\langle\nabla_XY,\nu\rangle &= -\langle Y,\nabla_X\nu\rangle\\
&= \frac{f}{|\nabla f|}\mathrm{ric}(X,Y)\\
&= 0.
\end{aligned}$$

It follows that $\nabla_XY\in\Gamma\left(\mathrm{Span}(\eta,\nu)^\perp\right)$, therefore this distribution is integrable and totally geodesic. To compute the curvature of both integral submanifolds, we notice that, from the above computations, $R_{\eta,\nu}\nu=\mathrm{Ric}(\eta)=\epsilon\eta$ and $R_{X,\nu}\nu=0=R_{X,\eta}\eta$ for all $X\in\ker(\mathrm{Ric})$, so that

$$\frac{\langle R_{\eta,\nu}\nu,\eta\rangle}{|\eta|^2}=\frac{\epsilon|\eta|^2}{|\eta|^2}=\epsilon$$

and, using the Gauß formula for curvature, for all $X,Y\in\Gamma\left(\mathrm{Span}(\eta,\nu)^\perp\right)$,

$$\mathrm{ric}_\Sigma(X,Y)=\mathrm{ric}(X,Y)-\langle R_{X,\frac{\eta}{|\eta|}}\frac{\eta}{|\eta|},Y\rangle-\langle R_{X,\nu}\nu,Y\rangle=0,$$

where we denoted by ric_Σ the Ricci curvature of the integral submanifold Σ of $\mathrm{Span}(\eta,\nu)^\perp$. Therefore, Σ is Ricci-flat and thus flat if 1-or 2-dimensional. On the whole, this shows that the holonomy group of M splits locally, therefore the universal cover of M is isometric to the Riemannian product $S(\epsilon)\times\tilde{\Sigma}$ of the simply-connected complete surface with curvature $\epsilon\in\{-1,0,1\}$ with some simply-connected Ricci-flat manifold $\tilde{\Sigma}$. In case $\epsilon=1$, the lift $\tilde{f}$ of f to $\mathbb{S}^2\times\tilde{\Sigma}$ is constant along the $\tilde{\Sigma}$-factor and satisfies the equation $(\nabla^{\mathbb{S}^2})^2f=-f\cdot\mathrm{Id}$, which is exactly the equation characterizing the eigenfunctions associated to the first positive Laplace eigenvalue [9, Theorem A]. Furthermore, the isometry group of $\mathbb{S}^2\times\tilde{\Sigma}$ embeds into the product group of both isometry groups of $\mathbb{S}^2$ and $\tilde{\Sigma}$ and the first factor must be trivial since $\tilde{f}$, as the restriction of a linear form from $\mathbb{R}^3$ onto $\mathbb{S}^2$, is not invariant under $\{\pm\mathrm{Id}\}$. Therefore, M is isometric to $\mathbb{S}^2\times\Sigma$ for some Ricci-flat Σ and f is the trivial extension of an eigenfunction associated to the first positive Laplace eigenvalue on $\mathbb{S}^2$.

In case $\epsilon = 0$, the manifold M is Ricci-flat and therefore is isometric to the Riemannian product of $\mathbb{R}$ with a Ricci-flat manifold $\overline{N}$ as we mentioned in the introduction; our supplementary assumptions only mean that $\overline{N}$ carries a nontrivial parallel vector field. Mind in particular that $\overline{N}$ is not necessarily isometric to the Riemannian product of $\mathbb{R}$ or $\mathbb{S}^1$ with some Ricci-flat manifold, even if this is obviously locally the case.
In case $\epsilon = -1$, the lift $\tilde{f}$ of f to $\mathbb{H}^2 \times \tilde{\Sigma}$ is constant along the $\tilde{\Sigma}$-factor and satisfies the equation $(\nabla^{\mathbb{H}^2})^2 f = f \cdot \mathrm{Id}$, which is exactly the Tashiro equation. Since the isometry group of $\mathbb{H}^2 \times \tilde{\Sigma}$ embeds into the product group of both isometry groups of $\mathbb{H}^2$ and $\tilde{\Sigma}$ and the first factor must be trivial since $\tilde{f}$ has no nontrivial symmetry [11, Theorem 2 p.252], we can deduce as above that M is isometric to $\mathbb{H}^2 \times \Sigma$ for some Ricci-flat Σ and f is the trivial extension of a solution to the Tashiro equation on $\mathbb{H}^2$. This concludes the proof of Theorem 2.4. □

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