Massless Dirac-Maxwell systems in asymptotically flat spacetimes

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Abstract

We consider the Cauchy problem of massless Dirac-Maxwell equations on an asymptotically flat background and give a global existence and uniqueness theorem for initial values small in an appropriate weighted Sobolev space.

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1 Introduction, terminology, state of the art, main result

Electrodynamics and its coupling to matter is, on a fundamental level, described by Dirac-Maxwell systems. Whereas the vast majority of physicists is convinced nowadays that the truly fundamental description is not classical but a quantum field theory, most quantization schemes do take some profit from a knowledge of the space of global solutions of the classical equations. Moreover, some quantization procedures (as the quantization via Fock spaces in a perturbative and geometric quantization in a nonperturbative manner) are even based conceptually on this space. Therefore classical global solvability of the equations is an important question in this context.
The setup for the entire article is the following: Consider a globally hyperbolic spin manifold (M, g) and a trivial $U(1)$-principal bundle $\pi : E \to M$. Let A be a $U(1)$-principal bundle connection form on π, or equivalently, an invariant $i\mathbb{R}$-valued one-form on E. We will assume in the following that M is simply-connected in which case it is well-known that we can regard A as a real-valued one-form on M. We denote the standard spinor bundle by $\sigma : \Sigma \to M$. Then any one-form A induces a covariant derivative ∇^A on σ by $\nabla^A_X(\psi) := \nabla_X\psi + iA(X)\psi$. Consequently, we have for the corresponding Dirac operators $D^A := i\sum_{j=0}^n \epsilon_j e_j \cdot \nabla^A_{e_j}$ and $D := D^0$ the relation $D^A\psi = D\psi - A \cdot \psi$. Now the **Dirac-Maxwell Lagrangian** $L^T_S(DM)$, for two Cauchy surfaces S, T of (M, g) is a functional on $\Gamma(\bigoplus_{l=1}^N \sigma) \oplus \Omega^1(M)$ defined as $L^T_S(DM)(\psi, A) =$

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$\int_{I^+(S)\cap I^-(T)} \mathcal{L}_{DM}(\psi \oplus A)$ where $\mathcal{L}_{DM}$ is the **Dirac-Maxwell Lagrangian density for N particles of charges** $\mathrm{sgn}(\mu_1)\sqrt{|\mu_1|}, \ldots, \mathrm{sgn}(\mu_N)\sqrt{|\mu_N|}$ **and masses** $m_1, \ldots, m_N$ denoted as $\mathcal{L}_{DM}$ and defined by (writing $\psi = (\psi^1, \ldots, \psi^N)$)

$$\mathcal{L}_{DM}(\psi \oplus A) := \frac{1}{4} tr(F^A \wedge F^A) + \sum_{l=1}^{N} \frac{1}{2}(\langle D^{\mu_l A}\psi^l, \psi^l\rangle + \langle \psi^l, D^{\mu_l A}\psi^l\rangle) - \sum_{l=1}^{N} m_l \langle \psi^l, \psi^l\rangle,$$

The critical values of the Lagrangian are exactly the preimages of zero under the operator P_{DM} given by

$$P_{DM}(\psi^1 \oplus ... \oplus \psi^N \oplus A) = (D^{\mu_1 A}\psi^1 - m_1\psi^1, \ldots, D^{\mu_N A}\psi^N - m_N\psi^N, d^*F - J_\psi),$$

where $J_\psi(X) := \sum_{l=1}^N \mu_l \cdot j_{ll}(X)$ and $j_{kl}(X) := \langle X \cdot \psi^k, \psi^l\rangle$. If ψ^k and ψ^l have equal mass and charge, then it is easy to see that $d^* j_{kl} = 0$, thus in particular J_ψ is divergence-free for $(\psi, A) \in P_{DM}^{-1}(0)$. In the sequel, we shall call a pair $(\psi = (\psi^1, \ldots, \psi^N), A)$ as above a solution to the **Dirac-Maxwell equation** if $(\psi, A) \in P_{DM}^{-1}(0)$, that is, if

$$D^{\mu_l A}\psi^l = m_l\psi^l, \ \ l = 1, \ldots, N \qquad \text{and} \qquad d^* dA = J_\psi.$$

The *massless* Dirac-Maxwell equation is the Dirac-Maxwell equation with $m_1 = \ldots = m_N = 0$.

Let us first shortly review the state of the art on this subject. Considering the fact that the massless Dirac-Maxwell equation is in dimension 4 conformally invariant, Christodoulou and Choquet-Bruhat [5] show existence of solutions of Dirac-Yang-Mills-Higgs solutions on four-dimensional Minkowski space with initial values small in weighted Sobolev spaces. The weights they use come from certain conformal rescalings. One could try to apply their result to Maxwell-Dirac Theory, but, as we are going to explain in the next paragraph, this is only possible if we extend their setting to a system of finitely many particles whose charge changes sign and of vanishing mass each. Psarelli [17], in contrast, treated the question of Dirac-Maxwell equations with or without mass on $\mathbb{R}^{1,3}$ (not in terms of connections modelling potentials, but in terms of curvature tensors modelling field strength[1]), with results of the form: If C is any compact subset of a Cauchy surface S of $\mathbb{R}^{1,3}$ then there is a number a depending on C such that, if some initial values I such that (among others) the spinor part is supported in C have Sobolev norm smaller than a, then there is a global solution with initial values I. In the massless case, this result is of course strictly weaker than the weighted Sobolev result. Georgiev [14] established the global existence for massless or massive Maxwell-Dirac solutions for initial data sufficiently small in some weighted Sobolev norm in $\mathbb{R}^{1,3}$. The aim of the present article is to generalize Georgiev's results to the case of asymptotically flat spacetimes.

Choquet-Bruhat and Christodoulou, in their approach [5] to the problem, consider a more general setting by adding another dynamical field, namely the *Higgs field* providing a mass of some sort[2]

[1] Recall, however, that the Aharanov-Bohm effect shows that rather than the electromagnetic fields, the potentials play the more fundamental role in electrodynamics

[2] i.e., the coupling can be chosen in such a way that any solution with constant Higgs field gives rise to a solution to the massive Dirac equation.

and governed itself by the Laplace-d'Alembert operator. Now, if we want to represent the Maxwell-Dirac equation in their setting we have to choose the Higgs field as a constant field. As a good decay of the Higgs field is imposed, the only possible choice is zero, obtaining massless Dirac-Maxwell theory. Also the other fields are assumed to have an extension to the conformal compactification, which is the Einstein cylinder in that case. But this leaves us with pure Maxwell theory, as will be proven in Proposition 2.3 at the end of Section 2. This problem was apparently known to the authors of [5] and was actually the reason for introducing the Higgs field in their setting [6]. Our method, as well as Choquet-Bruhat's and Christodoulou's, needs a good decay of the electromagnetic fields (we consider the fields in a conformally compactified spacetime), thus we assume neutrality of the setting, that is, we consider from the beginning more than one particle and assume the **total charge** $\int_S J_\psi(\nu)$ to vanish, where S is a Cauchy hypersurface with future-directed unit normal ν; note that the total charge is independent of the choice of the Cauchy surface due to j_l being divergence-free. If all μ_l are nonzero, this assumption is of course only satisfiable if there are at least two μ_l of opposite sign. Considering the fact that the constraint equation in Theorem 1.1 describe the transition of the Dirac-Maxwell system to a symmetric hyperbolic system (whose local initial value problem is well-posed), the obstruction of Proposition 2.3 imply that a solution to the constraint equations for a constellation with nonvanishing total charge cannot be extended conformally.

Choquet-Bruhat's and Christodoulou's method naturally uses the well-known Penrose embedding of Minkowski space into the Einstein cylinder. In order to explain this embedding, let us first review some geometric notions as well as introduce some new terminology:
We call a continuous and piecewise C^1 curve c in a time-oriented Lorentzian manifold P **causal future** iff c' is causal future on the C^1 pieces. We call a subset A of P **causally convex** if any causal curve intersects A in the image of a (possibly empty) interval. We call a subset S of P **Cauchy surface** iff any C^0 inextendible causal future curve intersects S exactly once and a subset A **strongly future precompact** iff for any Cauchy surface S of P, the subset $J^+(S) \cap \overline{A}$ is compact.
Now, let (M, g) and (N, h) be globally hyperbolic Lorentzian manifolds, where g, h are supposed to be C^k metrics for some $k \in \mathbb{N} \setminus \{0\}$ (this reduced regularity is essential for our purposes!). An open conformal embedding $f \in C^k(M, N)$ is said to **C^k-extend g conformally** or to be a **C^k-conformal extension of (M, g)** iff $f(M)$ is causally convex and strongly future precompact. A globally hyperbolic manifold (M, g) is, called **C^k-extendable** for $k \in \mathbb{N} \cup \{\infty\}$ iff there is a C^k-conformal extension of (M, g) into a globally hyperbolic manifold.
Whereas Choquet-Bruhat and Christodoulou work with the Penrose embedding which is a C^∞-conformal extension of the entire spacetime, it turns out that, in order to generalize the result by Choquet-Bruhat and Christodoulou, we have to generalize our notion of conformal compactification in a twofold way. First, only the timelike future of a Cauchy surface will be conformally embeddable with open image; furthermore, we have to relax the required regularity of the metric of the target manifolds from C^∞ to C^k. The reason for the second generalization is that we want to include maximal Cauchy developments (g, Φ) of initial values for Einstein-Klein-Gordon theories that satisfy decay conditions at spatial infinity only for finitely many derivatives (controlled by a single weighted Sobolev norm). Thus one cannot control higher derivatives at future null infinity. Consequently, as in order to show critical regularity, we have to admit conformal extensions of (M, g) that are not smooth but only C^k, we need to show a version of the usual existence theorem

for symmetric hyperbolic systems for coefficients of finite regularity. This is done in the Appendix. The second modification is due to the fact that the extension via the Penrose embedding into the Einstein cylinder can, of course, be generalized in a straightforward manner to every compact perturbation of the Minkowski metric. But compact perturbations of Minkowski metric are not very interesting metrics from a physical point of view, as (with interactions like Maxwell theory satisfying the dominant energy condition) a nonzero energy-momentum tensor necessarily entails a positive mass of the metric. And a positive mass of the metric, in turn, is an obstacle to a smooth extension in spatial direction due to the Penrose peeling criterion, for a discussion of this problem see [15, pp. 180-181]. Thus we have necessarily a singularity in the surrounding metric at the spacelike infinity i_0, so we have to restrict to the timelike future of a fixed Cauchy surface.

Results by Anderson and Chruściel (cf. [2, Theorems 5.2, 6.1 & 6.2]), improving earlier results by Friedrich [13] imply that, apart from the — physically less interesting — class of compact perturbations of Minkowski space, there is a rich and more realistic class of manifolds which is C^4-extendable in the sense above, namely the class of all static initial values with Schwarzschildian ends and small initial values in an appropriate Sobolev space — see also the article of Corvino on this topic [8]. This space of initial values is quite rich, which can be seen by the conformal gluing technique of Corvino and Schoen [9]. This holds in any even dimension. And in the case of a four-dimensional space-time manifold, there is, in fact, an even larger class of initial values satisfying the conditions of our global existence theorem which is given by a smallness condition to the Einstein initial values in a weighted Sobolev space encoding a good asymptotic decay towards Schwarzschild initial data, cf. the remark following Theorem 6.2 in [2] and the remarks following Theorem 2.6 in [10]. The maximal Cauchy development of any such initial data set carries even a Cauchy temporal function t such that for all level sets $S_a := t^{-1}(\{a\})$ of t, $I^\pm(S_a)$ are both C^4-extendable and thus satisfy even the stronger assumption of Theorem 1.2. [3]

The central insight presented in this article is that the above mentioned weakened notion of conformal extension suffices to establish — however slightly less explicit — weighted Sobolev spaces of initial values allowing for a global solution. More precisely, our main theorem is the following:

Theorem 1.1 (Main theorem) *Let (M, g) be a 4-dimensional globally hyperbolic spacetime with a Cauchy hypersurface S' such that $I^+(S')$ is C^4-extendable in a globally hyperbolic spacetime (N, h) and let $u \in C^\infty(M)$ with $e^{2u}g = h$. Let P_{DM} be the massless Dirac-Maxwell operator for a finite number of fermion fields. Then, for any Cauchy hypersurface $S \subset I^+(S')$ of N, there is a weighted $W^{4,\infty}$ neighborhood U of 0 in $\pi|_S$ such that there is a solution (ψ, A) of $P_{DM}(\psi, A) = 0$ in all of $I^+(S)$ for every initial value in U with zero total charge w.r.t. S and satisfying the constraint*

[3] This is a remarkable fact as it is a first approach to the question whether *Einstein-Dirac-Maxwell theory is stable around zero*, as the stability theorems imply that Einstein-Maxwell theory is stable around zero initial values for given small Dirac fields, and our main result implies that Maxwell-Dirac Theory is stable around zero for maximal Cauchy developments of small Einstein initial values.

equation

$$
\begin{aligned}
0 &= \frac{1}{\beta}A_1(\frac{\partial}{\partial t}) - \sum_{j=1}^{3}(\nabla_{e_j}A_0)(e_j) \qquad (1.1)\\
0 &= -(\nabla^{\tan})^*\nabla^{\tan}A_0(\frac{\partial}{\partial t}) - \sum_{j=1}^{3}\nabla_{e_j}A_1(e_j) - \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})A_1(\frac{\partial}{\partial t}) + \frac{1}{\beta}A_1(\mathrm{grad}_{g_t}(\beta(t,\cdot)))\\
&\quad + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A_0(\frac{\partial}{\partial t}) + \frac{1}{2}g_t(\nabla^{\tan}A_0, \frac{\partial g_t}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A_0^\sharp) + \sum_{l=1}^{N}\mu_l j_{\psi_0^l}(\frac{\partial}{\partial t}), \qquad (1.2)
\end{aligned}
$$

where $(\nabla^{\tan})^*\nabla^{\tan} := \sum_{j=1}^{n-1}\nabla_{\nabla^{S_t}_{e_j}e_j} - \nabla_{e_j}\nabla_{e_j}$, $A_0 := A_{|S_0} \in \Gamma(T^*M_{|S_0})$, $A_1 := \frac{\nabla A}{\partial t}_{|S_0} \in \Gamma(T^*M_{|S_0})$ *and* $\psi_0^l := \psi^l_{|S_0} \in \Gamma(\sigma_{|S_0})$, *where the metric* $h = e^{2u}g = -\beta dt^2 + g_t$ *is the one of the extension to which also the orthonormal basis* $(e_j)_j$ *refers,* t *a time function with* $S = S_0 = t^{-1}(\{0\})$ *and the spinors for two conformally related metrics are identified as usual.*

Remark: In case $\beta = 1$, which can be assumed without loss of generality by the existence of Fermi coordinates w.r.t. h in a neighbourhood of S, the constraint equations (1.1) simplify to

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\left(A(\frac{\partial}{\partial t})\right) + d_S^*(A_S) + (n-1)H\cdot A(\frac{\partial}{\partial t})\\
0 &= -\Delta_S\left(A(\frac{\partial}{\partial t})\right) + d_S^*\left(\frac{\nabla A}{\partial t}_S\right) - 3g_t(\nabla^S A_S, W) + A(d_S^* W) + 2|W|^2 A(\frac{\partial}{\partial t}) + \mathrm{ric}^M\left(\frac{\partial}{\partial t}, A^\sharp\right)\\
&\quad + \sum_{l=1}^{N}\mu_l j_{\psi_0^l}(\frac{\partial}{\partial t}),
\end{aligned}
$$

where $A_S := \iota_S^* A \in \Gamma(T^*S)$, $\frac{\nabla A}{\partial t}_S := \iota_S^*\frac{\nabla A}{\partial t} \in \Gamma(T^*S)$, $W := \frac{1}{2}g_t^{-1}\frac{\partial g_t}{\partial t}$ is the Weingarten map of $\iota_S : S \hookrightarrow M$, $H := \frac{1}{n-1}\mathrm{tr}(W)$ is its mean curvature and $\psi_0^l := \psi^l_{|S} \in \Gamma(\sigma_{|S})$.

We can derive as an immediate corollary for the case that M has a Cauchy temporal function t all of whose level sets are "extendable in both directions". Here it is important to note that every conformal extension I induces a constraint equation C_I as above. Then we obtain:

Theorem 1.2 *Let* (M, g) *be a 4-dimensional globally hyperbolic manifold with a Cauchy temporal function* t *such that for all level sets* $S_a := t^{-1}(\{a\})$ *of* t, $I^\pm(S_a)$ *are both* C^4*-extendable by a conformal extension* $I^\pm(a)$*. Then for every Cauchy surface* S *such that* $t|_S$ *is bounded, and for any initial values satisfying the neutrality and the constraint equations* $C_{I^-(e)}, C_{I^+(f)}$ *for* $e > \sup t(S)$, $f < \inf t(S)$ *and small in the respective Sobolev spaces, there is a global solution on* M *to the massless Dirac-Maxwell system above extending those initial values.* □

The method we use allows to treat also other equations than the massless Dirac-Maxwell equations, as long as they display an appropriate conformal behaviour and are gauge-equivalent to a semilinear symmetric hyperbolic system admitting a global solution (cf. Appendix). In particular, Dirac-Yang-Mills-Higgs systems as in Choquet-Bruhat's and Christodoulou's article should be treatable in a similar manner.

For the physically interested reader, we make precise what would have to be done to connect our setting to proper QED. First of all, one should build up the n-particle space as the vector space generated as an exterior product by all classical solutions that are totally antisymmetric under permutations of different spinor fields of equal mass and charge to obtain the usual fermionic commutation relations. Expanding in a basis of $\mathrm{Span}(\psi^1, \dots, \psi^N)$ which is orthonormal w.r.t. the conserved L^2 scalar product $(\psi, \phi) := \int_S j_{\psi,\phi}(\nu)$ (where ν is the normal vector field of a Cauchy surface S), we see we can w.r.o.g. assume that the spinor fields form a $(\cdot, \cdot)$-orthogonal system. If we have initial values at S in appropriate Sobolev spaces satisfying this condition, it will be true for the corresponding solution any other Cauchy surface due to the divergence-freeness of the $j_{\psi,\phi}$. The neutrality condition $\int_S J_\psi(\nu) = 0$ is in the case of an orthonormal system of spinors equivalent to the condition $\sum_{l=1}^N \mu_l = 0$. Moreover, in that case, J_ψ can be seen as the expectation value of the quantum-mechanical Dirac current operator, cf. [12, Sec. 3]. In the end, one would also need to quantize the bosonic potential A. Furthermore, one should consider the sum of all n-particle spaces to include phenomena like particle creation, particle annihilation, and also possibly the Dirac sea.

In a subsequent work, we will furthermore examine the question whether the solutions of the constraint equations of fixed regularity intersected with any open ball around 0 always form an infinite-dimensional Banach manifold.

The article is structured as follows: The second section recalls well-known facts on transformations under which the Dirac-Maxwell equations display some sort of covariance, proves Proposition 2.3 and derives the constraint equations used in Theorem 1.1. The third section is devoted to a proof of the main theorem, and the last section is an appendix transferring standard textbook tools for symmetric hyperbolic systems to the case of coefficients of finite (i.e., C^k) regularity needed here, a result that should be known more or less by experts on the fields already and for which we do not claim originality by any means.

2 Invariances of the Dirac-Maxwell equations

Let us first recall important well-known invariances of the Dirac-Maxwell equation:

Lemma 2.1 *Let (ψ, A) be a solution of the Dirac-Maxwell equations on a spin spacetime (M^n, g).*

1. *(Gauge invariance) For any $f \in C^\infty(M, \mathbb{R})$, the pair $(\psi' := (e^{-i\mu_1 f}\psi^1, \dots, e^{-i\mu_N f}\psi^N), A' := A + df)$ solves again the Dirac-Maxwell equations on (M^n, g).*

2. *(Conformal invariance) If $n = 4$, then for any $u \in C^\infty(M, \mathbb{R})$, the pair $(\overline{\varphi} := e^{-\frac{3}{2}u}\overline{\psi}, A)$ solves $D_{\overline{g}}^{\mu_l A}\overline{\varphi^l} = m_l e^{-u}\overline{\varphi^l}$ and $d_{\overline{g}}^* dA = \sum_{l=1}^n \mu_l j_{\overline{\varphi^l}}$ on $(M^n, \overline{g} := e^{2u}g)$, where $\psi \mapsto \overline{\psi}$, $S_g M \otimes E \to S_{\overline{g}} M \otimes E$, denotes the natural unitary isomorphism induced by the conformal*

change of metric. In particular, in dimension 4, the Dirac-Maxwell equations are scaling-invariant and the massless Dirac-Maxwell equations are even conformally invariant.

Proof. Both statements follow from elementary computations. For the sake of simplicity, we perform the proof only for $N = 1$ and $q = 1$.
1. By definition of the Dirac operator, we have $D^{A'} = D^A - df\cdot$,

$$\begin{aligned} D^{A'}\psi' &= (D^A - df\cdot)(e^{-if}\psi) \\ &= i\cdot(-ie^{-if}df)\cdot\psi + e^{-if}D^A\psi - e^{-if}df\cdot\psi \\ &= m\psi' \end{aligned}$$

and $d^*dA' = d^*dA + d^*d^2f = d^*dA = j_\psi = j_{\psi'}$.
2. First, we compute, for all tangential vector fields X, Y, Z and every 2-form ω on M^n:

$$\begin{aligned} (\nabla^{\overline{g}}_X\omega)(Y,Z) &= X(\omega(Y,Z)) - \omega(\nabla^{\overline{g}}_X Y, Z) - \omega(Y, \nabla^{\overline{g}}_X Z) \\ &= X(\omega(Y,Z)) - \omega\left(\nabla^g_X Y + X(u)Y + Y(u)X - g(X,Y)\mathrm{grad}_g(u), Z\right) \\ &\quad -\omega\left(Y, \nabla^g_X Z + X(u)Z + Z(u)X - g(X,Z)\mathrm{grad}_g(u)\right) \\ &= (\nabla^g_X\omega)(Y,Z) - 2X(u)\omega(Y,Z) - Y(u)\omega(X,Z) + Z(u)\omega(X,Y) \\ &\quad +g(X,Y)\omega(\mathrm{grad}_g(u),Z) - g(X,Z)\omega(\mathrm{grad}_g(u),Y). \end{aligned}$$

We deduce that, for the divergence, we have, in a local g-ONB $(e_j)_{0\leq j\leq n-1}$ of TM and for every $X \in \Gamma(M, TM)$,

$$\begin{aligned} (d^*_{\overline{g}}\omega)(X) &= -\sum_{j=0}^{n-1}\varepsilon_j(\nabla^{\overline{g}}_{\overline{e_j}}\omega)(\overline{e_j}, X) \\ &= -e^{-2u}\sum_{j=0}^{n-1}\varepsilon_j(\nabla^{\overline{g}}_{e_j}\omega)(e_j, X) \\ &= -e^{-2u}\sum_{j=0}^{n-1}\varepsilon_j\Big((\nabla^g_{e_j}\omega)(e_j,X) - 2e_j(u)\omega(e_j,X) - e_j(u)\omega(e_j,X) + X(u)\underbrace{\omega(e_j,e_j)}_{0} \\ &\qquad + g(e_j,e_j)\omega(\mathrm{grad}_g(u),X) - g(e_j,X)\omega(\mathrm{grad}_g(u),e_j)\Big) \\ &= e^{-2u}\Big((d^*_g\omega)(X) - (n-4)\omega(\mathrm{grad}_g(u),X)\Big), \end{aligned}$$

that is, $d^*_{\overline{g}}\omega = e^{-2u}\left(d^*_g\omega - (n-4)\mathrm{grad}_g(u)\lrcorner\omega\right)$. If in particular $n = 4$, then $d^*_{\overline{g}}\omega = e^{-2u}d^*_g\omega$, so that $d^*_{\overline{g}}dA = e^{-2u}d^*_g dA$. On the other hand, the operator D^A is conformally covariant, that is, $D^A_{\overline{g}}(e^{-\frac{n-1}{2}u}\overline{\psi}) = e^{-\frac{n+1}{2}u}\overline{D^A_g\psi}$, in particular we have

$$\begin{aligned} D^A_{\overline{g}}\overline{\varphi} &= D^A_{\overline{g}}(e^{-\frac{n-1}{2}u}\overline{\psi}) \\ &= e^{-\frac{n+1}{2}u}\overline{D^A_g\psi} \\ &= -me^{-u}\overline{\varphi}. \end{aligned}$$

It remains to notice that, for every $X \in TM$,

$$
\begin{aligned}
j_{\overline{\varphi}}(X) &= \langle X \cdot_{\overline{g}} \overline{\varphi}, \overline{\varphi} \rangle \\
&= e^{-(n-1)u} \langle X \cdot_{\overline{g}} \overline{\psi}, \overline{\psi} \rangle \\
&= e^{-(n-1)u} e^{u} \langle \overline{X \cdot_g \psi}, \overline{\psi} \rangle \\
&= e^{-(n-2)u} \langle X \cdot_g \psi, \psi \rangle \\
&= e^{-(n-2)u} j_{\psi}(X),
\end{aligned}
$$

that is, $j_{\overline{\varphi}} = e^{-(n-2)u} j_{\psi}$. We deduce that, for $n = 4$, we have $d^*_{\overline{g}} dA = e^{-2u} j_{\psi} = j_{\overline{\varphi}}$, which concludes the proof. $\square$

The **Dirac-wave operator** P_{DW} is defined by

$$
P_{DW}(\psi^1 \oplus ... \oplus \psi^N \oplus A) := (D^A\psi^1 - m_1\psi^1, \ldots, D^A\psi^N - m_N\psi^N, \square A - J_{\psi}),
$$

and the **Dirac-wave equation** is just the equation $P_{DW}(\psi, A) = 0$, where $\square := dd^* + d^*d$.

Proposition 2.2 (Lorenz gauge) *Let (M, g) be as above.*

i) For any solution (ψ, A) of the Dirac-wave equation, $\square(d^*A) = 0$ holds on M. In particular $d^*A = 0$ on M iff $(d^*A)_{|S_0} = 0 = \big(\frac{\partial}{\partial t} d^*A\big)_{|S_0}$.

ii) Given any solution (ψ, A) to the Dirac-wave equation, the equations $(d^*A)_{|S_0} = 0 = \big(\frac{\partial}{\partial t} d^*A\big)_{|S_0}$ are equivalent to

$$
0 = \frac{1}{\beta} A_1(\frac{\partial}{\partial t}) - \sum_{j=1}^{3} (\nabla_{e_j} A_0)(e_j) \tag{2.1}
$$

$$
\begin{aligned}
0 = {} & -(\nabla^{\tan})^* \nabla^{\tan} A_0(\frac{\partial}{\partial t}) - \sum_{j=1}^{3} \nabla_{e_j} A_1(e_j) - \frac{1}{2\beta} \mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t}) A_1(\frac{\partial}{\partial t}) + \frac{1}{\beta} A_1(\mathrm{grad}_{g_t}(\beta(t,\cdot))) \\
& + \frac{1}{2\beta} \nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))} A_0(\frac{\partial}{\partial t}) + \frac{1}{2} g_t(\nabla^{\tan} A_0, \frac{\partial g_t}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A_0^\sharp) + \sum_{l=1}^{N} \mu_l j_{\psi_0^l}(\frac{\partial}{\partial t}),
\end{aligned} \tag{2.2}
$$

where $A_0 := A_{|S_0} \in \Gamma(T^*M_{|S_0})$, $A_1 := \frac{\nabla A}{\partial t}_{|S_0} \in \Gamma(T^*M_{|S_0})$ and $\psi_0^l := \psi^l_{|S_0} \in \Gamma(\sigma_{|S_0})$.

Proof. Let (ψ, A) solve the Dirac-wave equation. Then $\square(d^*A) = d^*(\square A) = d^*J_{\psi}$. But a direct calculation leads to

$$
d^* j_{\psi}^k = i \left(\langle D^A\psi^k, \psi^k \rangle - \langle \psi^k, D^A\psi^k \rangle \right) = -2\mathrm{Im}(\langle D^A\psi^k, \psi^k \rangle),
$$

hence $d^*J_{\psi} = 0$ as soon as $D^A\psi^k = m_k\psi^k$ with $m_k \in \mathbb{R}$ (or, more generally, if $D^A\psi = H\psi$ for some Hermitian endomorphism-field H of σ). This shows $\square(d^*A) = 0$ and i).

Next we express the equations $(d^*A)_{|S_0} = 0 = \left(\frac{\partial}{\partial t}d^*A\right)_{|S_0}$ solely in terms of the initial data A_0, A_1 and ψ_0. It is already obvious that the first equation $(d^*A)_{|S_0} = 0$ only depends on A_0 (and its tangential derivatives along S_0) and A_1, however the second equation $\left(\frac{\partial}{\partial t}d^*A\right)_{|S_0} = 0$, which contains a derivative of second order in t of A, requires the wave equation $\Box A = J_\psi$ in order to yield a relationship between the initial data.
Denoting by $(e_j)_{1\leq j\leq 3}$ a local o.n.b. of TS_0 and letting $e_0 := \frac{1}{\sqrt{\beta}}\frac{\partial}{\partial t}$ (the future-oriented unit normal field on S_0), we have

$$\begin{aligned} d^*A &= -\sum_{j=0}^{3}\varepsilon_j(\nabla_{e_j}A)(e_j) \\ &= (\nabla_{e_0}A)(e_0) - \sum_{j=1}^{3}(\nabla_{e_j}A)(e_j) \\ &= \frac{1}{\beta}\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t}) - \sum_{j=1}^{3}(\nabla_{e_j}A)(e_j). \end{aligned}$$

As a first consequence, if we restrict that identity to S_0, we obtain

$$(d^*A)_{|S_0} = \frac{1}{\beta}A_1(\frac{\partial}{\partial t}) - \sum_{j=1}^{3}(\nabla_{e_j}A_0)(e_j).$$

Note here that the second term is in general not the divergence of the pull-back of A_0 on S_0 since the second fundamental form of S_0 in M may be non-vanishing. Differentiating further, we also obtain

$$\begin{aligned} \frac{\partial}{\partial t}d^*A &= \frac{\partial}{\partial t}\left(\frac{1}{\beta}\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t})\right) - \sum_{j=1}^{3}\frac{\partial}{\partial t}\left((\nabla_{e_j}A)(e_j)\right) \\ &= \frac{1}{\beta}\left\{-\frac{1}{\beta}\frac{\partial\beta}{\partial t}\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t}) + \frac{\nabla^2 A}{\partial t^2}(\frac{\partial}{\partial t}) + \frac{\nabla A}{\partial t}(\frac{\nabla}{\partial t}\frac{\partial}{\partial t})\right\} \\ &\quad -\sum_{j=1}^{3}\frac{\nabla}{\partial t}\nabla_{e_j}A(e_j) - \sum_{j=1}^{3}\nabla_{e_j}A(\frac{\nabla e_j}{\partial t}), \end{aligned}$$

where

$$\begin{aligned} \sum_{j=1}^{3}\frac{\nabla}{\partial t}\nabla_{e_j}A(e_j) &= \sum_{j=1}^{3}\nabla_{e_j}\frac{\nabla A}{\partial t}(e_j) + \nabla_{[\frac{\partial}{\partial t},e_j]}A(e_j) + (R_{\frac{\partial}{\partial t},e_j}A)(e_j) \\ &= \sum_{j=1}^{3}\nabla_{e_j}\frac{\nabla A}{\partial t}(e_j) + \nabla_{[\frac{\partial}{\partial t},e_j]}A(e_j) - A(R_{\frac{\partial}{\partial t},e_j}e_j) \\ &= \sum_{j=1}^{3}\nabla_{e_j}\frac{\nabla A}{\partial t}(e_j) + \nabla_{[\frac{\partial}{\partial t},e_j]}A(e_j) - \mathrm{ric}^M(\frac{\partial}{\partial t}, A^\sharp). \end{aligned}$$

Using the equation $\Box A = J_\psi$, we express $\frac{\nabla^2 A}{\partial t^2}$ in terms of ψ and of tangential (up to second order) and normal (up to first order) derivatives of A. Since the metric g has the form $g = -\beta dt^2 \oplus g_t$, we can split the rough d'Alembert operator $\Box^\nabla$ (associated to an arbitrary connection ∇ on the bundle under consideration) under the form

$$\begin{aligned}
\Box^\nabla &= \sum_{j=0}^{n-1} \varepsilon_j (\nabla_{\nabla^M_{e_j} e_j} - \nabla_{e_j}\nabla_{e_j}) \\
&= (\frac{1}{\sqrt{\beta}}\frac{\nabla}{\partial t})^2 - \frac{1}{\sqrt{\beta}}\nabla_{\nabla^M_{\frac{\partial}{\partial t}} \frac{1}{\sqrt{\beta}}\frac{\partial}{\partial t}} + \sum_{j=1}^{n-1}\nabla_{\nabla^\perp_{e_j} e_j} + \underbrace{\sum_{j=1}^{n-1}\nabla_{\nabla^{S_t}_{e_j} e_j} - \nabla_{e_j}\nabla_{e_j}}_{=:(\nabla^{\tan})^*\nabla^{\tan}} \\
&= \frac{1}{\beta}(\frac{\nabla}{\partial t})^2 + \frac{1}{\sqrt{\beta}}\frac{\partial}{\partial t}(\frac{1}{\sqrt{\beta}})\frac{\nabla}{\partial t} - \frac{1}{\sqrt{\beta}}\frac{\partial}{\partial t}(\frac{1}{\sqrt{\beta}})\frac{\nabla}{\partial t} - \frac{1}{\beta}\nabla_{\nabla^M_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}} \\
&\quad + \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\frac{\nabla}{\partial t} + (\nabla^{\tan})^*\nabla^{\tan} \\
&= \frac{1}{\beta}(\frac{\nabla}{\partial t})^2 - \frac{1}{2\beta^2}\frac{\partial \beta}{\partial t}\frac{\nabla}{\partial t} - \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))} + \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\frac{\nabla}{\partial t} + (\nabla^{\tan})^*\nabla^{\tan} \\
&= \frac{1}{\beta}\Big((\frac{\nabla}{\partial t})^2 + \frac{1}{2}\{\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t}) - \frac{1}{\beta}\frac{\partial \beta}{\partial t}\}\frac{\nabla}{\partial t}\Big) + (\nabla^{\tan})^*\nabla^{\tan} - \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}, \qquad (2.3)
\end{aligned}$$

where, as usual, $(e_j)_{0\le j\le n-1}$ denotes a local ONB of TM with $e_0 = \frac{1}{\sqrt{\beta}}\frac{\partial}{\partial t}$ and $\varepsilon_j = g(e_j, e_j) \in \{\pm 1\}$, the Levi-Civita connections of (M, g) and (S, g_t) are denoted respectively by ∇^M and ∇^{S_t} and where we have made use of the following identities (which are easy to check using Koszul's identity):

$$\nabla^M_X Y = \nabla^{S_t}_X Y + \nabla^\perp_X Y = \nabla^{S_t}_X Y + \frac{1}{2\beta}\frac{\partial g_t}{\partial t}(X, Y)\frac{\partial}{\partial t}$$

for all $X, Y \in TS_t = T(\{t\} \times S)$ and

$$\nabla^M_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \frac{1}{2\beta}\frac{\partial \beta}{\partial t}\frac{\partial}{\partial t} + \frac{1}{2}\mathrm{grad}_{g_t}(\beta(t,\cdot)).$$

As a consequence, (2.3) gives

$$\frac{1}{\beta}\frac{\nabla^2 A}{\partial t^2} = \big(\Box - (\nabla^{\tan})^*\nabla^{\tan}\big)A + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A + \frac{1}{2\beta}\left(\frac{1}{\beta}\frac{\partial \beta}{\partial t} - \mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\right)\frac{\nabla A}{\partial t}.$$

If $\Box A = J_\psi$, then we deduce that

$$\frac{1}{\beta}\frac{\nabla^2 A}{\partial t^2}(\frac{\partial}{\partial t}) = J_\psi(\frac{\partial}{\partial t}) - (\nabla^{\tan})^*\nabla^{\tan}A(\frac{\partial}{\partial t}) + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A(\frac{\partial}{\partial t}) + \frac{1}{2\beta}\left(\frac{1}{\beta}\frac{\partial \beta}{\partial t} - \mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\right)\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t}).$$

Using again the above identities connecting the Levi-Civita connections of S_t and M, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}d^*A &= -\frac{1}{\beta^2}\frac{\partial\beta}{\partial t}\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t}) + \frac{1}{\beta}\frac{\nabla A}{\partial t}(\frac{1}{2\beta}\frac{\partial\beta}{\partial t}\frac{\partial}{\partial t} + \frac{1}{2}\mathrm{grad}_{g_t}(\beta(t,\cdot)))\\
&\quad + J_\psi(\frac{\partial}{\partial t}) - (\nabla^{\tan})^*\nabla^{\tan}A(\frac{\partial}{\partial t}) + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A(\frac{\partial}{\partial t}) + \frac{1}{2\beta}\left(\frac{1}{\beta}\frac{\partial\beta}{\partial t} - \mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\right)\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t})\\
&\quad + \mathrm{ric}^M(\frac{\partial}{\partial t}, A^\sharp) - \sum_{j=1}^3 \nabla_{e_j}\frac{\nabla A}{\partial t}(e_j) + \nabla_{[\frac{\partial}{\partial t}, e_j]}A(e_j) + \nabla_{e_j}A(\frac{\nabla e_j}{\partial t})\\
&= -(\nabla^{\tan})^*\nabla^{\tan}A(\frac{\partial}{\partial t}) - \sum_{j=1}^3 \nabla_{e_j}\frac{\nabla A}{\partial t}(e_j) - \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t}) + \frac{1}{2\beta}\frac{\nabla A}{\partial t}(\mathrm{grad}_{g_t}(\beta(t,\cdot)))\\
&\quad + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A(\frac{\partial}{\partial t}) - \sum_{j=1}^3 \nabla_{[\frac{\partial}{\partial t}, e_j]}A(e_j) + \nabla_{e_j}A(\frac{\nabla e_j}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A^\sharp) + J_\psi(\frac{\partial}{\partial t}).
\end{aligned}$$

Now using $\frac{\nabla g}{\partial t} = 0$ as well as $\nabla_{e_j}\frac{\partial}{\partial t} = \frac{1}{2\beta}e_j(\beta)\frac{\partial}{\partial t} + \frac{1}{2}g_t^{-1}\frac{\partial g_t}{\partial t}(e_j,\cdot)$, we have

$$\begin{aligned}
\sum_{j=1}^3 \nabla_{[\frac{\partial}{\partial t}, e_j]}A(e_j) + \nabla_{e_j}A(\frac{\nabla e_j}{\partial t}) &= \sum_{j=1}^3 \nabla_{\frac{\nabla e_j}{\partial t} - \nabla_{e_j}\frac{\partial}{\partial t}}A(e_j) + \nabla_{e_j}A(\frac{\nabla e_j}{\partial t})\\
&= \underbrace{\sum_{j=1}^3 \nabla_{\frac{\nabla e_j}{\partial t}}A(e_j) + \nabla_{e_j}A(\frac{\nabla e_j}{\partial t})}_{0} - \sum_{j=1}^3 \nabla_{\nabla_{e_j}\frac{\partial}{\partial t}}A(e_j)\\
&= -\sum_{j=1}^3 \frac{1}{2\beta}e_j(\beta)\frac{\nabla A}{\partial t}(e_j) + \frac{1}{2}\nabla_{g_t^{-1}\frac{\partial g_t}{\partial t}(e_j,\cdot)}A(e_j)\\
&= -\frac{1}{2\beta}\frac{\nabla A}{\partial t}(\mathrm{grad}_{g_t}(\beta(t,\cdot))) - \frac{1}{2}\sum_{j=1}^3 \nabla_{g_t^{-1}\frac{\partial g_t}{\partial t}(e_j,\cdot)}A(e_j)\\
&= -\frac{1}{2\beta}\frac{\nabla A}{\partial t}(\mathrm{grad}_{g_t}(\beta(t,\cdot))) - \frac{1}{2}g_t(\nabla^{\tan}A, \frac{\partial g_t}{\partial t}),
\end{aligned}$$

so that we get

$$\begin{aligned}
\frac{\partial}{\partial t}d^*A &= -(\nabla^{\tan})^*\nabla^{\tan}A(\frac{\partial}{\partial t}) - \sum_{j=1}^3 \nabla_{e_j}\frac{\nabla A}{\partial t}(e_j) - \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})\frac{\nabla A}{\partial t}(\frac{\partial}{\partial t}) + \frac{1}{\beta}\frac{\nabla A}{\partial t}(\mathrm{grad}_{g_t}(\beta(t,\cdot)))\\
&\quad + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A(\frac{\partial}{\partial t}) + \frac{1}{2}g_t(\nabla^{\tan}A, \frac{\partial g_t}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A^\sharp) + J_\psi(\frac{\partial}{\partial t}).
\end{aligned}$$

Restricting that equation onto S_0, we come to

$$\begin{aligned}
\left(\frac{\partial}{\partial t}d^*A\right)_{|S_0} &= -(\nabla^{\tan})^*\nabla^{\tan}A_0(\frac{\partial}{\partial t}) - \sum_{j=1}^3 \nabla_{e_j}A_1(e_j) - \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})A_1(\frac{\partial}{\partial t}) + \frac{1}{\beta}A_1(\mathrm{grad}_{g_t}(\beta(t,\cdot)))\\
&\quad + \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A_0(\frac{\partial}{\partial t}) + \frac{1}{2}g_t(\nabla^{\tan}A_0, \frac{\partial g_t}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A_0^\sharp) + J_{\psi_0}(\frac{\partial}{\partial t}).
\end{aligned}$$

This yields the second equation and concludes the proof. $\square$

Proposition 2.3 *Let $(\psi = (\psi^1, \ldots, \psi^N), A)$ be any classical solution to the Dirac-Maxwell equation such that, along a given (smooth, spacelike) Cauchy hypersurface S with future-directed unit normal ν, the 1-form $dA(\nu, \cdot)$ is compactly supported. Then $\int_{S'} J_\psi(\nu') = 0$ for all Cauchy hypersurfaces S' of M with future unit normal vector ν'. In particular, for $N = 1$ and $\mu_1 \neq 0$, we can conclude $\psi_1 = 0$.*

Proof. Let (ψ, A) be any classical (i.e., sufficiently smooth) 1-particle solution to the Dirac-Maxwell equation, that is, $D^A\psi = m\psi$ and $d^*dA = j_\psi$. Let $S \subset M$ be any smooth spacelike Cauchy hypersurface and ν be the future-directed unit normal vector field along S. We first compute the codifferential along S of the 1-form $\nu \lrcorner dA = dA(\nu, \cdot)$. Let $\{e_j\}_{1 \leq j \leq n-1}$ be any local g-orthonormal frame on S, then

$$\begin{aligned}
d^*_S(\nu \lrcorner dA) &= -\sum_{j=1}^{n-1} e_j \lrcorner \nabla^S_{e_j}(\nu \lrcorner dA) \\
&= -\sum_{j=1}^{n-1} e_j \lrcorner \left(\nabla^M_{e_j}(\nu \lrcorner dA) - dA(\nu, \nabla^M_X \nu)\nu^\flat\right) \\
&= -\sum_{j=1}^{n-1} e_j \lrcorner \nabla^M_{e_j}(\nu \lrcorner dA) \\
&= -\sum_{j=1}^{n-1} e_j \lrcorner \left((\nabla^M_{e_j} dA)(\nu, \cdot) + dA(\nabla^M_{e_j}\nu, \cdot)\right) \\
&= -\sum_{j=1}^{n-1} (\nabla^M_{e_j} dA)(\nu, e_j) - \sum_{j=1}^{n-1} dA(\nabla^M_{e_j}\nu, e_j),
\end{aligned}$$

where the last sum vanishes since $(X, Y) \mapsto g(\nabla^M_X \nu, Y)$ is symmetric. We are left with

$$d^*_S(\nu \lrcorner dA) = -(d^*_M dA)(\nu) = -j_\psi(\nu).$$

As a consequence, if $\nu \lrcorner dA$ has compact support on S, then by the divergence theorem,

$$\int_S j_\psi(\nu) d\sigma_g = -\int_S d^*_S(\nu \lrcorner dA) d\sigma_g = 0.$$

Since $j_\psi(\nu) \geq 0$, we obtain $j_\psi(\nu) = 0$ on S and hence $\psi_{|S} = 0$ by positive-definiteness of the Hermitian inner product $(\varphi, \phi) \mapsto \langle \nu \cdot \varphi, \phi \rangle$. Since ψ is uniquely determined by its values along a Cauchy hypersurface, we obtain $\psi = 0$ on M. $\square$

3 Proof of the main theorem

In a first geometric step, for a Cauchy surface S' of (M, g), we extend $(I^+(S'), g)$ C^k-conformally to a globally hyperbolic manifold (N, h) by a map F. Then we use the generalization of the C^1

extension criterion in Taylor for C^k coefficients to show that there is a C^1 solution in N and thus in M, as follows: Let S be the image in N of a Cauchy surface in M. In our main theorem, we assume the Lorenz gauge condition (see Proposition 2.2) and therefore can use the first prolongation (for the definition, see end of Appendix) $\tilde{P}_{DW}$ of the Dirac-wave operator P_{DW} instead of P_{DM}. Due to the lifetime estimate in Theorem 4.6, there is a positive number δ such that for initial values a on a given Cauchy hypersurface Z of N with $\|a\|_{H^s(N,h)} < \delta$ there is a global solution on M. Now, in a second step, we have to manage the "initial jump" from S to Z, that is, we have to define sufficient conditions on S such that, starting with initial conditions on S giving rise to solutions reaching Z and satisfying a smallness condition, we get global solutions on $I_+(S)$.

Due to the unavoidable divergence of the conformal structure, we have to "avoid spatial infinity" in all computations, in the following sense: We transport sufficient smallness conditions down to $S_\infty := S$ only away from the boundary of the image while closer to the boundary we only transport them halfway down from one hypersurface S_n to the next hypersurface S_{n+1}. The hypersurfaces will be level sets of some Cauchy temporal function T of (N, h), *not* Cauchy surfaces of (M, g). More exactly, we choose a compact exhaustion of S, i.e. a sequence of open sets C_n in S such that $\overline{C_n}$ is compact, such that $\overline{C_n} \subset C_{n+1}$ and $\bigcup_{i=1}^\infty C_i = M$. Furthermore, we define $K_n := D^+(C_n)$ as their future domains of dependence. Then, for all $n \in \mathbb{N} \setminus \{0\}$, we choose any number $s \in (0, \pi/2)$ and define $Q_n := K_n \cap S$, $S_1 := T^{-1}(s) \cap F(M)$, and we define inductively $S_{n+1} := T^{-1}(r_{n+1}) \cap F(M)$ for r_{n+1} so small that

$$S \cap J^-(S_{n+1} \setminus K_{n+1}) \subset S \setminus K_n. \tag{3.1}$$

This can be done, as $\overline{K_n \cap S} \subset K_{n+1}$ and thus, by compactness of $D_n := J^+(\overline{K_n \cap S}) \cap \partial K_{n+1}$, we have $\tau_n := \min\{T(x) | x \in D_n\} > 0$ and we can choose $r_{n+1} \in (0, \tau_n)$. Thus we can arrange that $r_{n+1} \to 0$. Obviously, the Q_j, $j \in \mathbb{N}$, form a monotonous precompact covering of S. We define the annulus regions $D_j := Q_j \setminus Q_{j-1}$ and $D_0 := Q_0$. Now we construct inductively a sequence b_n such that if $\|u_0|_{D_j}\|_{C^4} < b_j$ then there is a global C^1 solution u on $F(M)$ of $\tilde{P}_{DW} u = 0$ with $u|_S = u_0$ (note that $\tilde{P}_{DW}$ is symmetric hyperbolic and therefore we can use the theorems from the appendix). This sequence b will be constructed via a corresponding sequence a for the H^4 norms which in turn is constructed as a limit of finite sequences $a^{(m)} \in \mathbb{R}^m$ that are stable in the sense that $a_n^{(m)} = a_n^{(m')}$ whenever $n \leq m-2, m'-2$, so that, for n fixed, the sequence $m \mapsto a_n^{(m)}$ is eventually constant, thus we can, indeed, define $a_i := \lim_{m\to\infty} a_i^{(m)}$. Inductively we prove that if there is a $C^1 H^4$ solution u of $\tilde{P}_{DW} u = 0$ between S and S_{n+1} with $\|u|_{D_i}\|_{H^4} < a_i$ for all $i \in \mathbb{N} \cap [0, n]$ and $\|u|_{E_{n+1}}\|_{H^4} < a_{n+1}^{(n)}$ (where $E_1 := S_1 \setminus K_0$ and, for $i > 1$, $E_i := I^-(S_{i-1} \setminus K_{i-1}) \cap S_i$) then u extends to a global C^1 solution on $F(M)$. The spatial regularity H^4 is chosen as the degree of the operator is 2 and as the critical regularity of the associated symmetric hyperbolic operator defined as a first prolongation is $\frac{n-1}{2} = 3/2$. The induction step is done by applying the theorem on lifetime (Theorem 4.6) in two regions: $I^-(E_n \setminus K_{n+1}) \cap I^+(S_{n+1})$ and $K_{n+1} \cap I^+(S)$. As announced above, to guarantee the smallness condition on the inner part there is a sufficient smallness condition on S while the smallness condition on the outer part transfers to a sufficient smallness condition on S_{n+1}. The value of $a_{m-1}^{(m+1)}$ is the minimum of $a_{m-1}^{(m)}$ and the value obtained by the smallness estimate necessary for $a_m^{(m)}$ but after that change the value of $a_m^{(j)}$ for higher j remains constant because of the condition 3.1. Thus the limit a_m is well-defined and positive. Thus we get a sequence

a_i and conditions $\|u_0\|_{H^4(D_i)} < a_i$, satisfied if $\|u_0\|_{C^4(D_i)} < b_i$ for appropriate constants b_i. Now, given initial values u_0 satisfying the smallness condition given by the sequence b, and given any point $q \in F(M)$, we want to show that q is contained in a domain of definition for a solution u of $\tilde{P}_{DW} u = 0$ with $u|_S = u_0$. To that purpose, we choose an i such that $q \in K_i$ and choose $f_i \in C^\infty(S, [0,1])$ with $f_i(Q_i) = \{1\}$ and $\mathrm{supp}(f_i) \subset S \setminus Q_{i+1}$. Then we solve the initial value problem for $u^{(i)} = f_i \cdot u_0$. Then we apply the ith step in the induction above.
Finally, we show higher regularity by considering the differentiated equation (which is a linear equation in the highest derivatives again). Consider the highest derivatives in a Sobolev Hilbert space as independent variables and show that they are in the same Sobolev Hilbert space as the coefficients, thereby gaining one order of (weak) differentiability. Finally we use Sobolev embeddings in the usual way.

□

4 Appendix: Modification of the extension criterion, existence time and regularity

Following [20, Ch. 16] but modifying the proof so as to allow for coefficients of finite regularity, we present the proof of local existence and uniqueness for solutions to symmetric hyperbolic systems. Although we could not find the existence theory for symmetric hyperbolic systems with coefficients of finite regularity in the literature, we do not claim originality of the following results but present them in full detail for the sake of self-containedness.

Definition 4.1 ([20, Sec. 16.2]) *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $N \in \mathbb{N}$, a first-order* **symmetric hyperbolic system on $\mathbb{R}^n$ with values in $\mathbb{K}^N$** *is a system of equations of the form*

$$\begin{cases} A_0(t,x,u)\frac{\partial u}{\partial t} & = L(t,x,u,\partial)u + g(t,x,u) \text{ on } \mathbb{R} \times \mathbb{R}^n \\ u(0) & = f, \end{cases} \tag{4.1}$$

where

- $L(t,x,u,\partial)v := \sum_{j=1}^n A_j(t,x,u)\partial_j v$ *for all* $v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{K}^N$, *with* $A_j : \mathbb{R} \times \mathbb{R}^n \times \mathbb{K}^N \to \mathrm{Mat}_{N\times N}(\mathbb{K})$ *such that* $A_j^* = A_j$ *(pointwise),*
- $A_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{K}^N \to \mathrm{Mat}_{N\times N}(\mathbb{K})$ *such that* $A_0^* = A_0$ *(pointwise) and* $A_0(t,x,u) \geq c \cdot \mathrm{I}$ *for some* $c > 0$,
- $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{K}^N \to \mathbb{K}^N$ *and*
- $f : \mathbb{R}^n \to \mathbb{K}^N$.

The same definition can be made when replacing $\mathbb{R}^n$ by an n-dimensional torus $\mathbb{T}^n$. The condition on A_0 means that A_0 is a pointwise Hermitian/symmetric matrix that is *uniformly* positive definite on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{K}^N$.

We want to prove the local existence and the uniqueness of solutions to first-order symmetric hyperbolic systems on $\mathbb{T}^n$. Later on, we shall consider the case of higher order symmetric hyperbolic system also on other manifolds.

We start by assuming low regularity on the data (we shall see below how the regularity of the solution depends on that of the data). The main theorem we want to prove is the following:

Theorem 4.2 *Consider a $\mathbb{K}^N$-valued first-order symmetric hyperbolic system on $\mathbb{T}^n$ as in* Definition 4.1 *and assume A_j, g to be C^1 in (t, x, u) and C^k in (x, u) for some $k > \frac{n}{2} + 1$. Then for any $f \in H^k$,*

1. *there is an $\eta \in \mathbb{R}_+^\times$ for which a unique solution $u \in C^1(]-\eta, \eta[\times\mathbb{T}^n) \cap C^0(]-\eta, \eta[, H^k(\mathbb{T}^n))$ to* (4.1) *exists;*

2. (extension criterion): *that solution u exists as long as $\|u(t)\|_{C^1(\mathbb{T}^n)}$ remains bounded.*

By C^0H^k, we mean continuous in the first variable $t \in I$ with values in the H^k-Sobolev space on $\mathbb{T}^n$ or $\mathbb{T}^n \times \mathbb{K}^N$. We shall mostly omit the interval I or the torus $\mathbb{T}^n$ in the notation. As usual, $H^k := W^{k,2}$. In the sequel, we shall often denote those spaces of functions with regularity R in t and with values in a Banach space S (mostly of functions in the other variables) with RS (e.g. C^0H^k, $L^\infty H^k$ etc.).

During the seven-step proof of Theorem 4.2, in several estimates, as multiplicative factors functions $C_i : \mathbb{R}^m \to \mathbb{R}$ will appear that take certain norms of the (approximate) solutions or of other maps as arguments. For simplicity, we will adopt the convention that these functions ('constants only depending on the norm') are taken to be *monotonously increasing*, and we try to number them consecutively by indices in every of the seven steps of the proof, which are the following:

1. Using mollifiers, perturb (4.1) by a small parameter $\varepsilon > 0$ in order to obtain a new system that can be interpreted as an ODE in the Banach space $H^k = H^k(\mathbb{T}^n)$.

2. For each value of the parameter $\varepsilon > 0$, solve the corresponding ODE locally about $0 \in \mathbb{R}$ and obtain a so-called approximate solution.

3. By a uniform (in the parameter ε) control of the pointwise H^k-norm of those approximate solutions, show that they all exist on a common interval $]-\eta, \eta[$ with $\eta > 0$.

4. Up to shrinking η a bit, extract of the families of approximate solutions a weak accumulation point and show that it is a C^1-solution to (4.1) on $]-\eta, \eta[\times\mathbb{T}^n$.

5. Show uniqueness of the local solution by controlling the rate of convergence of the approximate solutions against the solution when $\varepsilon \to 0$.

6. Improve the regularity of the solution to C^0H^k. This proves 1.

7. Show that in fact $\|u(t)\|_{H^k(\mathbb{T}^n)}$ remains bounded as long as $\|u(t)\|_{C^1(\mathbb{T}^n)}$ does. Assuming the solution u stops existing at $T > 0$, use a precise control of the length of the existence interval in the theorem of Picard-Lindelöf to prove that all approximate solutions - for an initial value fixed "shortly before" T - can be extended beyond T; this also implies (using uniqueness) that the solution can be extended beyond T, contradiction.

Let J_ε be the convolution with $\theta_\varepsilon = \varepsilon^{-n}\theta(\frac{\cdot}{\varepsilon})$, where $\theta \in C^\infty(\mathbb{R}^n, [0,\infty[)$, $\mathrm{supp}(\theta) \subset \overline{B}_1(0)$, $\int_{\mathbb{R}^n} \theta dx = 1$ and $\theta \circ (-\mathrm{Id}) = \theta$; the last condition is needed for the self-adjointness of J_ε in L^2 and higher Sobolev spaces. The operator J_ε is a smoothing operator approximating the identity in the following sense: $J_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \mathrm{Id}$ pointwise in $W^{k,q}(\mathbb{R}^n)$ for every $(k,q) \in \mathbb{N} \times [1,\infty[$ and also pointwise in $C^0(I, C^k(\mathbb{T}^n))$ for any open interval I. We shall often make use of $[J_\varepsilon, \partial^\alpha] = 0$ for every multi-index α and of the following facts: $J_\varepsilon \colon W^{k,q}(\mathbb{R}^n) \to W^{k,q}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ has norm $\|J_\varepsilon\| \leq 1$, $J_\varepsilon \colon C^k_b(\mathbb{R}^n) \to C^k_b(\mathbb{R}^n)$ has norm $\|J_\varepsilon\| \leq 1$, the operator $J_\varepsilon \colon C^0_b(\mathbb{R}^n) \to C^k_b(\mathbb{R}^n)$ has norm $\|J_\varepsilon\| \leq C(\varepsilon)$, the operator $J_\varepsilon \colon C^0(I, C^k(\mathbb{T}^n)) \to C^0(I, C^k(\mathbb{T}^n))$ has norm $\|J_\varepsilon\| \leq 1$. It is also interesting to notice that J_ε is an operator $\mathrm{Lip}(I, H^{k-1}) \to \mathrm{Lip}(I, H^k)$ with $\|J_\varepsilon u\|_{C^{0,1}(I,H^k)} \leq C(\varepsilon)\|u\|_{C^{0,1}(I,H^{k-1})}$, where $I \subset \mathbb{R}$ is a bounded open interval and $\mathrm{Lip}(I, H^l) = C^{0,1}(I, H^l)$. Namely for any $f \in H^{k-1}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, one has

$$\begin{aligned} \|\partial^\alpha J_\varepsilon f\|^2_{L^2} &= \|f * \partial^\alpha \theta_\varepsilon\|^2_{L^2} \\ &\leq \|f\|^2_{L^2} \cdot \underbrace{\|\partial^\alpha \theta_\varepsilon\|^2_{L^1}}_{C(\varepsilon)^2} \\ &\leq C(\varepsilon)^2 \|f\|^2_{H^{k-1}}, \end{aligned}$$

so that $\|J_\varepsilon f\|_{H^k} \leq C(\varepsilon)\|f\|_{H^{k-1}}$, which shows the claim.

Proposition 4.3 $\|\mathrm{Id} - J_\varepsilon\|_{\mathcal{L}(H^1,L^2)} \leq C \cdot \varepsilon$.[4]

[4]The statement holds as well for $\mathbb{R}^n$ instead of $\mathbb{T}^n$ with the same proof *mutatis mutandis*.

Proof. For any $f \in H^1(\mathbb{T}^n)$, we have

$$\begin{aligned}
\|J_\varepsilon f - f\|_{L^2}^2 &\leq \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} |f(x-y) - f(x)| \cdot \theta_\varepsilon(y) dy \right)^2 dx \\
&\leq \int_{\mathbb{T}^n} \left(\int_{B_\varepsilon(0)} |f(x-y) - f(x)|^2 dy \right) \|\theta_\varepsilon\|_{L^2(B_\varepsilon)}^2 dx \\
&\leq \varepsilon^{-n} \|\theta\|_{L^2}^2 \cdot \int_{\mathbb{T}^n} \left(\int_{B_\varepsilon(0)} |f(x-y) - f(x)|^2 dy \right) dx \\
&\leq \varepsilon^{-n} \|\theta\|_{L^2}^2 \cdot \int_{\mathbb{T}^n} \int_{B_\varepsilon(0)} \left(\int_0^1 |d_{x-ty} f(y)|^2 dt \right) dy dx \\
&\leq \varepsilon^2 \cdot \varepsilon^{-n} \|\theta\|_{L^2}^2 \cdot \int_{\mathbb{T}^n} \int_{B_\varepsilon(0)} \left(\int_0^1 |d_{x-ty} f|^2 dt \right) dy dx \\
&\leq \varepsilon^{2-n} \|\theta\|_{L^2}^2 \cdot \int_{B_\varepsilon(0)} \int_0^1 \|df\|_{L^2(\mathbb{T}^n)}^2 dt dy \qquad \text{(Fubini)} \\
&\leq \varepsilon^{2-n} \|\theta\|_{L^2}^2 \cdot \mathrm{Vol}(B_\varepsilon(0)) \cdot \|df\|_{L^2(\mathbb{T}^n)}^2 \\
&\leq C \cdot \|\theta\|_{L^2}^2 \cdot \varepsilon^2 \cdot \|df\|_{L^2(\mathbb{T}^n)}^2 \\
&\leq C \cdot \varepsilon^2 \cdot \|f\|_{H^1(\mathbb{T}^n)}^2,
\end{aligned}$$

which concludes the proof of the proposition. □

In the proof of Theorem 4.2, we use the following inequalities, see e.g. [20, Prop. 13.3.7], [18, Thm. 2.2.2, 2.2.3 & Lemma 2.2.6] and [11, Thm. 2.3.6 & 2.3.7].

Lemma 4.4 (Moser) *Let $k, n \in \mathbb{N} \setminus \{0\}$.*

i) **(First Moser estimate)** *There exists a constant $C = C(k, n) \in \mathbb{R}_+^\times$ such that, for all $f, g \in L^\infty(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$,*

$$\|f \cdot g\|_{H^k} \leq C \cdot \left(\|f\|_{L^\infty} \|g\|_{H^k} + \|f\|_{H^k} \|g\|_{L^\infty} \right). \tag{4.2}$$

ii) **(Second Moser estimate)** *There exists a constant $C = C(k, n) \in \mathbb{R}_+^\times$ such that, for all $f \in W^{1,\infty}(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$, $g \in L^\infty(\mathbb{R}^n) \cap H^{k-1}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$,*

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_{L^2} \leq C \cdot \left(\|\nabla f\|_{H^{k-1}} \|g\|_{L^\infty} + \|\nabla f\|_{L^\infty} \|g\|_{H^{k-1}} \right). \tag{4.3}$$

iii) **(Third Moser estimate)** *Let $F \in C^\infty(\mathbb{K}^N, \mathbb{K}^L)$ with $F(0) = 0$. Then there is a constant $C \in \mathbb{R}_+^\times$, which only depends on k, n, F and on $\|f\|_{L^\infty}$, such that, for any $f \in L^\infty(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$,*

$$\|\partial^\alpha F(f)\|_{L^2} \leq C(\|f\|_{L^\infty}) \cdot \|\nabla^{|\alpha|} f\|_{L^2}. \tag{4.4}$$

In [20, Prop. 13.3.9], there is the following alternative (and weaker) version of (4.4): for every $F \in C^\infty(\mathbb{K}^N, \mathbb{K}^L)$ with $F(0) = 0$, there exists a constant $C > 0$ depending only on k, n, F and on $\|f\|_{L^\infty}$ such that, for all $f \in L^\infty(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$,

$$\|F(f)\|_{H^k} \leq C(\|f\|_{L^\infty}) \cdot (1 + \|f\|_{H^k}). \tag{4.5}$$

Note that all estimates from Lemma 4.4 remain true when replacing $\mathbb{R}^n$ by the n-dimensional torus $\mathbb{T}^n$. Moreover, since $\mathbb{T}^n$ has finite volume, the assumption $F(0) = 0$ can be dropped for the weaker third Moser estimate (4.5), however not for (4.4) and $\alpha = 0$.

Lemma 4.5 *Let $A \in C^1(\mathbb{R}^n)$, $p \in [1, \infty[$ and $\varepsilon > 0$. Then there exists a constant $C = C(n, p) > 0$ such that, for any $v \in L^p(\mathbb{R}^n)$,*

i) $\|[A, J_\varepsilon]v\|_{L^p} \leq \left\{ \begin{array}{c} C \cdot \|A\|_{C^0} \cdot \|v\|_{L^p} \\ C \cdot \varepsilon \cdot \|A\|_{C^1} \cdot \|v\|_{L^p} \end{array} \right. .$

ii) $\|[A, J_\varepsilon]v\|_{W^{1,p}} \leq C \cdot \|A\|_{C^1} \cdot \|v\|_{L^p}$.

iii) $\|[A, J_\varepsilon]\frac{\partial v}{\partial x_j}\|_{L^p} \leq C \cdot \|A\|_{C^1} \cdot \|v\|_{L^p}$.

Proof. See e.g. [20, Ex. 13.1.1 - 13.1.3]. □

Step 1: We mollify the symmetric hyperbolic system in order to obtain an ODE in H^k.

Claim 1: *For any sufficiently small $\varepsilon > 0$, the equation $A_0(t, x, J_\varepsilon u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L(t, x, J_\varepsilon u_\varepsilon)J_\varepsilon u_\varepsilon + J_\varepsilon g(t, x, J_\varepsilon u_\varepsilon)$ is an ODE in H^k that is strongly locally Lipschitz in u_ε, that is, there exists a Lipschitz constant (in x) on all products $[0, T] \times \overline{B}_R(0)$, where $\overline{B}_R(0)$ is the closed R-ball about $0 \in H^k$.*

Proof. Consider the map $F\colon \mathbb{R} \times H^k \to H^k$,

$$F(t, v)(x) := A_0^{-1}(t, x, (J_\varepsilon v)(x)) \cdot \Big(J_\varepsilon\big[y \mapsto L(t, y, (J_\varepsilon v)(y))(J_\varepsilon v)(y)\big](x) + J_\varepsilon\big[y \mapsto g(t, y, (J_\varepsilon v)(y))\big](x)\Big)$$

for all $(t, v) \in \mathbb{R} \times H^k$ and every $x \in \mathbb{T}^n$. As in [20], we shortly write

$$F(t, v) = A_0^{-1}(t, x, J_\varepsilon v) \cdot (J_\varepsilon L(t, x, J_\varepsilon v)J_\varepsilon v + J_\varepsilon g(t, x, J_\varepsilon v))$$

for every $v \in H^k$. We show that F is C^1 (in the Fréchet sense) with bounded differential on each subset of the form $[0, T] \times \overline{B}_R(0)$ in $I \times H^k$. We only treat the case of one term in the definition of F, the others being handled in a similar manner. Namely consider the map $(t, v) \mapsto J_\varepsilon g(t, x, J_\varepsilon v)$ from $\mathbb{R} \times H^k \to H^k$. Then for any $v \in H^k$ and $h \in H^k$, we have

$$\begin{aligned} J_\varepsilon g(t, x, J_\varepsilon(v + h)) - J_\varepsilon g(t, x, J_\varepsilon v) &= J_\varepsilon\left(g(t, x, J_\varepsilon(v + h)) - g(t, x, J_\varepsilon v)\right) \\ &= J_\varepsilon\left(x \mapsto g'_u(t, x, (J_\varepsilon v)(x)) \cdot (J_\varepsilon h)(x) + |(J_\varepsilon h)(x)| \cdot \epsilon((J_\varepsilon h)(x))\right), \end{aligned}$$

where $\epsilon(w) \underset{w \to 0}{\longrightarrow} 0$. The map $h \mapsto J_\varepsilon\left(x \mapsto g'_u(t, x, (J_\varepsilon v)(x)) \cdot (J_\varepsilon h)(x)\right)$ is linear and bounded $H^k \to H^k$:

$$\begin{aligned} \|J_\varepsilon\left(x \mapsto g'_u(t, x, (J_\varepsilon v)(x)) \cdot (J_\varepsilon h)(x)\right)\|_{H^k} &\leq C_1(\varepsilon)\,\|x \mapsto g'_u(t, x, (J_\varepsilon v)(x)) \cdot (J_\varepsilon h)(x)\|_{L^2} \\ &\leq C_1(\varepsilon)\,\|x \mapsto g'_u(t, x, (J_\varepsilon v)(x))\|_{L^\infty} \cdot \|J_\varepsilon h\|_{L^2} \\ &\leq C_1(\varepsilon, t) \cdot \|h\|_{H^k}, \end{aligned}$$

where we have used the compactness of $\mathbb{T}^n$ and the fact that g'_u is continuous. Furthermore, the map $h \mapsto J_\varepsilon\,(x \mapsto |(J_\varepsilon h)(x)| \cdot \epsilon((J_\varepsilon h)(x)))$ is of the form $\mathrm{o}(\|h\|_{H^k})$ since

$$\begin{aligned}\frac{\|J_\varepsilon\,(x \mapsto |(J_\varepsilon h)(x)| \cdot \epsilon((J_\varepsilon h)(x)))\|_{H^k}}{\|h\|_{H^k}} &\leq \frac{C_1(\varepsilon)}{\|h\|_{H^k}}\,(\||J_\varepsilon h| \cdot \epsilon((J_\varepsilon h))\|_{L^2}) \\ &\leq \|\epsilon((J_\varepsilon h))\|_{L^\infty},\end{aligned}$$

where $\|\epsilon((J_\varepsilon h))\|_{L^\infty} \leq C_2(\varepsilon, t)\|h\|_{L^1} \underset{\|h\|_{H^k}\to 0}{\longrightarrow} 0$ because of $\mathbb{T}^n$ being compact. Finally, the map

$$H^k \to \mathcal{B}(H^k, H^k), \qquad v \mapsto J_\varepsilon\,(x \mapsto g'_u(t, x, (J_\varepsilon v)(x)) \cdot (J_\varepsilon \bullet)(x))$$

is continuous and bounded on each ball in H^k: this follows from the same kind of estimates as above as well as the continuity of g'_u on $\mathbb{R} \times \mathbb{T}^n \times \mathbb{K}^N$. This shows the claim. □

Step 2: This is mainly classical ODE theory, applicable as soon as the nonlinearity is continuous (in (t, x)) and locally Lipschitz (in the usual sense) in x.

Claim 2: *For any $f \in H^k$ and any sufficiently small $\varepsilon > 0$, the system*

$$\begin{cases} A_0(t, x, J_\varepsilon u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t} & = J_\varepsilon L(t, x, J_\varepsilon u_\varepsilon) J_\varepsilon u_\varepsilon + J_\varepsilon g(t, x, J_\varepsilon u_\varepsilon) \\ u_\varepsilon(0) & = f \end{cases} \tag{4.6}$$

has a unique solution $u_\varepsilon \in C^1(]-\eta_\varepsilon, \eta_\varepsilon[, H^k)$ for some $\eta_\varepsilon > 0$.

Proof. straightforward consequence of the theorem of Picard-Lindelöf. □

Step 3: "Standard estimates" based on Moser(-Trudinger) estimates and on Bihari's inequality [4].

Claim 3: *Under the assumptions of* Claim 2 *and with $k > \frac{n}{2} + 1$, there exists an $\eta > 0$ and a $K \in [0, \infty[$ such that $\|u_\varepsilon(t)\|_{H^k} \leq K$ for all $t \in]-\eta, \eta[$. In particular, the number η_ε from* Claim 2 *may be chosen independently on ε.*

Proof. We introduce the new L^2-Hermitian inner product $(\cdot, \cdot)_{L^2,\varepsilon} := (A_{0\varepsilon}\cdot, \cdot)_{L^2}$ on $\mathbb{T}^n$, where $A_{0\varepsilon} := A_0(t, x, J_\varepsilon u_\varepsilon)$. Note that $(\cdot, \cdot)_{L^2,\varepsilon}$ depends on $\varepsilon > 0$ and also implicitely on t; but by assumption on A_0 and because we only consider compact sets of the form $[0, T] \times \mathbb{T}^n$, the norms $\|\cdot\|_{L^2,\varepsilon}$ and $\|\cdot\|_{L^2}$ are equivalent; more precisely, for any $T \in [0, \infty[$, there exists $C = C(T, \|u_\varepsilon\|_{C^0([0,T],L^\infty)}) \in]0, \infty[$ such that $c\|\cdot\|^2_{L^2} \leq \|\cdot\|^2_{L^2,\varepsilon} \leq C\|\cdot\|^2_{L^2}$, where $c > 0$ is the constant from Definition 4.1. We pick an arbitrary $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ and estimate $\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2,\varepsilon}$ using (4.6). First, because $A_{0\varepsilon}$

is pointwise Hermitian,

$$\begin{aligned}
\frac{d}{dt}\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2,\varepsilon} &= \frac{d}{dt}\left(A_{0\varepsilon}\partial^\alpha u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} \\
&= \Re e\left(\frac{\partial A_{0\varepsilon}}{\partial t}\cdot\partial^\alpha u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} + 2\Re e\left(A_{0\varepsilon}\cdot\frac{\partial\partial^\alpha u_\varepsilon}{\partial t}, \partial^\alpha u_\varepsilon\right)_{L^2} \\
&= \Re e\left(\frac{\partial A_{0\varepsilon}}{\partial t}\cdot\partial^\alpha u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} + 2\Re e\left(A_{0\varepsilon}\cdot\partial^\alpha\frac{\partial u_\varepsilon}{\partial t}, \partial^\alpha u_\varepsilon\right)_{L^2} \\
&= \Re e\left(\frac{\partial A_{0\varepsilon}}{\partial t}\cdot\partial^\alpha u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} + 2\Re e\left(\partial^\alpha(A_{0\varepsilon}\cdot\frac{\partial u_\varepsilon}{\partial t}), \partial^\alpha u_\varepsilon\right)_{L^2} \\
&\quad +2\Re e\left([A_{0\varepsilon},\partial^\alpha]\frac{\partial u_\varepsilon}{\partial t}, \partial^\alpha u_\varepsilon\right)_{L^2} \\
&\overset{(4.6)}{=} \Re e\left(\frac{\partial A_{0\varepsilon}}{\partial t}\cdot\partial^\alpha u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} + 2\Re e\left(\partial^\alpha J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} + 2\Re e\left(\partial^\alpha g_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} \\
&\quad +2\Re e\left([A_{0\varepsilon},\partial^\alpha]\frac{\partial u_\varepsilon}{\partial t}, \partial^\alpha u_\varepsilon\right)_{L^2}, \qquad (4.7)
\end{aligned}$$

where we have denoted $L_\varepsilon := L(t,x,J_\varepsilon u_\varepsilon)$ and $g_\varepsilon := J_\varepsilon g(t,x,J_\varepsilon u_\varepsilon)$. The first term in the r.h.s. of (4.7) can easily be estimated:

$$\begin{aligned}
\left|\Re e\left(\frac{\partial A_{0\varepsilon}}{\partial t}\cdot\partial^\alpha u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2}\right| &\leq \|\frac{\partial A_{0\varepsilon}}{\partial t}\cdot\partial^\alpha u_\varepsilon(t)\|_{L^2}\cdot\|\partial^\alpha u_\varepsilon(t)\|_{L^2} \\
&\leq \|\frac{\partial A_{0\varepsilon}}{\partial t}\|_{L^\infty}\cdot\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2} \\
&\leq C_1(\|J_\varepsilon u_\varepsilon(t)\|_{L^\infty}, \|\frac{\partial J_\varepsilon u_\varepsilon}{\partial t}(t)\|_{L^\infty})\cdot\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2} \\
&\leq C_2(\|u_\varepsilon(t)\|_{L^\infty}, \|\frac{\partial u_\varepsilon}{\partial t}(t)\|_{L^\infty})\cdot\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2} \\
&\overset{(4.6)}{\leq} C_3(\|u_\varepsilon(t)\|_{L^\infty}, \|u_\varepsilon(t)\|_{C^1})\cdot\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2} \\
&\leq C_4(\|u_\varepsilon(t)\|_{H^k})\cdot\|u_\varepsilon(t)\|^2_{H^k},
\end{aligned}$$

where we have used the continuous embedding $H^k(\mathbb{T}^n)\hookrightarrow C^1(\mathbb{T}^n)$ (valid because of $k > \frac{n}{2}+1$) as well as $\|J_\varepsilon\|_{\mathcal{L}(C^l,C^l)} \leq 1$ for any $l\in\mathbb{N}$. Let us consider the second term in the r.h.s. of (4.7). Since we may choose the mollifier J_ε such that $J_\varepsilon^* = J_\varepsilon$ in L^2 (choose e.g. $\theta\in C_c^\infty(\mathbb{R}^n)$ with $\theta\circ(-\mathrm{Id}) = \theta$), we have

$$\begin{aligned}
2\Re e\left(\partial^\alpha J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} &= 2\Re e\left(J_\varepsilon\partial^\alpha L_\varepsilon J_\varepsilon u_\varepsilon, \partial^\alpha u_\varepsilon\right)_{L^2} \\
&= 2\Re e\left(\partial^\alpha L_\varepsilon J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2} \\
&= 2\Re e\left(L_\varepsilon\partial^\alpha J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2} + 2\Re e\left([\partial^\alpha, L_\varepsilon]J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2} \\
&= \left((L_\varepsilon + L_\varepsilon^*)\partial^\alpha J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2} + 2\Re e\left([\partial^\alpha, L_\varepsilon]J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2},
\end{aligned}$$

where L_ε^* is the formal adjoint of the differential operator L_ε. Now, since by assumption $A_j = A_j^*$ pointwise, we have $L_\varepsilon^* = -\sum_{j=1}^n \partial_j(A_j(t,x,J_\varepsilon u_\varepsilon)\cdot)$, so that

$$L_\varepsilon + L_\varepsilon^* = -\sum_{j=1}^n \partial_j A_j(t,x,J_\varepsilon u_\varepsilon)$$

is of zero order (this is one of the main places where symmetric hyperbolicity is used), so that

$$\begin{aligned}
|\left((L_\varepsilon + L_\varepsilon^*)\partial^\alpha J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2}| &\le \|(L_\varepsilon + L_\varepsilon^*)\partial^\alpha J_\varepsilon u_\varepsilon\|_{L^2} \cdot \|\partial^\alpha J_\varepsilon u_\varepsilon\|_{L^2} \\
&\le \sum_{j=1}^n \|\partial_j A_j(t,x,J_\varepsilon u_\varepsilon)\|_{L^\infty} \cdot \|\partial^\alpha J_\varepsilon u_\varepsilon\|_{L^2}^2 \\
&\le C_5(\|J_\varepsilon u_\varepsilon(t)\|_{C^1}) \cdot \|u_\varepsilon(t)\|_{H^k}^2 \\
&\le C_6(\|u_\varepsilon(t)\|_{H^k}) \cdot \|u_\varepsilon(t)\|_{H^k}^2. \qquad (4.8)
\end{aligned}$$

With

$$\begin{aligned}
[\partial^\alpha, L_\varepsilon]v &= \sum_{j=1}^n \partial^\alpha(A_j(t,x,J_\varepsilon u_\varepsilon)\partial_j v) - A_j(t,x,J_\varepsilon u_\varepsilon)\partial_j(\partial^\alpha v) \\
&= \sum_{j=1}^n \partial^\alpha(A_j(t,x,J_\varepsilon u_\varepsilon)\partial_j v) - A_j(t,x,J_\varepsilon u_\varepsilon)\partial^\alpha(\partial_j v) \\
&= \sum_{j=1}^n [\partial^\alpha, A_j(t,x,J_\varepsilon u_\varepsilon)]\partial_j v,
\end{aligned}$$

we have

$$\begin{aligned}
\|[\partial^\alpha, L_\varepsilon]v\|_{L^2} &\le \sum_{j=1}^n \|[\partial^\alpha, A_j(t,x,J_\varepsilon u_\varepsilon)]\partial_j v\|_{L^2} \\
&\overset{(4.3)}{\le} C_7 \cdot \sum_{j=1}^n (\|\nabla A_j(t,x,J_\varepsilon u_\varepsilon)\|_{H^{k-1}} \cdot \|\partial_j v\|_{L^\infty} + \|\nabla A_j(t,x,J_\varepsilon u_\varepsilon)\|_{L^\infty} \cdot \|\partial_j v\|_{H^{k-1}}) \\
&\le C_8 \cdot \sum_{j=1}^n (\|A_j(t,x,J_\varepsilon u_\varepsilon)\|_{H^k} \cdot \|v\|_{C^1} + \|A_j(t,x,J_\varepsilon u_\varepsilon)\|_{C^1} \cdot \|v\|_{H^k}) \\
&\overset{(4.5)}{\le} C_8 \cdot (C_9(\|J_\varepsilon u_\varepsilon(t)\|_{L^\infty}) \cdot (1 + \|J_\varepsilon u_\varepsilon(t)\|_{H^k}) \cdot \|v\|_{C^1} + C_{10}(\|J_\varepsilon u_\varepsilon(t)\|_{C^1}) \cdot \|v\|_{H^k}) \\
&\le C_{11}(\|u_\varepsilon(t)\|_{H^k}) \cdot \|v\|_{H^k},
\end{aligned}$$

so that

$$\begin{aligned}
|2\Re e\left([\partial^\alpha, L_\varepsilon]J_\varepsilon u_\varepsilon, \partial^\alpha J_\varepsilon u_\varepsilon\right)_{L^2}| &\le 2\|[\partial^\alpha, L_\varepsilon]J_\varepsilon u_\varepsilon\|_{L^2} \cdot \|\partial^\alpha J_\varepsilon u_\varepsilon\|_{L^2} \\
&\le C_{11}(\|u_\varepsilon(t)\|_{H^k}) \cdot \|J_\varepsilon u_\varepsilon(t)\|_{H^k} \cdot \|J_\varepsilon u_\varepsilon(t)\|_{H^k} \\
&\le C_{11}(\|u_\varepsilon(t)\|_{H^k}) \cdot \|u_\varepsilon(t)\|_{H^k}^2,
\end{aligned}$$

which gives, together with (4.8) and using $[J_\varepsilon, \partial^\alpha] = 0$,

$$2|\Re e\,(\partial^\alpha J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, \partial^\alpha u_\varepsilon)_{L^2}| \leq C_{12}(\|u_\varepsilon(t)\|_{H^k}).$$

For the third term in the r.h.s. of (4.7), we have

$$\begin{aligned} 2|\Re e\,(\partial^\alpha g_\varepsilon, \partial^\alpha u_\varepsilon)_{L^2}| &\leq 2\|\partial^\alpha g_\varepsilon\|_{L^2} \cdot \|\partial^\alpha u_\varepsilon\|_{L^2} \\ &\overset{(4.5)}{\leq} C_{13}(\|u_\varepsilon(t)\|_{H^k}) \cdot (1 + \|J_\varepsilon u_\varepsilon(t)\|_{H^k} \cdot \|u_\varepsilon(t)\|_{H^k}) \\ &\leq C_{14}(\|u_\varepsilon(t)\|_{H^k}). \end{aligned}$$

The last term in the r.h.s. of (4.7) can be estimated as follows:

$$\begin{aligned} 2|\Re e\left([A_{0\varepsilon}, \partial^\alpha]\frac{\partial u_\varepsilon}{\partial t}, \partial^\alpha u_\varepsilon\right)_{L^2}| &\overset{(4.6)}{=} 2|\Re e\,\big([A_{0\varepsilon}, \partial^\alpha] A_{0\varepsilon}^{-1}(J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon), \partial^\alpha u_\varepsilon\big)_{L^2}| \\ &\leq 2\|[A_{0\varepsilon}, \partial^\alpha] A_{0\varepsilon}^{-1}(J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon)\|_{L^2} \cdot \|\partial^\alpha u_\varepsilon\|_{L^2} \\ &\overset{(4.3)}{\leq} C_{15} \cdot \|u_\varepsilon\|_{H^k} \cdot \Big(\|\nabla A_{0\varepsilon}\|_{H^{k-1}} \cdot \|A_{0\varepsilon}^{-1} \cdot (J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon)\|_{L^\infty} \\ &\qquad + \|\nabla A_{0\varepsilon}\|_{L^\infty} \cdot \|A_{0\varepsilon}^{-1} \cdot (J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon)\|_{H^{k-1}}\Big) \\ &\leq C_{16}(\|u_\varepsilon\|_{H^k}), \end{aligned}$$

where, in the last step, we have used the Moser estimates (4.2) and (4.5). On the whole, $|\frac{d}{dt}\|\partial^\alpha u_\varepsilon(t)\|^2_{L^2,\varepsilon}| \leq C(\|u_\varepsilon\|_{H^k})$, so that, using the equivalence of the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^2,\varepsilon}$ on some (fixed) compact set $[-T, T] \times \mathbb{T}^n$, we deduce that, setting $\|v\|^2_{H^k,\varepsilon} := \sum_{|\alpha| \leq k} \|\partial^\alpha v\|^2_{L^2,\varepsilon}$

$$|\frac{d}{dt}\|u_\varepsilon(t)\|^2_{H^k,\varepsilon}| \leq C_{17}(\|u_\varepsilon(t)\|_{H^k,\varepsilon}).$$

By Bihari's inequality [4], we deduce that there exists a function K, defined and continuous on a sufficiently small interval $]-\eta, \eta[$ about 0, such that $\|u_\varepsilon(t)\|_{H^k,\varepsilon} \leq K(t)$ for all $t \in]-\eta_\varepsilon, \eta_\varepsilon[$. Up to making $\eta > 0$ smaller, we may assume that $K(t) \leq K' < \infty$ for all $t \in]-\eta, \eta[$, so that $\|u_\varepsilon(t)\|_{H^k,\varepsilon} \leq K'$ and hence also $\|u_\varepsilon(t)\|_{H^k} \leq K$ for some $K \in]0, \infty[$. The last statement of Claim 3 follows from the extension criterion for ODE's (valid by Steps 1 and/or 2), stating that, by $\|u_\varepsilon(t)\|_{H^k} \leq K < \infty$, the solution u_ε can be *a fortiori* extended onto $]-\eta, \eta[$, QED. □

Step 4: The preceding uniform estimate shows boundedness of approximate solutions in certain Sobolev spaces; use weak $*$-compactness to deduce the existence of an accumulation point. Then apply the interpolation inequalities (allowing compact embeddings into fractional Sobolev spaces) to deduce that the solution is actually $C^0C^1 \cap C^1C^0$, that is, C^1 (use uniform continuity because of compactness of $[-\eta, \eta] \times \mathbb{T}^n$).

Claim 4: *The family* $(u_\varepsilon)_\varepsilon \subset C^1(]-\eta, \eta[, H^k)$ *from* Claim 3*, when restricted to any compact interval* $I \subset]-\eta, \eta[$*, admits a weak (in a particular sense) limit point* $u \in C^1(I \times \mathbb{T}^n) \cap L^\infty(I, H^k) \cap \mathrm{Lip}(I, H^{k-1})$ *which solves* (4.1).

Proof. From Step 3 we have the existence of an $\eta > 0$ and a $K \in]0, \infty[$ such that, for all sufficiently small $\varepsilon > 0$, the approximate solution u_ε lies in $C^1(]-\eta, \eta[, H^k)$ with $\|u_\varepsilon\|_{C^0(]-\eta,\eta[,H^k)} \le K$. Hence fixing an arbitrary compact interval $I \subset]-\eta, \eta[$, we have $\|u_\varepsilon\|_{C^0(I,H^k)} \le K$, in particular the family $(u_\varepsilon)_\varepsilon$ is bounded in $C^0(I, H^k)$ and thus in $L^\infty(I, H^k)$. Using (4.6) and Moser estimates, the norm $\|\frac{\partial u_\varepsilon}{\partial t}\|_{C^0 H^{k-1}}$ can be uniformly in ε estimated in terms of $\|u_\varepsilon\|_{C^0(I,H^k)}$ and hence the family $(\frac{\partial u_\varepsilon}{\partial t})_\varepsilon$ is bounded in $C^0(I, H^{k-1})$, so that $(u_\varepsilon)_\varepsilon$ is bounded in $C^1(I, H^{k-1})$ and therefore in $\mathrm{Lip}(I, H^{k-1})$. Now $L^\infty(I, H^k) = L^1(I, H^k)'$ (topological dual), $\mathrm{Lip}(I, H^{k-1}) = W^{1,\infty}(I, H^{k-1})$ by Rademacher's theorem and the latter space in turn can be identified with a closed subspace of $L^\infty(I, H^{k-1}) \oplus L^\infty(I, H^{k-1}) = L^1(I, H^{k-1})' \oplus L^1(I, H^{k-1})'$ via $f \mapsto (f, f')$. Since the unit ball of the dual space of any Banach space is weakly $*$-compact, there exists a sequence $\varepsilon_p \to 0$, a $u \in L^\infty(I, H^k) \cap \mathrm{Lip}(I, H^{k-1})$, such that $(u_{\varepsilon_p})_p$ converges to u $*$-weakly in both spaces. On the other hand, since $k > \frac{n}{2} + 1$ and, for any $\sigma \in]0, k - \frac{n}{2} - 1[$, the embedding $H^{k-\sigma} \subset C^1$ is compact, we can assume up to taking subsequences that $(u_{\varepsilon_p})_p$ converges in C^0C^1 to a $\overline{u} \in C^0C^1$; in fact $\overline{u} = u$ since both can be seen as sitting in the space $L^\infty(I, C^1)$ and both convergences imply the convergence in a weaker sense. Similarly, for any $\sigma \in]0, k - \frac{n}{2} - 1[$, the embedding $H^{k-1-\sigma} \subset C^0$ is compact, hence so is $C^1H^{k-1} \subset C^1C^0$, so that we may assume that $(u_{\varepsilon_p})_p$ converges in C^1C^0 to some $\hat{u} \in C^1C^0$ and again $\hat{u} = u$. Since u is the limit of $(u_{\varepsilon_p})_p$ in the C^1-topology and $J_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \mathrm{Id}$ pointwise in C^0C^1, we deduce that u solves (4.1). □

Step 5: Look at the pointwise (in t) L^2-norm of the difference between an exact C^1 solution to (4.1) and an approximate solution for any $\varepsilon > 0$. Estimate that norm on I using standard estimates and Bihari's inequality. The key point at the end is to show that $\|\mathrm{Id} - J_\varepsilon\|_{\mathcal{L}(H^1,L^2)} \le C_1 \cdot \varepsilon$ for some constant $C_1 > 0$.

Claim 5: *Given any $h \in H^k(\mathbb{T}^n)$ and any $\varepsilon > 0$, let $u_\varepsilon \in C^1(I, H^{k,2}(\mathbb{T}^n))$ solve*

$$\begin{cases} A_0(t, x, J_\varepsilon u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t} & = J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon \text{ on } I \\ u_\varepsilon(0) & = h \end{cases} \tag{4.9}$$

with u_ε is bounded uniformly in $\varepsilon > 0$ in the C^0H^k-norm for all ε. Let $u \in C^1$ solve (4.1) *and consider $v_\varepsilon := u - u_\varepsilon$. Then there is a function $a(t) := C(\|u_\varepsilon(t)\|_{C^1}, \|u(t)\|_{C^1})$ for all t such that*

$$\|v_\varepsilon(t)\|^2_{L^2} \le \exp\left(\int_0^t a(s)ds\right) \cdot \left(\|\underbrace{f - h}_{v_\varepsilon(0)}\|^2_{L^2} + \int_0^t C_2(\|u_\varepsilon(s)\|_{H^k}) \cdot \varepsilon \cdot e^{-\int_0^s a(\tau)d\tau} ds\right).$$

In particular, it follows from the boundedness of $(\|u_\varepsilon\|_{C^0H^k})_\varepsilon$ in Claim 3 that u is unique.

Proof. We estimate $\|v_\varepsilon(t)\|^2_{L^2}$ for all $t \in I$. First, with the notations introduced above, we write

$$\begin{aligned} \frac{\partial v_\varepsilon}{\partial t} &= A_0^{-1}\frac{\partial u}{\partial t} - A_{0\varepsilon}^{-1}\frac{\partial u_\varepsilon}{\partial t} \\ &= A_0^{-1}L(t, x, u, \partial)u - A_{0\varepsilon}^{-1}J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + A_0^{-1}g(t, x, u) - A_{0\varepsilon}^{-1}g_\varepsilon \\ &= A_0^{-1}L(t, x, u, \partial)v_\varepsilon + (A_0^{-1} - A_{0\varepsilon}^{-1})L(t, x, u, \partial)u_\varepsilon + A_{0\varepsilon}^{-1}(L(t, x, u, \partial)u_\varepsilon - J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon) \\ &\quad + (A_0^{-1} - A_{0\varepsilon}^{-1})g(t, x, u) + A_{0\varepsilon}^{-1}(g(t, x, u) - g_\varepsilon). \end{aligned} \tag{4.10}$$

We start looking at the difference

$$
\begin{aligned}
L(t,x,u,\partial)u_\varepsilon - J_\varepsilon L(t,x,J_\varepsilon u_\varepsilon,\partial)J_\varepsilon u_\varepsilon \quad &= \quad (L(t,x,u,\partial) - L(t,x,u_\varepsilon,\partial))u_\varepsilon \\
&\quad +(\mathrm{Id}-J_\varepsilon)L(t,x,u_\varepsilon,\partial)u_\varepsilon + J_\varepsilon L(t,x,u_\varepsilon,\partial)(\mathrm{Id}-J_\varepsilon)u_\varepsilon \\
&\quad +J_\varepsilon\left(L(t,x,u_\varepsilon,\partial) - L(t,x,J_\varepsilon u_\varepsilon,\partial)\right)J_\varepsilon u_\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
g(t,x,u) - J_\varepsilon g(t,x,J_\varepsilon u_\varepsilon) \quad &= \quad g(t,x,u) - g(t,x,u_\varepsilon) + g(t,x,u_\varepsilon) - g(t,x,J_\varepsilon u_\varepsilon) \\
&\quad +g(t,x,J_\varepsilon u_\varepsilon) - J_\varepsilon g(t,x,J_\varepsilon u_\varepsilon).
\end{aligned}
$$

Since $A_j, g \in C^1(I \times \mathbb{T}^n)$, we may write, for all $w_1, w_2 \in \mathbb{K}^N$,

$$
g(t,x,w_1) - g(t,x,w_2) = \int_0^1 \partial_z g(t,x,(1-s)w_2 + sw_1)(w_1 - w_2)ds =: G(w_1,w_2)(w_1 - w_2)
$$

where $\partial_z g$ denotes the derivative of $w \mapsto g(t,x,w)$ and similarly for the first-order operator

$$
\begin{aligned}
L(t,x,w_1,\partial) - L(t,x,w_2,\partial) \quad &= \quad \sum_{j=1}^n \int_0^1 \partial_z A_j(t,x,(1-s)w_2 + sw_1)(w_1 - w_2)ds \frac{\partial}{\partial x_j} \\
&=: \quad M(t,x,w_1,w_2)(w_1 - w_2).
\end{aligned}
$$

In the same way, we can write

$$
\begin{aligned}
A_0^{-1}(t,x,u) - A_0^{-1}(t,x,J_\varepsilon u_\varepsilon) \quad &= \quad A_0^{-1}(t,x,u) - A_0^{-1}(t,x,u_\varepsilon) + A_0^{-1}(t,x,u_\varepsilon) - A_0^{-1}(t,x,J_\varepsilon u_\varepsilon) \\
&= \quad \int_0^1 d_{(t,x,(1-s)u_\varepsilon + su)}(A_0^{-1})(u - u_\varepsilon)ds \\
&\quad + \int_0^1 d_{(t,x,(1-s)J_\varepsilon u_\varepsilon + su_\varepsilon)}(A_0^{-1})(u_\varepsilon - J_\varepsilon u_\varepsilon)ds \\
&=: \quad M_0(t,x,u,u_\varepsilon)(v_\varepsilon) + N_0(t,x,u_\varepsilon)(\mathrm{Id} - J_\varepsilon)(u_\varepsilon).
\end{aligned}
$$

It is very important to notice that G, M, M_0 and N_0 depend only pointwise on $u, u_\varepsilon \ldots$, so that they can be estimated in terms of the C^0-norms of $u, u_\varepsilon \ldots$ only. Now, we split the r.h.s. of (4.10) according to their dependence on v_ε and obtain

$$
\frac{\partial v_\varepsilon}{\partial t} = A_0^{-1} L(t,x,u,\partial)v_\varepsilon + A(t,x,u,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon + R_\varepsilon,
$$

where

$$
\begin{aligned}
A(t,x,u,u_\varepsilon,\nabla u_\varepsilon) \quad &:= \quad M_0(t,x,u,u_\varepsilon)(v_\varepsilon)(L(t,x,u,\partial)u_\varepsilon + g(t,x,u)) \\
&\quad +A_{0\varepsilon}^{-1}(M(t,x,u,u_\varepsilon)(v_\varepsilon)u_\varepsilon + G(u,u_\varepsilon)(v_\varepsilon))
\end{aligned}
$$

and

$$
\begin{aligned}
R_\varepsilon \quad := \quad & N_0(t,x,u_\varepsilon)(\mathrm{Id} - J_\varepsilon)(u_\varepsilon)L(t,x,u,\partial)u_\varepsilon + A_{0\varepsilon}^{-1}(\mathrm{Id} - J_\varepsilon)L(t,x,u_\varepsilon,\partial)u_\varepsilon \\
& +A_{0\varepsilon}^{-1} J_\varepsilon L(t,x,u_\varepsilon,\partial)(\mathrm{Id} - J_\varepsilon)u_\varepsilon + A_{0\varepsilon}^{-1} J_\varepsilon M(t,x,u_\varepsilon,J_\varepsilon u_\varepsilon)(\mathrm{Id} - J_\varepsilon)(u_\varepsilon)(J_\varepsilon u_\varepsilon) \\
& +N_0(t,x,u_\varepsilon)(\mathrm{Id} - J_\varepsilon)(u_\varepsilon)(g(t,x,u)) + A_{0\varepsilon}^{-1} G(u_\varepsilon, J_\varepsilon u_\varepsilon)(\mathrm{Id} - J_\varepsilon)u_\varepsilon + A_{0\varepsilon}^{-1}(\mathrm{Id} - J_\varepsilon)g(t,x,J_\varepsilon u_\varepsilon).
\end{aligned}
$$

Next we estimate $\|R_\varepsilon(t)\|_{L^2(\mathbb{T}^n)}$ term by term. We estimate the first term as follows:

$$\begin{aligned}
\|N_0(t,x,u_\varepsilon)(\mathrm{Id}-J_\varepsilon)(u_\varepsilon)L(t,x,u_\varepsilon,\partial)u_\varepsilon\|_{L^2(\mathbb{T}^n)}(t) &\leq \|N_0(t,x,u_\varepsilon)\|_{L^\infty}\cdot\|(\mathrm{Id}-J_\varepsilon)(u_\varepsilon)\|_{L^2}\cdot\|L(t,x,u_\varepsilon,\partial)u_\varepsilon\|_{L^\infty}\\
&\leq C_2(\|u_\varepsilon(t)\|_{C^0})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot\|u_\varepsilon(t)\|_{H^k}\cdot C_3(\|u_\varepsilon(t)\|_{C^1})\\
&\leq C_4(\|u_\varepsilon(t)\|_{C^1})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot\|u_\varepsilon(t)\|_{H^k}.
\end{aligned}$$

For the second term

$$\begin{aligned}
\|A_{0\varepsilon}^{-1}(\mathrm{Id}-J_\varepsilon)L(t,x,u_\varepsilon,\partial)u_\varepsilon\|_{L^2(\mathbb{T}^n)}(t) &\leq C_5(\|u_\varepsilon(t)\|_{C^0})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}\cdot\|L(t,x,u_\varepsilon,\partial)u_\varepsilon\|_{H^{k-1}(\mathbb{T}^n)}(t)\\
&\overset{(4.2)}{\leq} C_5(\|u_\varepsilon(t)\|_{C^0})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}\cdot\sum_{j=1}^n\|A_j(t,x,u_\varepsilon)\|_{L^\infty}\cdot\|\partial_j u_\varepsilon\|_{H^{k-1}}\\
&\qquad\qquad+\|A_j(t,x,u_\varepsilon)\|_{H^{k-1}}\cdot\|\partial_j u_\varepsilon\|_{L^\infty}\\
&\overset{(4.5)}{\leq} C_5(\|u_\varepsilon(t)\|_{C^0})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}\cdot\sum_{j=1}^n C_6(\|u_\varepsilon(t)\|_{L^\infty})\cdot\|u_\varepsilon\|_{H^k}\\
&\qquad\qquad+\|u_\varepsilon(t)\|_{C^1}\cdot C_7(\|u_\varepsilon(t)\|_{L^\infty})(1+\|u_\varepsilon\|_{H^{k-1}})\\
&\leq C_8(\|u_\varepsilon(t)\|_{C^1})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}\cdot(1+\|u_\varepsilon\|_{H^k}).
\end{aligned}$$

In the same way, using also $\|J_\varepsilon\|_{\mathcal{L}(L^2,L^2)}\leq 1$ and $[\partial_j,J_\varepsilon]=0$,

$$\begin{aligned}
\|A_{0\varepsilon}^{-1}J_\varepsilon L(t,x,u_\varepsilon,\partial)(\mathrm{Id}-J_\varepsilon)u_\varepsilon\|_{L^2} &\leq C_9(\|u_\varepsilon(t)\|_{C^0})\cdot\|L(t,x,u_\varepsilon,\partial)(\mathrm{Id}-J_\varepsilon)u_\varepsilon\|_{L^2}\\
&\leq C_9(\|u_\varepsilon(t)\|_{C^0})\cdot\sum_{j=1}^n\|A_j(t,x,u_\varepsilon)\|_{L^\infty}\cdot\|\partial_j(\mathrm{Id}-J_\varepsilon)u_\varepsilon\|_{L^2}\\
&\leq C_{10}(\|u_\varepsilon(t)\|_{C^0})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}\cdot\|u_\varepsilon\|_{H^k}
\end{aligned}$$

and, as $\|M(t,x,u_\varepsilon,J_\varepsilon u_\varepsilon)(u_\varepsilon-J_\varepsilon u_\varepsilon)\|_{L^2}\leq C_{11}(\|u_\varepsilon\|_{L^\infty})\cdot\|u_\varepsilon-J_\varepsilon u_\varepsilon\|_{L^2}$, we obtain

$$\begin{aligned}
\|A_{0\varepsilon}^{-1}J_\varepsilon M(t,x,u_\varepsilon,J_\varepsilon u_\varepsilon)(u_\varepsilon-J_\varepsilon u_\varepsilon)J_\varepsilon u_\varepsilon\|_{L^2} &\leq C_{12}(\|u_\varepsilon(t)\|_{C^0})\cdot\|M(t,x,u_\varepsilon,J_\varepsilon u_\varepsilon)(u_\varepsilon-J_\varepsilon u_\varepsilon)\cdot J_\varepsilon u_\varepsilon\|_{L^2}\\
&\leq C_{13}(\|u_\varepsilon(t)\|_{C^0})\cdot\|(\mathrm{Id}-J_\varepsilon)u_\varepsilon\|_{L^2}\\
&\leq C_{13}(\|u_\varepsilon(t)\|_{C^1})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot\|u_\varepsilon\|_{H^k}.
\end{aligned}$$

As before, estimating N_0 and g, we have

$$\begin{aligned}
\|N_0(t,x,u_\varepsilon)(\mathrm{Id}-J_\varepsilon)(u_\varepsilon)(g(t,x,u))\|_{L^2} &\leq \|N_0(t,x,u_\varepsilon)\|_{L^\infty}\cdot\|(\mathrm{Id}-J_\varepsilon)(u_\varepsilon)\|_{L^2}\cdot\|g(t,x,u)\|_{L^\infty}\\
&\leq C_{14}(\|u_\varepsilon(t)\|_{C^0})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot\|u_\varepsilon\|_{H^k}.
\end{aligned}$$

For the last two terms, we obtain

$$\begin{aligned}
\|A_{0\varepsilon}^{-1}G(u_\varepsilon,J_\varepsilon u_\varepsilon)(u_\varepsilon-J_\varepsilon u_\varepsilon)\|_{L^2} &\leq C_{15}(\|u_\varepsilon(t)\|_{L^\infty})\cdot\|u_\varepsilon-J_\varepsilon u_\varepsilon\|_{L^2}\\
&\leq C_{15}(\|u_\varepsilon(t)\|_{C^1})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot\|u_\varepsilon\|_{H^k}
\end{aligned}$$

and

$$\begin{aligned}
\|A_{0\varepsilon}^{-1}(\mathrm{Id}-J_\varepsilon)g(t,x,J_\varepsilon u_\varepsilon)\|_{L^2} &\leq C_{16}(\|u_\varepsilon(t)\|_{L^\infty})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot\|g(t,x,J_\varepsilon u_\varepsilon)\|_{H^k}\\
&\overset{(4.5)}{\leq} C_{17}(\|u_\varepsilon(t)\|_{L^\infty})\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^k,L^2)}\cdot(1+\|u_\varepsilon(t)\|_{H^k}).
\end{aligned}$$

Note that $\|\mathrm{Id} - J_\varepsilon\|_{\mathcal{L}(H^k,L^2)} \leq \|\mathrm{Id} - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}$. On the whole, we obtain

$$\|R_\varepsilon(t)\|_{L^2} \leq C_{18}(\|u_\varepsilon(t)\|_{C^1}) \cdot (1 + \|u_\varepsilon(t)\|_{H^k}) \cdot \|\mathrm{Id} - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}.$$

We deduce that

$$\begin{aligned}\frac{d}{dt}\|v_\varepsilon(t)\|^2_{L^2,\varepsilon} &= \left(\frac{\partial A_{0\varepsilon}}{\partial t} \cdot v_\varepsilon, v_\varepsilon\right)_{L^2} + 2\Re e\left(A_{0\varepsilon}\frac{\partial v_\varepsilon}{\partial t}, v_\varepsilon\right)_{L^2} \\ &= \left(\frac{\partial A_{0\varepsilon}}{\partial t} \cdot v_\varepsilon, v_\varepsilon\right)_{L^2} \\ &\quad +2\Re e\left\{(L(t,x,u,\partial)v_\varepsilon, v_\varepsilon)_{L^2} + (A(t,x,u,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon, v_\varepsilon)_{L^2,\varepsilon} + (R_\varepsilon, v_\varepsilon)_{L^2,\varepsilon}\right\},\end{aligned}$$

with $|\left(\frac{\partial A_{0\varepsilon}}{\partial t} \cdot v_\varepsilon, v_\varepsilon\right)_{L^2}| \leq C_{19}(\|u_\varepsilon(t)\|_{C^1}) \cdot \|v_\varepsilon(t)\|^2_{L^2}$ and

$$\begin{aligned}\|A(t,x,u,u_\varepsilon,\nabla u_\varepsilon)v_\varepsilon\|_{L^2,\varepsilon} &\leq \|M_0(t,x,u,u_\varepsilon)(v_\varepsilon)(L(t,x,u,\partial)u_\varepsilon + g(t,x,u))\|_{L^2,\varepsilon} \\ &\quad +\|A_{0\varepsilon}^{-1}(M(t,x,u,u_\varepsilon)(v_\varepsilon)u_\varepsilon + G(u,u_\varepsilon)(v_\varepsilon))\|_{L^2,\varepsilon} \\ &\leq C_{20}(\|u(t)\|_{C^0}, \|u_\varepsilon(t)\|_{C^1}) \cdot \|v_\varepsilon(t)\|_{L^2} \\ &\quad +C_{21}(\|u_\varepsilon(t)\|_{C^0}) \cdot (\|M(t,x,u,u_\varepsilon)(v_\varepsilon)u_\varepsilon\|_{L^2} + \|G(u,u_\varepsilon)v_\varepsilon(t)\|_{L^2}) \\ &\leq C_{20}(\|u(t)\|_{C^0}, \|u_\varepsilon(t)\|_{C^1}) \cdot \|v_\varepsilon(t)\|_{L^2} \\ &\quad +C_{22}(\|u_\varepsilon(t)\|_{C^0}, \|u(t)\|_{C^0}) \cdot \|u_\varepsilon(t)\|_{C^1} \cdot \|v_\varepsilon\|_{L^2} + C_{23}(\|u_\varepsilon(t)\|_{C^0}, \|u(t)\|_{C^0}) \cdot \|v_\varepsilon(t)\|_{L^2} \\ &\leq C_{24}(\|u_\varepsilon(t)\|_{C^1}, \|u(t)\|_{C^0}) \cdot \|v_\varepsilon(t)\|_{L^2,\varepsilon}\end{aligned}$$

as well as

$$\begin{aligned}\Re e\,(L(t,x,u,\partial)v_\varepsilon, v_\varepsilon)_{L^2} &= ((L + L^*)(t,x,u,\partial)v_\varepsilon, v_\varepsilon)_{L^2} \\ &= -\sum_{j=1}^{n}(\partial_j A_j(t,x,u) \cdot v_\varepsilon, v_\varepsilon)_{L^2}\end{aligned}$$

because $A_j^* = A_j$, so that

$$|\Re e\,(L(t,x,u,\partial)v_\varepsilon, v_\varepsilon)_{L^2} \leq C_{25}(\|u_\varepsilon(t)\|_{C^1}) \cdot \|v_\varepsilon\|^2_{L^2,\varepsilon}.$$

Noticing that $2\Re e\,(R_\varepsilon, v_\varepsilon)_{L^2,\varepsilon} \leq \|R_\varepsilon\|^2_{L^2,\varepsilon} + \|v_\varepsilon\|^2_{L^2,\varepsilon}$, we obtain

$$\frac{d}{dt}\|v_\varepsilon(t)\|^2_{L^2,\varepsilon} \leq C_{26}(\|u_\varepsilon(t)\|_{C^1}, \|u(t)\|_{C^1}) \cdot \|v_\varepsilon\|^2_{L^2,\varepsilon} + \|R_\varepsilon(t)\|^2_{L^2,\varepsilon}.$$

Then by Grönwall's lemma, for every $t \in I \cap \mathbb{R}_+$,

$$\|v_\varepsilon(t)\|^2_{L^2,\varepsilon} \leq \exp\left(\int_0^t a(s)ds\right) \cdot \left(\|\underbrace{f - h}_{v_\varepsilon(0)}\|^2_{L^2,\varepsilon} + \int_0^t \|R_\varepsilon(s)\|^2_{L^2,\varepsilon} \cdot e^{-\int_0^s a(\tau)d\tau} ds\right).$$

Note that, since $\|u_\varepsilon(t)\|_{C^1} \leq C_{27} \cdot \|u_\varepsilon(t)\|_{H^k} \leq C_{28}$ uniformly in ε and $t \in I$, we may choose a to be constant. For the same reason (and by the estimate above), $\|R_\varepsilon(t)\|^2_{L^2,\varepsilon} \leq C_{29} \cdot \|\mathrm{Id} - J_\varepsilon\|^2_{\mathcal{L}(H^{k-1},L^2)}$.

This combined with the choice $h := f$ and the equivalence of the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^2,\varepsilon}$ yields

$$\begin{aligned}\|v_\varepsilon(t)\|_{L^2}^2 &\leq e^{at}\cdot\int_0^t \|R_\varepsilon(s)\|_{L^2,\varepsilon}^2 e^{-as}ds\\ &\leq C_{29}\cdot a^{-1}\cdot(e^{at}-1)\cdot\|\mathrm{Id}-J_\varepsilon\|_{\mathcal{L}(H^1,L^2)}^2.\end{aligned}$$

It follows from the Proposition 4.3 that $\|v_\varepsilon(t)\|_{L^2}^2 \leq C_{30}\cdot(e^{at}-1)\cdot\varepsilon$ (recall that $k-1>\frac{n}{2}\geq\frac{1}{2}$). This implies on the one hand that any solution to (4.1) – with given initial condition f – is the pointwise (in t) limit when $\varepsilon\to 0$ of the uniquely determined family $(u_\varepsilon)_\varepsilon$, so that any two such solutions must coincide on their common interval of definition. On the other hand, this inequality gives the C^0L^2-rate of convergence for $(u_\varepsilon)_\varepsilon$ to u. □

Step 6: By what seems to be a well-known result from functional analysis (see e.g. [1, Lemma 4.1]), the fact that the solution to (4.1) belongs to certain Sobolev spaces implies its continuity $I\to H^k$ where H^k is endowed with the weak topology. To show the strong continuity of the solution, it suffices to show the continuity of its (pointwise) H^k-norm. Estimate the t-derivative of that norm by inserting again a J_ε, using standard estimates, Grönwall and making $\varepsilon\to 0$ to show that the norm of the solution is actually Lipschitz.

Claim 6: *The solution u from Claim 4 actually lies not only in $L^\infty(I,H^k)$ as proven in Claim 4 but also in $C^0(I,H^k)$.*

Proof. So the continuity in the weak sense follows from [1, Lemma 4.1] applied to $Y = H^k$ and $X = H^{k-1}$ (note that $Y\subset X$ densely and so does $X'\subset Y'$). To show the strong continuity, it suffices to show that $t\mapsto\|u(t)\|_{H^k}$ is continuous. Note here that one cannot directly estimate $\frac{d}{dt}\|u(t)\|_{H^k}^2$ as before since the differential operator L does not preserve H^k. As in the proof of [20, Prop. 16.1.4], we avoid this difficulty by inserting a J_ε before u. Setting $(\cdot\,,\cdot)_{L^2,0} := (A_0\cdot\,,\cdot)_{L^2}$, we pick any multiindex α with $|\alpha|\leq k$. Recalling that $u\in\mathrm{Lip}(I,H^{k-1})$, we have $J_\varepsilon u\in\mathrm{Lip}(I,H^k)$, in particular, the function $t\mapsto\|\partial^\alpha J_\varepsilon u(t)\|_{L^2,0}^2$ is differentiable almost everywhere. We start computing the derivative of $t\mapsto\|\partial^\alpha J_\varepsilon u(t)\|_{L^2,0}^2$:

$$\begin{aligned}\frac{d}{dt}\|\partial^\alpha J_\varepsilon u(t)\|_{L^2,0}^2 &= \left(\frac{\partial A_0}{\partial t}\cdot\partial^\alpha J_\varepsilon u,\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(\frac{\partial}{\partial t}\partial^\alpha J_\varepsilon u,\partial^\alpha J_\varepsilon u\right)_{L^2,0}\\
&= \left(\frac{\partial A_0}{\partial t}\cdot\partial^\alpha J_\varepsilon u,\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(\partial^\alpha J_\varepsilon\frac{\partial u}{\partial t},\partial^\alpha J_\varepsilon u\right)_{L^2,0}\\
&= \left(\frac{\partial A_0}{\partial t}\cdot\partial^\alpha J_\varepsilon u,\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(\partial^\alpha J_\varepsilon A_0\frac{\partial u}{\partial t},\partial^\alpha J_\varepsilon u\right)_{L^2}\\
&\quad +2\Re e\left([A_0,J_\varepsilon]\partial^\alpha\frac{\partial u}{\partial t},\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(J_\varepsilon[A_0,\partial^\alpha]\frac{\partial u}{\partial t},\partial^\alpha J_\varepsilon u\right)_{L^2}\\
&\overset{(4.1)}{=} \left(\frac{\partial A_0}{\partial t}\cdot\partial^\alpha J_\varepsilon u,\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left([A_0,J_\varepsilon]\partial^\alpha\frac{\partial u}{\partial t},\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(J_\varepsilon[A_0,\partial^\alpha]\frac{\partial u}{\partial t},\partial^\alpha J_\varepsilon u\right)_{L^2}\\
&\quad +2\Re e\left(\partial^\alpha J_\varepsilon L(t,x,u,\partial)u,\partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(\partial^\alpha J_\varepsilon g(t,x,u),\partial^\alpha J_\varepsilon u\right)_{L^2}.\end{aligned}$$

We begin with estimating the last term. First, if $|\alpha| \geq 1$, we have

$$\begin{aligned}
|2\Re e\left(\partial^\alpha J_\varepsilon g(t,x,u), \partial^\alpha J_\varepsilon u\right)_{L^2}| &\leq 2\|\partial^\alpha J_\varepsilon g(t,x,u)\|_{L^2} \cdot \|\partial^\alpha J_\varepsilon u\|_{L^2} \\
&\leq 2\|\partial^\alpha g(t,x,u)\|_{L^2} \cdot \|\partial^\alpha u\|_{L^2} \\
&\overset{(4.4)}{\leq} C_1(\|u(t)\|_{C^0}) \cdot \|\partial^\alpha u\|_{L^2}^2 \\
&\leq C_2(\|u(t)\|_{C^1}) \cdot \|u(t)\|_{H^k}^2.
\end{aligned}$$

For $\alpha = 0$, we can only apply (4.5) and obtain

$$\begin{aligned}
|2\Re e\left(J_\varepsilon g(t,x,u), J_\varepsilon u\right)_{L^2}| &\leq C_3(\|u(t)\|_{C^1}) \cdot (1 + \|u(t)\|_{L^2}) \cdot \|u(t)\|_{L^2} \\
&\leq C_4(\|u(t)\|_{C^1}) \cdot \left(1 + \|u(t)\|_{L^2}^2\right),
\end{aligned}$$

which actually suffices for the proof of Step 6 (as well as for the extension criterion in Step 7). Next we decompose the last but one term as follows:

$$\begin{aligned}
2\Re e\left(\partial^\alpha J_\varepsilon L(t,x,u,\partial)u, \partial^\alpha J_\varepsilon u\right)_{L^2} &= 2\Re e\left(J_\varepsilon \partial^\alpha L(t,x,u,\partial)u, \partial^\alpha J_\varepsilon u\right)_{L^2} \\
&= 2\Re e\left(J_\varepsilon L\partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left(J_\varepsilon [\partial^\alpha, L]u, \partial^\alpha J_\varepsilon u\right)_{L^2}.
\end{aligned}$$

The second term on the r.h.s. can be easily estimated with the help of Moser estimates:

$$\begin{aligned}
|2\Re e\left(J_\varepsilon[\partial^\alpha, L]u, \partial^\alpha J_\varepsilon u\right)_{L^2}| &\leq 2\|J_\varepsilon[\partial^\alpha, L]u\|_{L^2} \cdot \|\partial^\alpha J_\varepsilon u\|_{L^2} \\
&\leq 2\|[\partial^\alpha, L]u\|_{L^2} \cdot \|\partial^\alpha u\|_{L^2} \\
&\overset{(4.3)}{\leq} C_5 \cdot \left(\sum_{j=1}^n \|\nabla A_j\|_{L^\infty} \cdot \|u(t)\|_{H^k} + \|\nabla A_j\|_{H^{k-1}} \cdot \|\nabla u\|_{L^\infty}\right) \|u(t)\|_{H^k} \\
&\leq C_5 \cdot (C_6(\|u(t)\|_{C^1})\|u(t)\|_{H^k} + C_7(\|u(t)\|_{C^1})\|u(t)\|_{H^k}\|u(t)\|_{C^1})\, \|u(t)\|_{H^k} \\
&\leq C_8(\|u(t)\|_{C^1}) \cdot \|u(t)\|_{H^k}^2.
\end{aligned}$$

Using symmetric hyperbolicity, we may estimate the term

$$\begin{aligned}
2\Re e\left(J_\varepsilon L\partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2} &= 2\Re e\left(LJ_\varepsilon \partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left([J_\varepsilon, L]\partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2} \\
&= \Re e\left((L + L^*)\partial^\alpha J_\varepsilon u, \partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left([J_\varepsilon, L]\partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2} \\
&= -\sum_{j=1}^n \Re e\left(\frac{\partial A_j}{\partial x_j} \cdot \partial^\alpha J_\varepsilon u, \partial^\alpha J_\varepsilon u\right)_{L^2} + 2\Re e\left([J_\varepsilon, L]\partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2},
\end{aligned}$$

where

$$\begin{aligned}
\left|\sum_{j=1}^n \Re e\left(\frac{\partial A_j}{\partial x_j} \cdot \partial^\alpha J_\varepsilon u, \partial^\alpha J_\varepsilon u\right)_{L^2}\right| &\leq \sum_{j=1}^n \|\frac{\partial A_j}{\partial x_j} \cdot \partial^\alpha J_\varepsilon u\|_{L^2}\| \cdot \|\partial^\alpha J_\varepsilon u\|_{L^2} \\
&\leq \sum_{j=1}^n \|\frac{\partial A_j}{\partial x_j}\|_{L^\infty} \cdot \|\partial^\alpha J_\varepsilon u\|_{L^2}^2 \\
&\leq C_9(\|u(t)\|_{C^1}) \cdot \|u(t)\|_{H^k}^2
\end{aligned}$$

and, with $[J_\varepsilon, L]v = \sum_{j=1}^n [J_\varepsilon, A_j]\frac{\partial v}{\partial x_j}$,

$$\begin{aligned}
|2\Re e\left([J_\varepsilon, L]\partial^\alpha u, \partial^\alpha J_\varepsilon u\right)_{L^2}| &\leq 2\sum_{j=1}^n \|[J_\varepsilon, A_j]\frac{\partial \partial^\alpha u}{\partial x_j}\|_{L^2}\cdot\|\partial^\alpha J_\varepsilon u\|_{L^2} \\
&\overset{(\text{Lemma4.5}.iii))}{\leq} C_{10}\cdot\sum_{j=1}^n \|A_j\|_{C^1}\cdot\|\partial^\alpha u\|_{L^2}^2 \\
&\leq C_{11}(\|u(t)\|_{C^1})\cdot\|u(t)\|_{H^k}^2.
\end{aligned}$$

We also have $|\left(\frac{\partial A_0}{\partial t}\cdot\partial^\alpha J_\varepsilon u, \partial^\alpha J_\varepsilon u\right)_{L^2}| \leq C_{12}(\|u(t)\|_{C^1})\cdot\|u(t)\|_{H^k}^2$,

$$\begin{aligned}
|2\Re e\left(J_\varepsilon[A_0,\partial^\alpha]\frac{\partial u}{\partial t}, \partial^\alpha J_\varepsilon u\right)_{L^2}| &\leq 2\|[A_0,\partial^\alpha]\frac{\partial u}{\partial t}\|_{L^2}\cdot\|u(t)\|_{H^k} \\
&\overset{(4.3)}{\leq} C_{13}\cdot\left(\|\nabla A_0\|_{H^{k-1}}\cdot\|\frac{\partial u}{\partial t}\|_{L^\infty} + \|\nabla A_0\|_{L^\infty}\cdot\|\frac{\partial u}{\partial t}\|_{H^{k-1}}\right)\cdot\|u(t)\|_{H^k} \\
&\overset{(4.5)}{\leq} (C_{14}(\|u(t)\|_{C^0})\cdot\|u(t)\|_{H^k}\cdot C_{15}(\|u(t)\|_{C^1}) + C_{16}(\|u(t)\|_{C^1})\cdot(1+\|u(t)\|_{H^k}))\cdot\|u(t)\|_{H^k} \\
&\leq C_{17}(\|u(t)\|_{C^1})\cdot\|u(t)\|_{H^k}\cdot(1+\|u(t)\|_{H^k})
\end{aligned}$$

and

$$\begin{aligned}
|2\Re e\left([A_0, J_\varepsilon]\partial^\alpha\frac{\partial u}{\partial t}, \partial^\alpha J_\varepsilon u\right)_{L^2}| &\leq 2\|[A_0, J_\varepsilon]\partial^\alpha\frac{\partial u}{\partial t}\|_{L^2}\cdot\|u(t)\|_{H^k} \\
&\overset{(4.5)}{\leq} C_{18}(\|u(t)\|_{C^1})\cdot\|u(t)\|_{H^k}\cdot(1+\|u(t)\|_{H^k}).
\end{aligned}$$

Bringing everything together, we deduce that, setting $\|v\|_{H^k,0}^2 := \sum_{|\alpha|\leq k}\|\partial^\alpha v\|_{L^2,0}^2$,

$$\frac{d}{dt}\|J_\varepsilon u(t)\|_{H^k,0}^2 \leq C_{19}(\|u(t)\|_{C^1})\cdot\left(1+\|u(t)\|_{H^k}^2\right). \tag{4.11}$$

That inequality *does not depend* on $\varepsilon > 0$. Since by construction of u we have the existence of a constant C_{20} such that $\|u(t)\|_{H^k} \leq C_{20}$ for all $t \in I$ (because $u \in L^\infty H^k$) and since $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^2,0}$ are equivalent, we deduce that $J_\varepsilon u : I \to H^k$ is C_{21}-Lipschitz continuous for a constant $C_{21} > 0$ independent of ε. Since for all $t \in I$ one has $(J_\varepsilon u)(t) \underset{\varepsilon\to 0}{\longrightarrow} u(t)$ (in the strong H^k-topology) and because the pointwise limit of a family of C_{21}-Lipschitz continuous family is again C_{21}-Lipschitz continuous, we obtain that $u : I \to H^k$ is C_{21}-Lipschitz continuous, in particular $u \in C^0H^k$. □

Step 7: Use in fact the preceding estimate of the t-derivative of $\|J_\varepsilon u(t)\|_{H^k}$ to deduce, using Grönwall and after letting $\varepsilon \to 0$, that $\|u(t)\|_{H^k}$ can be controlled in terms of a continuous function of $\|u(t)\|_{C^1}$. Conclude the proof of Theorem 4.2.

Claim 7: *The solution $u \in C^0H^k$ constructed above exists as long as $\|u(t)\|_{C^1}$ remains bounded: if, for a given $T \in (0,\infty)$, there is a constant C such that $\|u(t)\|_{C^1} \leq C$ for all $t \in [0,T[$, then there exists a $\delta > 0$ such that the solution u can be extended to a solution in $C^0([0,T+\delta], H^k)$.*

Proof. Since by assumption $\|u(t)\|_{C^1} \leq C < \infty$ for all $t \in [0,T[$, there exists a constant C' such that $C'^{-1}\|u(t)\|_{H^k} \leq \|u(t)\|_{H^k,0} \leq C'\|u(t)\|_{H^k}$ for all $t \in [0,T[$ and inequality (4.11) yields

$$\frac{d}{dt}\|J_\varepsilon u(t)\|^2_{H^k,0} \leq C_1 \cdot (1 + \|u(t)\|^2_{H^k,0}),$$

which can be rewritten in integral form: for every $\tau > 0$,

$$\frac{\|(J_\varepsilon u)(t+\tau)\|^2_{H^k,0} - \|(J_\varepsilon u)(t)\|^2_{H^k,0}}{\tau} = \frac{1}{\tau}\int_0^\tau \frac{d}{ds}\|J_\varepsilon u(s)\|^2_{H^k,0}ds \leq \frac{C_1}{\tau} \cdot \int_0^\tau 1 + \|u(s)\|^2_{H^k,0}ds.$$

Using the pointwise convergence $J_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{Id}$ in H^k (and $H^k_{,0}$) and letting then $\tau \to 0^+$ lead to

$$\frac{d}{dt}\|u(t)\|^2_{H^k,0} \leq C_1 \cdot (1 + \|u(t)\|^2_{H^k,0})$$

and therefore $\|u(t)\|^2_{H^k,0} \leq (1 + \|u(0)\|^2_{H^k,0}) \cdot e^{C_1 t} - 1$ for all $t \in [0,T[$, in particular there is a constant $K > 0$ with $\|u(t)\|_{H^k} \leq K < \infty$ for all $t \in [0,T[$. The latter inequality implies that u can be extended beyond T, namely as follows. Consider a small interval of the form $]T - \hat{\eta}, T + \hat{\eta}[$ for some $\hat{\eta} > 0$. Because A_j and g are continuous, satisfy the "strong" local Lipschitz condition and because the time of existence for solutions to ODE's depends *continuously* on the norm of the initial condition (see e.g. proof of [16, Theorem 6.2.1]), up to making $\hat{\eta}$ a bit smaller, there exists an $\eta > 0$ such that, for any $\hat{u}_0 \in H^k$ with $\|\hat{u}_0\|_{H^k} \leq K$ and for any $\hat{t}_0 \in]T-\hat{\eta}, T+\hat{\eta}[$, the solution to the approximate symmetric hyperbolic equation (4.9) starting in $\hat{u}_0$ at time $\hat{t}_0$ exists on $[\hat{t}_0, \hat{t}_0 + \eta[$, and this *independently on* $\varepsilon > 0$ (use again Step 3). Taking $\check{\eta} := \min(\eta, \hat{\eta}) > 0$, we can look at the initial condition $u(T - \frac{\check{\eta}}{2})$ at time $T - \frac{\check{\eta}}{2}$ and obtain the existence of a family of approximate solutions starting in $u(T - \frac{\check{\eta}}{2})$ at time $T - \frac{\check{\eta}}{2}$ and existing on $[T - \frac{\check{\eta}}{2}, T + \frac{\check{\eta}}{2}[$. Restricting to any compact interval in $[T - \frac{\check{\eta}}{2}, T + \frac{\check{\eta}}{2}[$ and applying the preceding results from Steps 4 to 6, we obtain the existence of a solution to the symmetric hyperbolic system starting in $u(T - \frac{\check{\eta}}{2})$ at time $T - \frac{\check{\eta}}{2}$ and existing beyond T. By uniqueness of solutions to symmetric hyperbolic systems, the latter solution coincides with the former on $[T - \frac{\check{\eta}}{2}, T[$ and in particular u can be extended beyond T, QED. □

Now we need an additional control on the lifetime of the solution under the additional assumptions of semilinearity (instead of merely quasilinearity) and the one of punctured nonlinearity, i.e., we assume that there is one regular solution (satisfied in our case, as the nonlinearity vanishes at the zero section):

Theorem 4.6 (Estimate on lifetime) *Consider a symmetric hyperbolic system of equations on $\mathbb{T}^n$ of the form* (4.1) *where $A_0, A_j, g \in C^k$ for some $k > \frac{n}{2} + 1$. Assume* (4.1) *to be* **semilinear***, i.e., that A_0 and A_j are constant in their last argument u, and furthermore assume that there is a sufficiently regular (say, C^0H^k) global solution v to* (4.1).
Then for every $C, T > 0$, there exists an $\varepsilon > 0$ such that every C^1-solution u to (4.1) *with $u(0) = f \in H^k$ and $\|f - v(0)\|_{H^k} \leq \varepsilon$ exists on $[0,T] \times \mathbb{T}^n$ and satisfies $\|u(s) - v(s)\|_{H^k} \leq C$ for all $s \leq T$.*

Remark: On the one hand, if $g(t,x,0) = 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{T}^n$, obviously 0 is a smooth solution. On the other hand, by defining $\tilde{g}(w) := g(v + w) + (L - A_0\partial_t)v$ for a solution v, one can consider the equation $\tilde{P}(w) = 0$ for $\tilde{P} := -A_0\partial_t + L + \tilde{g}$. Obviously, $\tilde{P}(w) = 0$ is a symmetric hyperbolic equation, where the nonlinearity $\tilde{g}$ satisfies $\tilde{g}(0) = 0$.

Proof. In view of the extension criterion in Theorem 4.2 and the remark above, we assume that $g(t, x, 0) = 0$ and estimate $\|u(t)\|^2_{H^k}$ by a function of t. We proceed as in the proof of Claim 3 above and first estimate $\frac{d}{dt}\|J_\varepsilon u(t)\|^2_{H^k,0}$ for any $\varepsilon > 0$, where $\|v\|^2_{H^k,0} := \sum_{|\alpha|\leq k} (A_0 \cdot \partial^\alpha v, \partial^\alpha v)_{L^2}$ for every $v \in H^k$ (both norms $\|\cdot\|_{H^k,0}$ and $\|\cdot\|_{H^k}$ are equivalent on any compact subset of $\mathbb{R}\times\mathbb{T}^n$); then we let ε tend to 0 and obtain a differential inequality which, by Bihari's inequality, implies the statement. Taking into account that all A_j, $0 \leq j \leq n$, only depend on (t, x) and that $g(t, x, 0) = 0$, we can mimic the proof of Step 3 and obtain, after letting $\varepsilon \to 0$, the estimate

$$\left|\frac{d}{dt}\|u(t)\|^2_{H^k,0}\right| \leq F(\|u(t)\|^2_{C^0}) \cdot \|u(t)\|^2_{H^k,0}$$

for some continuous real-valued function F on $[0, \infty)$. In particular, up to changing F, we obtain

$$\left|\frac{d}{dt}\|u(t)\|^2_{H^k,0}\right| \leq F(\|u(t)\|^2_{H^k,0}) \cdot \|u(t)\|^2_{H^k,0}.$$

By Bihari's inequality, this proves the statement. Namely, letting $y(t) := \|u(t)\|^2_{H^k,0}$, we have the inequality $y' \leq yF(y)$ so that, assuming $y > 0$ (otherwise y vanishes identically because of Theorem 4.7 below) and setting $z := \ln(y)$, we obtain

$$\int_{z(0)}^{z(t)} \frac{ds}{F(e^s)} \leq t$$

for every $t \geq 0$. Because $F(e^s) \underset{s\to-\infty}{\longrightarrow} F(0) \geq 0$, we have $\int_{z(0)}^{z(t)} \frac{ds}{F(e^s)} \underset{y(0)\searrow 0}{\longrightarrow} \infty$ which implies that, for any $T, D > 0$, there exists an $\varepsilon > 0$ such that, for any y fulfilling $y' \leq yF(y)$ with $y(0) < \varepsilon$, the function $z(t)$ exists on $[0, T]$ and satisfies $z(t) \leq D$. This concludes the proof. □

Finally, we need (global) uniqueness for solutions to symmetric hyperbolic systems.

Theorem 4.7 *Consider a $\mathbb{K}^N$-valued first-order symmetric hyperbolic system on $\mathbb{T}^n$ or $\mathbb{R}^n$ as in* Definition 4.1 *and assume $A_j, g \in C^1$. Let I be an open interval with $0 \in I$. Let $u_1, u_2 \in C^1(I\times\mathbb{T}^n)$ (resp. $u_1, u_2 \in C^1(I \times \mathbb{R}^n)$) be any solutions to*

$$A_0(t, x, u_j)\frac{\partial u_j}{\partial t} = L(t, x, u_j, \partial)u_j + g(t, x, u_j) \text{ for } t \in I$$

with $u_j(0) = f \in C^0$. Then $u_1 = u_2$.

Proof. We show that $u_1 - u_2$ solves a *linear* symmetric hyperbolic system. We write

$$\begin{aligned}\frac{\partial(u_1-u_2)}{\partial t} &= A_0^{-1}(t,x,u_1)\cdot L(t,x,u_1,\partial)u_1 + A_0^{-1}(t,x,u_1)\cdot g(t,x,u_1)\\ &\quad -A_0^{-1}(t,x,u_2)\cdot L(t,x,u_2,\partial)u_2 - A_0^{-1}(t,x,u_2)\cdot g(t,x,u_2)\\ &= A_0^{-1}(t,x,u_1)\cdot L(t,x,u_1,\partial)(u_1-u_2) + A_0^{-1}(t,x,u_1)\cdot L(t,x,u_1,\partial)u_2 - A_0^{-1}(t,x,u_2)\cdot L(t,x,u_2,\partial)u_2\\ &\quad +(A_0^{-1}g)(t,x,u_1) - (A_0^{-1}g)(t,x,u_2)\\ &= A_0^{-1}(t,x,u_1)\cdot L(t,x,u_1,\partial)(u_1-u_2) + (A_0^{-1}(t,x,u_1) - A_0^{-1}(t,x,u_2))\cdot L(t,x,u_1,\partial)u_2\\ &\quad +A_0^{-1}(t,x,u_2)\cdot (L(t,x,u_1,\partial) - L(t,x,u_2,\partial))u_2 + (A_0^{-1}g)(t,x,u_1) - (A_0^{-1}g)(t,x,u_2).\end{aligned}$$

Now, because $A_0, A_j, g \in C^1$, we may write

$$\begin{aligned}A_0^{-1}(t,x,u_1) - A_0^{-1}(t,x,u_2) &= M(t,x,u_2,u_2)\cdot(u_1-u_2)\\ L(t,x,u_1,\partial) - L(t,x,u_2,\partial) &= \sum_{j=1}^{n} B_j(y,x,u_1,u_2)\cdot(u_1-u_2)\frac{\partial}{\partial x_j}\\ (A_0^{-1}g)(t,x,u_1) - (A_0^{-1}g)(t,x,u_2) &= N(t,x,u_1,u_2)\cdot(u_1-u_2),\end{aligned}$$

therefore

$$\begin{aligned}\frac{\partial(u_1-u_2)}{\partial t} &= A_0^{-1}(t,x,u_1)\cdot L(t,x,u_1,\partial)(u_1-u_2) + M(t,x,u_2,u_2)\cdot(u_1-u_2)L(t,x,u_1,\partial)u_2\\ &\quad +A_0^{-1}(t,x,u_2)\cdot\sum_{j=1}^{n} B_j(y,x,u_1,u_2)\cdot(u_1-u_2)\frac{\partial u_2}{\partial x_j} + N(t,x,u_1,u_2)\cdot(u_1-u_2),\end{aligned}$$

that is,

$$A_0(t,x,u_1)\cdot\frac{\partial(u_1-u_2)}{\partial t} = L(t,x,u_1,\partial)(u_1-u_2) + B(t,x,u_1,u_2)\cdot(u_1-u_2),$$

where B is of zero order. Hence $u_1 - u_2$ solves a linear symmetric hyperbolic system of first order with vanishing initial condition along the Cauchy hypersurface $\{0\}\times\mathbb{T}^n$ (resp. $\{0\}\times\mathbb{R}^n$) of the globally hyperbolic spacetime $I\times\mathbb{T}^n$ (resp. $I\times\mathbb{R}^n$). An elementary energy estimate for such systems (see e.g. [3, Theorem 5.3]) implies that $u_1 - u_2 = 0$ on $I\times\mathbb{T}^n$ (resp. $I\times\mathbb{R}^n$). $\square$

Now we want to transfer the previous local results to the framework of Lorentzian manifolds. Let (M^n, g) be any globally hyperbolic spacetime and $S \subset M$ be any spacelike Cauchy hypersurface with induced Riemannian metric g_S. Let $E \xrightarrow{\pi} M$ be any vector bundle. A **differential operator** P **of order** $k \in \mathbb{N}$ on π is a fibre-bundle-morphism from the kth jet bundle $J^k\pi$ of π to π. It is called **semilinear** if $[\ldots[P, f\cdot], f\cdot, \ldots, f\cdot] =: \sigma_P(df)$ is a vector bundle endomorphism for all scalar functions f on M, where f appears k times in the brackets. Generalizing [3, Definition 5.1] to the nonlinear case, we define a **semilinear symmetric hyperbolic operator** of first order acting π as a semilinear first-order-differential operator P acting on sections of π such that, denoting by $\sigma_P : T^*M \to \mathrm{End}(E)$ its principal symbol, there is an (definite or indefinite) inner product $\langle\cdot,\cdot\rangle$ on E such that for any $\xi \in T^*M$, the endomorphism $\sigma_P(\xi)$ of E is symmetric/Hermitian and positive-definite in case ξ is future-directed causal. It is easy to see that, locally, P is described exactly by Definition 4.1, where t is a local time-function on M. Theorems 4.2 and 4.7 imply the following

Corollary 4.8 *Let (M^n, g) be any globally hyperbolic spacetime and $S \subset M$ be any spacelike Cauchy hypersurface with induced Riemannian metric g_S. Let $E \xrightarrow{\pi} M$ be any vector bundle with (definite or indefinite) inner product and P be any semilinear symmetric hyperbolic operator of first order acting on sections of π. Let $k \in \mathbb{N}$ with $k > \frac{n-1}{2} + 1$. Then for any $f \in H^{k,2}(S, g_S)$, there exists an open neighbourhood U of S in M such that a unique solution $u \in \Gamma_{C^1}(U, E)$ to $Pu = 0$ with $u_{|_S} = f$ exists.*

Proof. Choose for any point $x \in S$ a neighbourhood B_x in S such that the domain of dependence A_x of B_x is contained in a submanifold chart domain for S. Then, via the embedding of B_x into a possibly large torus, we can express the equation $Pu = h(u)$ locally in each A_x as a symmetric hyperbolic system as in Definition 4.1. Consider for each x a cut-off function which is 1 on B_x and has support contained in a chart neighbourhood of the torus. We cut-off the initial data using that function and get the existence of a solution in a small strip around B_x. There is a small neighbourhood of x whose domain of dependence C_x is contained in that strip. The solutions obtained that way coincide on the intersection of any two such domains. Patching all such domains C_x together, we obtain a small open neighbourhood of S in M carrying a solution to the original equation. $\square$

Corollary 4.9 *Let (M^n, g) be any globally hyperbolic spacetime with compact Cauchy hypersurface $S \subset M$. Let $k \in \mathbb{N}$ with $k > \frac{n-1}{2} + 1$. Let $E \xrightarrow{\pi} M$ be any vector bundle with (definite or indefinite) inner product and P be any C^k semilinear symmetric hyperbolic operator of first order acting on sections of π with $P = L + h$, where L is linear and h is of order zero with $h(0) = 0$. Then we have the following* **estimate on lifetime** *for the solution u of $Pu = 0$: for each $T > 0$, there is an $\varepsilon > 0$ such for all initial values u_0 on S with H^k-norm smaller than ε, the lifetime for the solution with that initial value is greater than T.*

Proof. First observe that for every coordinate patch, a global solution is given by 0. Then use finitely many times the estimates given in Theorem 4.6. $\square$

Symmetric hyperbolic operators of second order on $E \xrightarrow{\pi} M$ are defined as follows: a differential operator P of second order on π is called **symmetric hyperbolic** if there exists a symmetric hyperbolic operator of first order Q – called **the first prolongation of** P – acting on sections of $\pi \oplus T^*M \otimes \pi$ such that $Pu = Q(u, \nabla u)$ for every section u of π. This fits to the restriction to charts — there, ∇u is expressed as $\partial u + \Gamma$ where Γ is an algebraic (actually, linear) expression in the u variable. Therefore a representation by Q as above entails an analogous expression in each chart. Furthermore, common textbook knowledge assures that every operator of the form

$$Pu = -\partial_t^2 u + \sum_{i,j=1}^{m} A_{ij}(t,x) \cdot \nabla_{ij} u + \sum_{i=1}^{m} B_i(t,x) \cdot \nabla_i u + c \cdot \partial_t u + d \cdot u$$

(with A_{ij} symmetric and uniformly positive) can be presented as $Pu = Q(u, \nabla u)$ as above, and the Laplace-d'Alembert equation on a compact subset can be brought into the form $Pu = 0$ for P

as above. If P is semilinear, so is Q; if $P = A + g$ with A linear and $g(0) = 0$, then $Q = B + h$ with $h(0) = 0$. The local-in-time existence result for second-order symmetric hyperbolic systems is based on Corollary 4.8. It is important to note that, if P has C^k coefficients, then so has Q. However, as the new operator Q includes a derivative of u, we loose one order of regularity for u, but as we do not care much for the weakest possible regularity condition on the initial values anyway, we treat the semilinear operator Q just like a quasilinear operator. However, notice that there is a folklore theorem mentioned in Taylor's book stating that semilinear symmetric hyperbolic systems of first order have a C^0-extension criterion, therefore we could avoid the loss of one derivative of u and obtain sharper statements for the necessary regularity of the initial values.

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