The spectrum of the Dirac operator on $\mathrm{SU}_2/\mathrm{Q}_8$

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17th May 2007

Abstract. We compute the fundamental Dirac operator for the three-parameter-family of homogeneous Riemannian metrics and the four different spin structures on $\mathrm{SU}_2/\mathrm{Q}_8$, where Q_8 denotes the group of quaternions. We deduce its spectrum for the Berger metrics and show the sharpness of Christian Bär's upper bound for the smallest Dirac eigenvalue in the particular case where $\mathrm{SU}_2/\mathrm{Q}_8$ is a homogeneous minimal hypersurface of S^4.

Mathematics Subject Classification: 53C27, 53C30, 58C40

Key words: Spin geometry, homogeneous manifolds, spectral theory

Throughout this paper and unless explicitly mentioned we denote by M the quotient of SU_2 by the right-action of the group of quaternions Q_8, i.e., the group with 8 elements defined by $\{\pm\mathrm{I}_2, \pm A_1, \pm A_2, \pm A_3\}$ with $A_1 := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, $A_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $A_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The manifold M is a 3-dimensional compact connected spin homogeneous space and at the same time the simplest example of homogeneous hypersurface in the round sphere with 3 different principal curvatures, see e.g. [6] and end of Section 2.
Using classical techniques (see e.g. [2]) we first compute the Dirac operator of M for any homogeneous metric and any spin structure:

Theorem 0.1

i) The manifold M carries a 3-parameter family of homogeneous Riemannian metrics which are given by the orthonormal bases $\{X_1 := a_1A_1, X_2 := a_2A_2, X_3 := a_3A_3\}$ of $\mathfrak{su}(2)$, where $a_1, a_2, a_3 \in \mathbb{R}^*$. Conversely, every homogeneous metric on M is of that form.

ii) The isotropy representation α of M is given in the basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ by

$$\begin{array}{ll} \alpha(\pm\mathrm{I}_2) = \mathrm{I}_3 & \alpha(\pm A_1) = \mathrm{diag}(1, -1, -1) \\ \alpha(\pm A_2) = \mathrm{diag}(-1, 1, -1) & \alpha(\pm A_3) = \mathrm{diag}(-1, -1, 1). \end{array}$$

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In particular the manifold M is orientable.

iii) The manifold M is spin and carries exactly 4 spin structures, each one corresponding to one of the following group homomorphisms $\mathrm{Q}_8 \xrightarrow{\varepsilon_j} \{-1,1\}$: $\varepsilon_0 \equiv 1$ and $\mathrm{Ker}(\varepsilon_j) = \{\pm \mathrm{I}_2, \pm A_j\}$ for $j \in \{1,2,3\}$.

iv) The finite dimensional Dirac operator D_n corresponding to the irreducible representation of SU_2 on the space V_n of homogeneous polynomials of degree n in two variables is non-trivial only if n is odd. In that situation

$$D_n = D'_n - \frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2 a_1 a_2 a_3} \mathrm{Id}$$

where D'_n is described by a $\frac{n+1}{2} \times \frac{n+1}{2}$ tridiagonal matrix. More precisely, there exists a basis $(v_0, \ldots, v_{\frac{n-1}{2}})$ in which D'_n can be expressed as

0) *in case M carries the spin structure given by ε_0,*

$$\begin{aligned} D'_n(v_k) &= (-1)^k a_1 (n-2k) v_k + (k+1)(a_2 + (-1)^k a_3) v_{k+1} \\ &\quad + (n-k+1)(a_2 - (-1)^k a_3) v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \Big(a_1 + \frac{n+1}{2}(a_2+a_3)\Big) v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3) v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 1 \ (4)$ and

$$\begin{aligned} D'_n(v_k) &= -(-1)^k a_1 (n-2k) v_k + (k+1)(a_2 - (-1)^k a_3) v_{k+1} \\ &\quad + (n-k+1)(a_2 + (-1)^k a_3) v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \Big(a_1 - \frac{n+1}{2}(a_2+a_3)\Big) v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3) v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 3 \ (4)$.

1) *in case M carries the spin structure given by ε_1,*

$$\begin{aligned} D'_n(v_k) &= (-1)^k a_1 (n-2k) v_k + (k+1)(a_2 + (-1)^k a_3) v_{k+1} \\ &\quad + (n-k+1)(a_2 - (-1)^k a_3) v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \Big(a_1 - \frac{n+1}{2}(a_2+a_3)\Big) v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3) v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 1 \ (4)$ and

$$\begin{aligned} D'_n(v_k) &= -(-1)^k a_1 (n-2k) v_k + (k+1)(a_2 - (-1)^k a_3) v_{k+1} \\ &\quad + (n-k+1)(a_2 + (-1)^k a_3) v_{k-1}, \qquad 0 \le k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \Big(a_1 + \frac{n+1}{2}(a_2+a_3)\Big) v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3) v_{\frac{n-3}{2}} \end{aligned}$$

if $n \equiv 3 \ (4)$.

2) *in case M carries the spin structure given by ε_2,*

$$\begin{aligned}
D'_n(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2-(-1)^k a_3)v_{k+1}\\
&\quad +(n-k+1)(a_2+(-1)^k a_3)v_{k-1}, \qquad 0 \leq k < \frac{n-1}{2}\\
D'_n(v_{\frac{n-1}{2}}) &= \Big(-a_1 + \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}}
\end{aligned}$$

if $n \equiv 1\ (4)$ and

$$\begin{aligned}
D'_n(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2+(-1)^k a_3)v_{k+1}\\
&\quad +(n-k+1)(a_2-(-1)^k a_3)v_{k-1}, \qquad 0 \leq k < \frac{n-1}{2}\\
D'_n(v_{\frac{n-1}{2}}) &= \Big(-a_1 - \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}}
\end{aligned}$$

if $n \equiv 3\ (4)$.

3) *in case M carries the spin structure given by ε_3,*

$$\begin{aligned}
D'_n(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2-(-1)^k a_3)v_{k+1}\\
&\quad +(n-k+1)(a_2+(-1)^k a_3)v_{k-1}, \qquad 0 \leq k < \frac{n-1}{2}\\
D'_n(v_{\frac{n-1}{2}}) &= \Big(-a_1 - \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}}
\end{aligned}$$

if $n \equiv 1\ (4)$ and

$$\begin{aligned}
D'_n(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2+(-1)^k a_3)v_{k+1}\\
&\quad +(n-k+1)(a_2-(-1)^k a_3)v_{k-1}, \qquad 0 \leq k < \frac{n-1}{2}\\
D'_n(v_{\frac{n-1}{2}}) &= \Big(-a_1 + \frac{n+1}{2}(a_2-a_3)\Big)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2+a_3)v_{\frac{n-3}{2}}
\end{aligned}$$

if $n \equiv 3\ (4)$.

We deduce the spectrum of the Dirac operator D of M for the so-called Berger metrics, which form a 2-parameter subfamily of homogeneous metrics:

Corollary 0.2 *With the notations of* Theorem 0.1*, assume furthermore that $a_2 = a_3$. Then the spectrum of the operator $D + \frac{2a_1^2+a_2^2}{2a_1}\mathrm{Id}$ on M for the metric induced by a_1, a_2 and the spin structure given by ε_j ($j \in \{0,1,2,3\}$) consists of the following family of eigenvalues:*

0. *for $j = 0$,*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2}$$
$$\,|\, k \in \{0,\ldots,\frac{n-5}{2}\} \text{ even}, a_1 + (n+1)a_2\Big\}$$

$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{a_1 \pm \sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$

$$\,|\, k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, a_1-(n+1)a_2, -na_1\Big\},$$

each eigenvalue having multiplicity $n+1$ *for the corresponding* n.

1. *for* $j=1$,

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{a_1 \pm \sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$

$$\,|\, k\in\{0,\ldots,\frac{n-5}{2}\}\text{ even}, a_1-(n+1)a_2\Big\}$$

$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{a_1 \pm \sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$

$$\,|\, k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, a_1+(n+1)a_2, -na_1\Big\},$$

each eigenvalue having multiplicity $n+1$ *for the corresponding* n.

2. *for* $j=2$ *and* $j=3$,

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{a_1 \pm \sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$

$$\,|\, k\in\{1,\ldots,\frac{n-3}{2}\}\text{ odd}, -na_1\Big\}$$

$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{a_1 \pm \sqrt{(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2}$$

$$\,|\, k\in\{0,\ldots,\frac{n-3}{2}\}\text{ even}\Big\},$$

each eigenvalue having multiplicity $n+1$ *for the corresponding* n.

In the case where $a_1=a_2=a_3$, i.e., M is a space-form with positive curvature, we reobtain the Dirac spectrum computed by Christian Bär in [3, Thm. 2], see Corollary 3.2.

On the other hand, considering M as embedded homogeneous hypersurface in the 4-dimensional round sphere S^4 one could ask if the following inequality due to Christian Bär [5, Cor. 4.3] is an equality:

$$\lambda_1(D^2) \leq \frac{9}{4}(\mathcal{H}^2+1), \tag{1}$$

where $\lambda_1(D^2)$ is the smallest eigenvalue of the Dirac Laplacian on M (for the induced metric and spin structure) and $\mathcal{H}$ is the mean curvature of M in S^4. This question takes its origin in the study of the equality case in Christian Bär's estimate [5, Cor. 4.3] for the smallest eigenvalue $\lambda_1(D^2)$ of the Dirac Laplacian. If this inequality is an equality, then the mean curvature of the hypersurface has to be constant, nevertheless the reverse statement has up to now neither been proved nor been contradicted. We give a partial answer to that question for M:

Corollary 0.3 *With the notations of* Theorem 0.1*, assume furthermore that M carries a homogeneous metric coming from a minimal embedding in S^4 and the spin structure described by ε_0. Then* (1) *is an equality.*

The paper is organized as follows. In the first section we describe the metrics and spin structures on M and thus prove Theorem 0.1 *i*)$-$*iii*). In the second one we compute the Dirac operator of M (Theorem 0.1 *iv*)) and the eigenvalue of D_1 (Corollary 2.9), which in the case where M is a hypersurface of S^4 turns out to coincide with the upper bound in (1), see Corollary 2.11. In the third section we prove Corollary 0.2 and derive the Dirac spectrum of M in case its metric either is of constant sectional curvature or comes from a minimal embedding in S^4, see Corollary 3.2. We deduce in Corollary 3.3 the existence of non-zero real Killing spinors in the first case and Corollary 0.3 in the other one.

Acknowledgement. This work provides a partial answer to a question set by Christian Bär, whom the author would like to thank for his interest and support. It's also a pleasure to thank Christian Bär and Bernd Ammann for their remarks.

1 Metrics and spin structures on M

The Lie-algebra of Q_8 being trivial the adjoint representation α of the homogeneous space M is nothing but the restriction of the adjoint map $\mathrm{SU}_2 \longrightarrow \mathrm{Aut}(\mathfrak{su}(2))$ to Q_8, where $\mathfrak{su}(2)$ denotes the Lie-algebra of SU_2. We define the scalar product $\langle \cdot\, , \cdot \rangle$ on $\mathfrak{su}(2)$ by declaring the following basis to be orthonormal:

$$\begin{aligned} X_1 &:= a_1 A_1 \\ X_2 &:= a_2 A_2 \\ X_3 &:= a_3 A_3, \end{aligned}$$

where $a_1, a_2, a_3 \in \mathbb{R}^*$ are fixed parameters. The map α is given in the basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ by

$$\begin{aligned} \alpha(\pm \mathrm{I}_2) &= \mathrm{I}_3 \\ \alpha(\pm A_1) &= \mathrm{diag}(1, -1, -1) \\ \alpha(\pm A_2) &= \mathrm{diag}(-1, 1, -1) \\ \alpha(\pm A_3) &= \mathrm{diag}(-1, -1, 1), \end{aligned}$$

therefore it obviously preserves $\langle \cdot\, , \cdot \rangle$ which hence induces a homogeneous metric on M. Using the form of α in the basis (A_1, A_2, A_3) computed above it is easy to prove that every homogeneous metric on M comes from such a scalar product on $\mathfrak{su}(2)$, i.e., it admits $\{a_1 A_1, a_2 A_2, a_3 A_3\}$ as orthonormal basis for suitable

$a_1, a_2, a_3 \in \mathbb{R}^*$. Note also that α preserves the orientation of $\mathfrak{su}(2)$, so that if we choose (X_1, X_2, X_3) as positively-oriented orthonormal basis of $\mathfrak{su}(2)$ then α is expressed in that basis by a map $\mathrm{Q}_8 \xrightarrow{\alpha} \mathrm{SO}_3$.

We now examine the spin structures on M considering the metric and the orientation given by (X_1, X_2, X_3). From [2, Lemma 3] the manifold M is spin if and only if its isotropy representation α lifts to Spin_3 through the non-trivial two-fold covering $\mathrm{Spin}_3 \xrightarrow{\xi} \mathrm{SO}_3$, and in that case spin structures on M are in one-to-one correspondence with those lifts, each one of those being uniquely determined by a group homomorphism $\mathrm{Q}_8 \xrightarrow{\varepsilon} \{-1, 1\}$. Here Q_8 already lies in $\mathrm{SU}_2 \cong \mathrm{Spin}_3$ so that M is obviously spin. Denoting by $\widehat{\alpha}$ the inclusion $\mathrm{Q}_8 \subset \mathrm{SU}_2$, every spin structure on M is uniquely described by a map $\widetilde{\alpha} : \mathrm{Q}_8 \longrightarrow \mathrm{SU}_2$ of the form $\widetilde{\alpha}(h) = \varepsilon(h)\widehat{\alpha}(h)$ for every $h \in \mathrm{Q}_8$, where $\varepsilon : \mathrm{Q}_8 \longrightarrow \{-1, 1\}$ is a group homomorphism. But there are exactly 4 such homomorphisms: the trivial one $\varepsilon_0 \equiv 1$ and the ε_j's, $j = 1, 2, 3$, with $\mathrm{Ker}(\varepsilon_j) = \{\pm \mathrm{I}_2, \pm A_j\}$. This proves Theorem 0.1 $i) - iii)$.
In the following we shall call the spin structure corresponding to $\varepsilon_j \cdot \widehat{\alpha}$ the ε_j-spin structure on M.

2 The Dirac operator on M

Let us denote by $\mathrm{Spin}_n \xrightarrow{\delta_n} \mathrm{Aut}(\Sigma_n)$ the spinor representation in dimension n. We recall the following theorem allowing the representation-theoretical computation of the fundamental Dirac operator on a homogeneous space, see e.g. [2, Thm. 2 & Prop. 1]:

Theorem 2.1 *Let $M := G/H$ be an n-dimensional Riemannian homogeneous spin manifold with G compact and simply-connected. Let $\mathfrak{p}$ be a supplementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$. Fix a p.o.n.b $(X_1, \ldots, X_n)$ of $\mathfrak{p}$ and let $\alpha : H \longrightarrow \mathrm{SO}_n$ be the isotropy representation of M expressed in the basis $(X_1, \ldots, X_n)$. Let $\widetilde{\alpha} : H \longrightarrow \mathrm{Spin}_n$ be the lift of α to Spin_n induced by the given spin structure of M and $\Sigma_{\widetilde{\alpha}} M \longrightarrow M$ be the spinor bundle of M associated with $\widetilde{\alpha}$. Let $\widehat{G}$ be the set of equivalence classes of irreducible unitary representations of G (in the following we shall always identify an element of $\widehat{G}$ with one of its representants).*

i) The space $L^2(M, \Sigma_{\widetilde{\alpha}} M)$ splits under the unitary left action of G into a direct Hilbert sum

$$\overline{\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)} \tag{2}$$

where V_γ is the space of the representation γ (i.e., $\gamma : G \longrightarrow \mathrm{U}(V_\gamma)$) and

$$\mathrm{Hom}_H(V_\gamma, \Sigma_n) := \Big\{ f \in \mathrm{Hom}(V_\gamma, \Sigma_n) \text{ s.t. } \\ \forall h \in H,\ f \circ \gamma(h) = (\delta_n \circ \widetilde{\alpha})(h) \circ f \Big\}.$$

ii) The Dirac operator D of M preserves each summand of (2); more precisely, if $(e_1, \ldots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$, then for every

$\gamma \in \widehat{G}$, the restriction of D to $V_\gamma \otimes \mathrm{Hom}_H(V_\gamma, \Sigma_n)$ is given by $\mathrm{Id} \otimes D_\gamma$, where, for every $A \in \mathrm{Hom}_H(V_\gamma, \Sigma_n)$,

$$D_\gamma(A) := -\sum_{k=1}^{n} e_k \cdot A \circ T_e\gamma(X_k) + \Big(\sum_{i=1}^{n} \beta_i e_i + \sum_{i<j<k} \alpha_{ijk} e_i \cdot e_j \cdot e_k\Big) \cdot A, \quad (3)$$

and

$$\begin{aligned} \beta_i &:= \frac{1}{2}\sum_{j=1}^{n} \langle [X_j, X_i]_{\mathfrak{p}}, X_j \rangle \\ \alpha_{ijk} &:= \frac{1}{4}\left(\langle [X_i, X_j]_{\mathfrak{p}}, X_k \rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i \rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j \rangle\right) \end{aligned}$$

(here and henceforth $X_{\mathfrak{p}}$ will denote the image of $X \in \mathfrak{g}$ under the projection $\mathfrak{g} \longrightarrow \mathfrak{p}$ with kernel $\mathfrak{h}$).

The following statement will be useful for taking the symmetries of M into account, see Examples 2.4 below.

Lemma 2.2 *Under the hypotheses of* Theorem 2.1 *let $\langle \cdot\,, \cdot \rangle'$ be a further homogeneous metric on M and $f : G \longrightarrow G$ be a Lie-group-homomorphism such that $f(H) \subset H$ and $f_* := [T_e f]$ is an orientation-preserving isometry $(T_{[e]}M, \langle \cdot\,, \cdot \rangle) \longrightarrow (T_{[e]}M, \langle \cdot\,, \cdot \rangle')$.*
Then the pull-back spin structure $f^*\mathrm{Spin}_{\widetilde{\alpha}}(TM)$ is described by

$$\begin{array}{rcl} H & \longrightarrow & \mathrm{Spin}_n \\ h & \longmapsto & \widehat{f}^{-1} \cdot \widetilde{\alpha} \circ f(h) \cdot \widehat{f} \end{array}$$

where $\widehat{f} \in \mathrm{Spin}_n$ satisfies $\xi(\widehat{f}) = f_*$.

Proof: The proof relies on the identity $f_* \circ \mathrm{Ad}(g) = \mathrm{Ad}(f(g)) \circ f_*$ for every $g \in G$, which implies in particular

$$\alpha(h) = f_*^{-1} \circ \alpha(f(h)) \circ f_*$$

for every $h \in H$.

□

Notes 2.3

1. Of course the homomorphism describing the pull-back spin structure in Lemma 2.2 is well-defined since $\widehat{f}$ is uniquely determined up to a sign.
2. One should pay attention that Lemma 2.2 can only be applied once p.o.n.b. $(X_1, \ldots, X_n)$ and $(X_1', \ldots, X_n')$ of $\mathfrak{p}$ w.r.t. $\langle \cdot\,, \cdot \rangle$ and $\langle \cdot\,, \cdot \rangle'$ respectively have been chosen. Then all the objects above should be expressed in those bases, see Examples 2.4 below.

Examples 2.4 Consider again $M := \mathrm{SU}_2/\mathrm{Q}_8$, fix $a_1, a_2, a_3 \in \mathbb{R}^*$ and as above set $X_k := a_k A_k$ for $k \in \{1, 2, 3\}$. We write $(M, \langle\cdot\,,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_j)$ for M endowed with the metric and the orientation given by (X_1, X_2, X_3) and the ε_j-spin structure ($j \in \{0, 1, 2, 3\}$).

1. Set $X_1' := X_1$, $X_2' := -X_2$ and $X_3' := -X_3$. Let $f(A_1) := A_1$, $f(A_2) := -A_2$ and $f(A_3) := -A_3$. Setting $f(I_2) := I_2$ and extending f linearly one obtains a Lie-group-homomorphism $\mathrm{SU}_2 \to \mathrm{SU}_2$ inducing an orientation-preserving isometry $(M, \langle\cdot\,,\cdot\rangle_{a_1,a_2,a_3}) \longrightarrow (M, \langle\cdot\,,\cdot\rangle_{a_1,-a_2,-a_3})$. The matrix of $f_* = f$ in the bases (X_1, X_2, X_3) and (X_1', X_2', X_3') respectively is the identity so that $\widehat{f} = 1$ can be chosen. Applying Lemma 2.2 the pull-back of the ε_j-spin structure by f is then described by

$$\mathrm{Q}_8 \longrightarrow \mathrm{SU}_2, \qquad h \longmapsto \varepsilon_j(h) f(h)$$

(remember that $-\mathrm{I}_2 \in \mathrm{Ker}(\varepsilon_j)$), i.e., the pull-back of the ε_0- (resp. ε_2-) spin structure is the ε_1- (resp. ε_3-) one. In other words, changing the sign of both a_2 and a_3 changes neither the metric nor the orientation, however it permutes the ε_0- (resp. ε_2-) spin structure with the ε_1- (resp. ε_3-) one. In particular the Dirac operator on e.g. $(M, \langle\cdot\,,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_0)$ coincides with that of $(M, \langle\cdot\,,\cdot\rangle_{a_1,-a_2,-a_3}, \varepsilon_1)$.

2. Let σ be a permutation of $\{0, 1, 2, 3\}$ with $\sigma(0) = 0$ and set $X_k' := a_{\sigma(k)} A_k$ for $k \in \{1, 2, 3\}$. Let $f(A_1) := A_{\sigma^{-1}(1)}$, $f(A_2) := A_{\sigma^{-1}(2)}$ and $f(A_3) := \varepsilon(\sigma) A_{\sigma^{-1}(3)}$ where $\varepsilon(\sigma) \in \{-1, 1\}$ is the signature of σ. Setting in the same way as just above $f(I_2) := I_2$ and extending f linearly one obtains a Lie-group-homomorphism $\mathrm{SU}_2 \to \mathrm{SU}_2$ inducing an orientation-preserving isometry $(M, \langle\cdot\,,\cdot\rangle_{a_1,a_2,a_3}) \longrightarrow (M, \langle\cdot\,,\cdot\rangle_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}})$. This time the matrix of $f_* = f$ in the bases (X_1, X_2, X_3) and (X_1', X_2', X_3') respectively is not the identity, however it coincides with the matrix of f in the basis (A_1, A_2, A_3) so that, per definition of the universal 2-fold covering map,

$$\widehat{f}^{-1} \cdot f(h) \cdot \widehat{f} = h$$

for any lift $\widehat{f}$ of f to SU_2 and every $h \in \mathrm{Q}_8$. The pull-back through f of the ε_j-spin structure is therefore the $(\varepsilon_j \circ f)$-one, that is, the $\varepsilon_{\sigma(j)}$-one. In other words, permuting the coefficients a_1, a_2, a_3 induces an orientation-preserving isometry permuting the spin structure in the reverse way, the ε_0-one staying unchanged under that transformation. In particular the Dirac operator on $(M, \langle\cdot\,,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_j)$ coincides with that of $(M, \langle\cdot\,,\cdot\rangle_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}}, \varepsilon_{\sigma^{-1}(j)})$.

3. It is well-known that, for any fixed metric and spin structure on M, the Dirac operators for the two different orientations are just opposite from one another (this is always the case in odd dimensions). For example, if one turns a_1 into $-a_1$ and lets a_2 and a_3 unchanged, then the Dirac operator on e.g. $(M, \langle\cdot\,,\cdot\rangle_{-a_1,a_2,a_3}, \varepsilon_0)$ coincides with minus that of $(M, \langle\cdot\,,\cdot\rangle_{a_1,-a_2,-a_3}, \varepsilon_0)$, i.e., with minus that of $(M, \langle\cdot\,,\cdot\rangle_{a_1,a_2,a_3}, \varepsilon_1)$.

Note that Examples 2.4 essentially exhausts all possible isometric transformations of M since the only Lie-group-automorphisms f of SU_2 preserving Q_8 are characterized by $f(A_k) = \epsilon(k)A_{\sigma(k)}$ for some permutation σ of $\{1,2,3\}$ and $\epsilon(k) \in \{-1,1\}$.

We come now to the computation of the Dirac operator on $M = \mathrm{SU}_2/\mathrm{Q}_8$. We begin with the part of the Dirac operator that does not depend on the representation γ of SU_2. Note also that this part only depends on the metric chosen on M and not on its spin structure.

Proposition 2.5 *For the metric on M given by a_1, a_2, a_3 we have $\beta_j = 0$ for every $j \in \{1,2,3\}$ and $\alpha_{123} = \frac{a_1^2a_2^2+a_2^2a_3^2+a_1^2a_3^2}{2a_1a_2a_3}$. In particular*

$$\sum_{j=1}^{3} \beta_j e_j \cdot + \alpha_{123} e_1 \cdot e_2 \cdot e_3 \cdot = -\frac{a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2}{2a_1a_2a_3}\mathrm{Id}.$$

Proof: We compute the Lie-brackets $[X_j, X_k]$ for all $1 \leq j < k \leq 3$. Since $A_1A_2 = -A_2A_1 = A_3$ we have

$$\begin{aligned} [X_1, X_2] &= a_1a_2[A_1, A_2] \\ &= 2a_1a_2A_3 \\ &= \frac{2a_1a_2}{a_3}X_3, \end{aligned}$$

and analogously $[X_2, X_3] = \frac{2a_2a_3}{a_1}X_1$, $[X_3, X_1] = \frac{2a_1a_3}{a_2}X_2$. We straightforward deduce that $\beta_1 = \beta_2 = \beta_3 = 0$. Furthermore,

$$\begin{aligned} \alpha_{123} &= \frac{1}{4}\left(\langle [X_1, X_2], X_3\rangle + \langle [X_2, X_3], X_1\rangle + \langle [X_3, X_1], X_2\rangle\right) \\ &= \frac{1}{4}\left(\frac{2a_1a_2}{a_3} + \frac{2a_2a_3}{a_1} + \frac{2a_1a_3}{a_2}\right) \\ &= \frac{a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2}{2a_1a_2a_3}. \end{aligned}$$

It remains to notice that, by convention, the complex volume form $i^{[\frac{3+1}{2}]}e_1 \cdot e_2 \cdot e_3 = -e_1 \cdot e_2 \cdot e_3$ acts by the identity on Σ_3. This concludes the proof.

□

We next determine the space of equivariant homomorphisms for each $\gamma \in \widehat{\mathrm{SU}_2}$ and each ε_j-spin structure on M. First recall that the irreducible unitary representations of SU_2 are given by its natural action on the $n+1$-dimensional vector spaces of all n-graded homogeneous complex polynomials in two variables: set, for any $n \in \mathbb{N}$ (we include $n = 0$)

$$V_n := \{P \in \mathbb{C}[z_1, z_2], \quad P = 0 \text{ or } P \text{ homogeneous and } d^\circ P = n\}.$$

Then SU_2 acts on V_n through

$$\begin{array}{rcl}\pi_n : \mathrm{SU}_2 & \longrightarrow & \mathrm{Aut}(V_n) \\ A & \longmapsto & (\pi_n(A) : P \mapsto P \circ R_A),\end{array}$$

where $P \circ R_A(z) := P(zA)$ for every $z = (z_1 \;\; z_2) \in \mathbb{C}^2$. From now on we shall always work with the following basis of V_n:

$$(P_k(z_1, z_2) := z_1^{n-k} z_2^k, \;\; 0 \leq k \leq n).$$

Identifying Spin_3 to SU_2 the spinor representation $\mathrm{Spin}_3 \xrightarrow{\delta_3} \mathrm{Aut}(\Sigma_3)$ is equivalent to the standard representation $\mathrm{SU}_2 \longrightarrow \mathrm{Aut}(\mathbb{C}^2)$. For every lift $\varepsilon_j \cdot \widehat{\alpha}$ of the isotropy representation α of M the space of equivariant homomorphisms for π_n and for the ε_j-spin structure - that we shall denote by $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n, \mathbb{C}^2)$ - is then given by

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n, \mathbb{C}^2) = \left\{ f \in \mathrm{Hom}(V_n, \mathbb{C}^2) \text{ s.t. } f \circ \pi_n(h) = \varepsilon_j(h) h \circ f \quad \forall\, h \in \mathrm{Q}_8 \right\}.$$

We fix the following basis $(F_0, \ldots, F_n, G_0, \ldots, G_n)$ of $\mathrm{Hom}(V_n, \mathbb{C}^2)$ (which is that of [2, p.73]): set, for every $k \in \{0, \ldots, n\}$,

$$F_k(P_l) := \begin{cases} (1 \;\; 0) & \text{if } l = k \text{ and } k \text{ even} \\ (0 \;\; 1) & \text{if } l = k \text{ and } k \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_k(P_l) := \begin{cases} (0 \;\; 1) & \text{if } l = k \text{ and } k \text{ even} \\ (1 \;\; 0) & \text{if } l = k \text{ and } k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

W.r.t. the bases $(P_0, \ldots, P_n)$ and $((1 \;\; 0), (0 \;\; 1))$ of V_n and $\mathbb{C}^2$ respectively the elements F_k and G_k are described by matrices of the form:

$$F_k = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}$$

if k is even and

$$F_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

if k is odd, where the "1" always stands in the $(k+1)^{\mathrm{st}}$ column.

Lemma 2.6 *Let M carry the ε_j-spin structure for $j \in \{0, 1, 2, 3\}$. Then* $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n, \mathbb{C}^2) = \{0\}$ *if n is even. Moreover*

0. *for $j = 0$ we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_0}(V_n, \mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k + F_{n-k}) & \text{if } n \equiv 1 \; (4) \\ \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k - G_{n-k}) & \text{if } n \equiv 3 \; (4). \end{cases}$$

1. *for* $j=1$ *we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_1}(V_n,\mathbb{C}^2)=\begin{cases}\bigoplus_{k=0}^{\frac{n-1}{2}}\mathbb{C}(F_k-F_{n-k}) & \text{if } n\equiv 1\ (4)\\[2ex] \bigoplus_{k=0}^{\frac{n-1}{2}}\mathbb{C}(G_k+G_{n-k}) & \text{if } n\equiv 3\ (4).\end{cases}$$

2. *for* $j=2$ *we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_2}(V_n,\mathbb{C}^2)=\begin{cases}\bigoplus_{k=0}^{\frac{n-1}{2}}\mathbb{C}(G_k+G_{n-k}) & \text{if } n\equiv 1\ (4)\\[2ex] \bigoplus_{k=0}^{\frac{n-1}{2}}\mathbb{C}(F_k-F_{n-k}) & \text{if } n\equiv 3\ (4).\end{cases}$$

3. *for* $j=3$ *we have*

$$\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_3}(V_n,\mathbb{C}^2)=\begin{cases}\bigoplus_{k=0}^{\frac{n-1}{2}}\mathbb{C}(G_k-G_{n-k}) & \text{if } n\equiv 1\ (4)\\[2ex] \bigoplus_{k=0}^{\frac{n-1}{2}}\mathbb{C}(F_k+F_{n-k}) & \text{if } n\equiv 3\ (4).\end{cases}$$

Proof: Since $-\mathrm{I}_2\in\mathrm{Ker}(\varepsilon_j)$ any element $f\in\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ must satisfy $f\circ\pi_n(-\mathrm{I}_2)=-f$, with $\pi_n(-\mathrm{I}_2)=(-1)^n\mathrm{Id}_{V_n}$, so that the condition reads

$$(-1)^n f=-f,$$

which requires $f=0$ as soon as n is even.
From now on, we assume that n is odd. We compute $\pi_n(A_j)$ for $j=1,2$ (remember that A_1 and A_2 generate Q_8): for every $k\in\{0,\ldots,n\}$ and $z\in\mathbb{C}^2$,

$$\begin{aligned}\{\pi_n(A_1)\}(P_k)(z) &= P_k\left((z_1\ \ z_2)\cdot\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\right)\\ &= P_k(-iz_1,iz_2)\\ &= (-iz_1)^{n-k}(iz_2)^k\\ &= (-1)^{n-k}i^n z_1^{n-k}z_2^k,\end{aligned}$$

i.e., $\{\pi_n(A_1)\}(P_k)=(-1)^{n-k}i^nP_k$. Analogously,

$$\begin{aligned}\{\pi_n(A_2)\}(P_k)(z) &= P_k\left((z_1\ \ z_2)\cdot\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\right)\\ &= P_k(iz_2,iz_1)\\ &= (iz_2)^{n-k}(iz_1)^k,\end{aligned}$$

i.e., $\{\pi_n(A_2)\}(P_k)=i^nP_{n-k}$. The conditions $f\circ\pi_n(A_l)=\varepsilon_j(A_l)A_l\circ f$ for $l=1,2$ then read

$$\left|\begin{array}{ll}f(P_k) & =(-1)^{k+\frac{n-1}{2}}i\varepsilon_j(A_1)(A_1\circ f)(P_k)\\ f(P_{n-k}) & =(-1)^{\frac{n+1}{2}}i\varepsilon_j(A_2)(A_2\circ f)(P_k)\end{array}\right. \tag{4}$$

for every $k \in \{0, 1, \dots, n\}$. From now on we denote by $\begin{pmatrix} f_{1k} \\ f_{2k} \end{pmatrix} := f(P_k) \in \mathbb{C}^2$.
We examine each case separately.

• *Case* $j = 0$: In that case the conditions (4) are equivalent to

$$\left| \begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n-1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n+1}{2}} i(A_2 \circ f)(P_k), \end{array} \right.$$

that is,

$$\left| \begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n-1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n+1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n-1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n-1}{2}} f_{1k}. \end{array} \right.$$

If $n \equiv 1$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 + F_n) + f_{21}(F_1 + F_{n-1}) + \dots + f_{1\frac{n-1}{2}}(F_{\frac{n-1}{2}} + F_{\frac{n+1}{2}})$$

and the result in that case.
If $n \equiv 3$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 - G_n) + f_{11}(G_1 - G_{n-1}) + \dots + f_{1\frac{n-1}{2}}(G_{\frac{n-1}{2}} - G_{\frac{n+1}{2}})$$

and the result in that case.

• *Case* $j = 1$: In that case the conditions (4) are equivalent to

$$\left| \begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n-1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n-1}{2}} i(A_2 \circ f)(P_k), \end{array} \right.$$

that is,

$$\left| \begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n-1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n+1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n+1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n+1}{2}} f_{1k}. \end{array} \right.$$

If $n \equiv 1$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 - F_n) + f_{21}(F_1 - F_{n-1}) + \ldots + f_{1\frac{n-1}{2}}(F_{\frac{n-1}{2}} - F_{\frac{n+1}{2}})$$

and the result in that case.
If $n \equiv 3$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 + G_n) + f_{11}(G_1 + G_{n-1}) + \ldots + f_{1\frac{n-1}{2}}(G_{\frac{n-1}{2}} + G_{\frac{n+1}{2}})$$

and the result in that case.
• *Case* $j = 2$: In that case the conditions (4) are equivalent to

$$\left| \begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n+1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n+1}{2}} i(A_2 \circ f)(P_k), \end{array} \right.$$

that is,

$$\left| \begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n+1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n-1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n-1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n-1}{2}} f_{1k}. \end{array} \right.$$

If $n \equiv 1$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 + G_n) + f_{11}(G_1 + G_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(G_{\frac{n-1}{2}} + G_{\frac{n+1}{2}})$$

and the result in that case.
If $n \equiv 3$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 - F_n) + f_{21}(F_1 - F_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(F_{\frac{n-1}{2}} - F_{\frac{n+1}{2}})$$

and the result in that case.

• *Case* $j = 3$: In that case the conditions (4) are equivalent to

$$\left| \begin{array}{ll} f(P_k) & = (-1)^{k+\frac{n+1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) & = (-1)^{\frac{n-1}{2}} i(A_2 \circ f)(P_k), \end{array} \right.$$

that is,

$$\left| \begin{array}{ll} f_{1k} & = (-1)^{k+\frac{n+1}{2}} f_{1k} \\ f_{2k} & = (-1)^{k+\frac{n-1}{2}} f_{2k} \\ f_{1n-k} & = (-1)^{\frac{n+1}{2}} f_{2k} \\ f_{2n-k} & = (-1)^{\frac{n+1}{2}} f_{1k}. \end{array} \right.$$

If $n \equiv 1$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = -(-1)^k f_{1k} \\ f_{2k} & = (-1)^k f_{2k} \\ f_{1n-k} & = -f_{2k} \\ f_{2n-k} & = -f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is even (resp. $f_{2k} = 0$ if k is odd) and $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{20}(G_0 - G_n) + f_{11}(G_1 - G_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(G_{\frac{n-1}{2}} - G_{\frac{n+1}{2}})$$

and the result in that case.

If $n \equiv 3$ (4) then those identities become

$$\left| \begin{array}{ll} f_{1k} & = (-1)^k f_{1k} \\ f_{2k} & = -(-1)^k f_{2k} \\ f_{1n-k} & = f_{2k} \\ f_{2n-k} & = f_{1k}, \end{array} \right.$$

hence $f_{1k} = 0$ if k is odd (resp. $f_{2k} = 0$ if k is even) and $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$ for every $0 \leq k \leq \frac{n-1}{2}$. We deduce that

$$f = f_{10}(F_0 + F_n) + f_{21}(F_1 + F_{n-1}) + \ldots + f_{2\frac{n-1}{2}}(F_{\frac{n-1}{2}} + F_{\frac{n+1}{2}})$$

and the result in that case. This concludes the proof.

□

It remains to compute the map $T_{\mathrm{I}_2}\pi_n$ for every (odd) n.

Lemma 2.7 *The endomorphisms* $T_{\mathrm{I}_2}\pi_n(X_j)$, $1 \leq j \leq 3$, *are given in the basis* $(P_0, \ldots, P_n)$ *of* V_n *by:*

$$\begin{array}{rcl} \{T_{\mathrm{I}_2}\pi_n(X_1)\}(P_k) & = & -ia_1(n-2k)P_k \\ \{T_{\mathrm{I}_2}\pi_n(X_2)\}(P_k) & = & ia_2\left((n-k)P_{k+1} + kP_{k-1}\right) \\ \{T_{\mathrm{I}_2}\pi_n(X_3)\}(P_k) & = & a_3\left(-(n-k)P_{k+1} + kP_{k-1}\right) \end{array}$$

for every $k \in \{0, \ldots, n\}$, *with the convention* $P_{-1} = P_{n+1} = 0$.

Proof: For every $X \in \mathfrak{su}_2$, $P \in V_n$ and $z \in \mathbb{C}^2$, we have

$$\begin{aligned}
\left(\{T_{\mathrm{I}_2}\pi_n(X)\}(P)\right)(z) &= \frac{d}{dt}|_{t=0}\left(P \circ R_{\exp(tX)}\right)(z) \\
&= \frac{d}{dt}|_{t=0}\left(P \circ R_{\exp(tX)}(z)\right) \\
&= \frac{d}{dt}|_{t=0}\left(P(z\exp(tX))\right) \\
&= d_z P(zX) \\
&= \frac{\partial P}{\partial z_1}(z)\,(zX)_1 + \frac{\partial P}{\partial z_2}(z)\,(zX)_2\,.
\end{aligned}$$

Since $zA_1 = (-iz_1\ \ iz_2)$, $zA_2 = (iz_2\ \ iz_1)$ and $zA_3 = (-z_2\ \ z_1)$ we have, for every $k \in \{0,\ldots,n\}$

$$\begin{aligned}
\{T_{\mathrm{I}_2}\pi_n(X_1)\}(P_k) &= a_1\{T_{\mathrm{I}_2}\pi_n(A_1)\}(P_k) \\
&= a_1\left(-iz_1\frac{\partial P_k}{\partial z_1}(z) + iz_2\frac{\partial P_k}{\partial z_2}(z)\right) \\
&= -ia_1\left((n-k)z_1 z_1^{n-k-1}z_2^k - kz_2 z_1^{n-k}z_2^{k-1}\right) \\
&= -ia_1\left((n-k)z_1^{n-k}z_2^k - kz_1^{n-k}z_2^k\right) \\
&= -ia_1(n-2k)P_k.
\end{aligned}$$

For X_2 we have

$$\begin{aligned}
\{T_{\mathrm{I}_2}\pi_n(X_2)\}(P_k) &= a_2\{T_{\mathrm{I}_2}\pi_n(A_2)\}(P_k) \\
&= a_2\left(iz_2\frac{\partial P_k}{\partial z_1}(z) + iz_1\frac{\partial P_k}{\partial z_2}(z)\right) \\
&= ia_2\left((n-k)z_1^{n-k-1}z_2^{k+1} + kz_1^{n-k+1}z_2^{k-1}\right) \\
&= ia_2\left((n-k)P_{k+1} + kP_{k-1}\right),
\end{aligned}$$

and for X_3 we obtain

$$\begin{aligned}
\{T_{\mathrm{I}_2}\pi_n(X_3)\}(P_k) &= a_3\{T_{\mathrm{I}_2}\pi_n(A_3)\}(P_k) \\
&= a_3\left(-z_2\frac{\partial P_k}{\partial z_1}(z) + z_1\frac{\partial P_k}{\partial z_2}(z)\right) \\
&= a_3\left(-(n-k)z_1^{n-k-1}z_2^{k+1} + kz_1^{n-k+1}z_2^{k-1}\right) \\
&= a_3\left(-(n-k)P_{k+1} + kP_{k-1}\right).
\end{aligned}$$

Note that the above expressions for $\{T_{\mathrm{I}_2}\pi_n(X_2)\}(P_k)$ and $\{T_{\mathrm{I}_2}\pi_n(X_3)\}(P_k)$ are also valid for $k = 0$ or $k = n$ with the convention $P_{-1} = P_{n+1} = 0$. The result follows.

$\square$

We now compute the component D_n of the Dirac operator of M acting on $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n, \mathbb{C}^2)$, see (3). We adopt henceforth the following convention: $F_k := G_k := 0$ as soon as $k \notin \{0,\ldots,n\}$.

The fix part of D_n has already been computed in Proposition 2.5, so that only the endomorphism D'_n of $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ given by

$$D'_n A = -\sum_{j=1}^{3} e_j \cdot A \circ T_{\mathrm{I}_2}\pi_n(X_j)$$

for every $A \in \mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$, remains to be made explicit.
First note that the Clifford product by e_j can be identified with the matrix multiplication by A_j for $j \in \{1,2,3\}$.
Furthermore, it is straightforward to show using Lemma 2.7 that, for every $k \in \{0,1,\ldots,n\}$,

$$\begin{aligned}
F_k \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)F_k \\
F_k \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\left((n-k+1)G_{k-1} + (k+1)G_{k+1}\right) \\
F_k \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\left(-(n-k+1)G_{k-1} + (k+1)G_{k+1}\right).
\end{aligned}$$

Those identities still hold for $k = 0$ or n using our convention above on the F_k's and G_k's. To obtain the corresponding identities on the G_k's one just has to exchange the roles of F_l and G_l for every l:

$$\begin{aligned}
G_k \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)G_k \\
G_k \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\left((n-k+1)F_{k-1} + (k+1)F_{k+1}\right) \\
G_k \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\left(-(n-k+1)F_{k-1} + (k+1)F_{k+1}\right).
\end{aligned}$$

We deduce the following set of identities:

$$\begin{aligned}
(F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)(F_k \mp F_{n-k}) \\
(F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\Big((k+1)(G_{k+1} \pm G_{n-k-1}) \\
&\qquad + (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
(F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\Big((k+1)(G_{k+1} \mp G_{n-k-1}) \\
&\qquad - (n-k+1)(G_{k-1} \mp G_{n-k+1})\Big) \\
(G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_1) &= -ia_1(n-2k)(G_k \mp G_{n-k}) \\
(G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_2) &= ia_2\Big((k+1)(F_{k+1} \pm F_{n-k-1}) \\
&\qquad + (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
(G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_3) &= a_3\Big((k+1)(F_{k+1} \mp F_{n-k-1}) \\
&\qquad - (n-k+1)(F_{k-1} \mp F_{n-k+1})\Big).
\end{aligned} \tag{5}$$

On the other hand, it is also a short calculation to show

$$\begin{aligned}
A_1 \cdot (F_k \pm F_{n-k}) &= (-1)^{k+1} i(F_k \mp F_{n-k}) \\
A_2 \cdot (F_k \pm F_{n-k}) &= i(G_k \pm G_{n-k}) \\
A_3 \cdot (F_k \pm F_{n-k}) &= (-1)^{k+1}(G_k \mp G_{n-k}) \\
A_1 \cdot (G_k \pm G_{n-k}) &= (-1)^k i(G_k \mp G_{n-k}) \\
A_2 \cdot (G_k \pm G_{n-k}) &= i(F_k \pm F_{n-k}) \\
A_3 \cdot (G_k \pm G_{n-k}) &= (-1)^k(F_k \mp F_{n-k}).
\end{aligned} \tag{6}$$

Bringing (5) and (6) together we deduce that

$$\begin{aligned}
& D_n'(F_k \pm F_{n-k}) = -\sum_{j=1}^{3} e_j \cdot (F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_j) \\
= \quad & -\sum_{j=1}^{3} A_j \cdot (F_k \pm F_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_j) \\
\overset{(5)}{=} \quad & ia_1(n-2k)A_1 \cdot (F_k \mp F_{n-k}) \\
& -ia_2A_2 \cdot \Big((k+1)(G_{k+1} \pm G_{n-k-1}) + (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
& -a_3A_3 \cdot \Big((k+1)(G_{k+1} \mp G_{n-k-1}) - (n-k+1)(G_{k-1} \mp G_{n-k+1})\Big) \\
\overset{(6)}{=} \quad & (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\
& +a_2\Big((k+1)(F_{k+1} \pm F_{n-k-1}) + (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
& +(-1)^k a_3\Big((k+1)(F_{k+1} \pm F_{n-k-1}) - (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
= \quad & (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\
& +(k+1)(a_2 + (-1)^k a_3)(F_{k+1} \pm F_{n-k-1}) \\
& +(n-k+1)(a_2 - (-1)^k a_3)(F_{k-1} \pm F_{n-k+1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& D_n'(G_k \pm G_{n-k}) = -\sum_{j=1}^{3} A_j \cdot (G_k \pm G_{n-k}) \circ T_{\mathrm{I}_2}\pi_n(X_j) \\
\overset{(5)}{=} \quad & ia_1(n-2k)A_1 \cdot (G_k \mp G_{n-k}) \\
& -ia_2A_2 \cdot \Big((k+1)(F_{k+1} \pm F_{n-k-1}) + (n-k+1)(F_{k-1} \pm F_{n-k+1})\Big) \\
& -a_3A_3 \cdot \Big((k+1)(F_{k+1} \mp F_{n-k-1}) - (n-k+1)(F_{k-1} \mp F_{n-k+1})\Big) \\
\overset{(6)}{=} \quad & -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\
& +a_2\Big((k+1)(G_{k+1} \pm G_{n-k-1}) + (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
& -(-1)^k a_3\Big((k+1)(G_{k+1} \pm G_{n-k-1}) - (n-k+1)(G_{k-1} \pm G_{n-k+1})\Big) \\
= \quad & -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\
& +(k+1)(a_2 - (-1)^k a_3)(G_{k+1} \pm G_{n-k-1}) \\
& +(n-k+1)(a_2 + (-1)^k a_3)(G_{k-1} \pm G_{n-k+1}).
\end{aligned}$$

Note that, for $k = \frac{n-1}{2}$, $F_{k+1} \pm F_{n-k-1} = \pm(F_k \pm F_{n-k})$ and the same holds for the G_k's, so that

$$\begin{aligned}
& D_n'(F_{\frac{n-1}{2}} \pm F_{\frac{n+1}{2}}) \\
= \quad & (-1)^{\frac{n-1}{2}} a_1(F_{\frac{n-1}{2}} \pm F_{\frac{n+1}{2}}) \\
& +\frac{n+1}{2}(a_2 + (-1)^{\frac{n-1}{2}} a_3)(F_{\frac{n+1}{2}} \pm F_{\frac{n-1}{2}})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{n+3}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n-3}{2}}\pm F_{\frac{n+3}{2}})\\
= \quad & \Big((-1)^{\frac{n-1}{2}}a_1\pm\frac{n+1}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)\Big)(F_{\frac{n-1}{2}}\pm F_{\frac{n+1}{2}})\\
&+\frac{n+3}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n-3}{2}}\pm F_{\frac{n+3}{2}})
\end{aligned}
$$

and in the same way

$$
\begin{aligned}
& D_n'(G_{\frac{n-1}{2}}\pm G_{\frac{n+1}{2}})\\
= \quad & -(-1)^{\frac{n-1}{2}}a_1(G_{\frac{n-1}{2}}\pm G_{\frac{n+1}{2}})\\
&+\frac{n+1}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n+1}{2}}\pm G_{\frac{n-1}{2}})\\
&+\frac{n+3}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n-3}{2}}\pm G_{\frac{n+3}{2}})\\
= \quad & \Big(-(-1)^{\frac{n-1}{2}}a_1\pm\frac{n+1}{2}(a_2-(-1)^{\frac{n-1}{2}}a_3)\Big)(G_{\frac{n-1}{2}}\pm G_{\frac{n+1}{2}})\\
&+\frac{n+3}{2}(a_2+(-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n-3}{2}}\pm G_{\frac{n+3}{2}}).
\end{aligned}
$$

Denoting by $(v_0,\ldots,v_{\frac{n-1}{2}})$ the basis of $\mathrm{Hom}_{\mathrm{Q}_8,\varepsilon_j}(V_n,\mathbb{C}^2)$ computed in Lemma 2.6 we conclude the proof of Theorem 0.1 *iv)*.

Note 2.8 From Theorem 0.1 *iv)* the matrix representing the operator D_n in the basis $(v_0,\ldots,v_{\frac{n-1}{2}})$ is not symmetric. Beware however that this basis does not take A_1,A_2,A_3 into account the same way and turns out not to be orthonormal.

We now make the eigenvalue of D_1 explicit:

Corollary 2.9 *Fix $j\in\{0,1,2,3\}$ and let $\epsilon_1,\epsilon_2,\epsilon_3\in\{-1,1\}$ be defined by $\epsilon_l:=-(-1)^{\delta_{j0}+\delta_{jl}}$ for $l\in\{1,2,3\}$. Then under the assumptions of* Theorem 0.1 *the following number is an eigenvalue of the Dirac operator of M for the spin structure given by ε_j and the metric induced by a_1,a_2,a_3:*

$$
\frac{-(\epsilon_2a_2-\epsilon_3a_3)^2a_1^2+2a_2a_3(\epsilon_2a_2+\epsilon_3a_3)a_1-a_2^2a_3^2}{2a_1a_2a_3}.
$$

If in particular $\epsilon_2\epsilon_3a_2a_3>0$ then there exists $a_1\in\mathbb{R}^*$ such that for the corresponding metric the Dirac operator of M has a non-zero kernel.

Proof: For $n=1$ the operator D_n' can be expressed from Theorem 0.1 as

$$
D_1'=(\epsilon_1a_1+\epsilon_2a_2+\epsilon_3a_3)\mathrm{Id}
$$

for the ϵ_l's defined above (beware that they depend on j). Therefore the corresponding Dirac operator D_n is given by

$$
\begin{aligned}
D_1 \quad &= \quad \Big(\epsilon_1a_1+\epsilon_2a_2+\epsilon_3a_3-\frac{a_1^2a_2^2+a_2^2a_3^2+a_1^2a_3^2}{2a_1a_2a_3}\Big)\mathrm{Id}\\
&= \quad \frac{-(\epsilon_2a_2-\epsilon_3a_3)^2a_1^2+2a_2a_3(\epsilon_2a_2+\epsilon_3a_3)a_1-a_2^2a_3^2}{2a_1a_2a_3}\mathrm{Id},
\end{aligned}
$$

from which the first statement follows.
An elementary computation shows that, if $\epsilon_2\epsilon_3 a_2 a_3 > 0$, then the numerator of the eigenvalue vanishes for

$$a_1 = \frac{a_2 a_3(\epsilon_2 a_2 + \epsilon_3 a_3) \pm 2(\epsilon_2\epsilon_3 a_2 a_3)^{\frac{3}{2}}}{(\epsilon_2 a_2 - \epsilon_3 a_3)^2}$$

in the case $\epsilon_2 a_2 \neq \epsilon_3 a_3$ and

$$a_1 = \frac{\epsilon_2 a_3}{4}$$

if $\epsilon_2 a_2 = \epsilon_3 a_3$. Note that none of those numbers can vanish because of $a_2 a_3 \neq 0$. This concludes the proof.

$\square$

Notes 2.10

1. It follows from Corollary 2.9 that, for any given spin structure on M, there exists a 2-parameter-family of Riemannian metrics for which M admits non-zero harmonic spinors. This is not a surprise since the existence of such metrics already follows from a purely theoretical result by Christian Bär [4]. However we can make some of those metrics explicit here.

2. There may exist non-zero harmonic spinors for other metrics on M and possibly without needing the condition $\epsilon_2\epsilon_3 a_2 a_3 > 0$ from Corollary 2.9, since we have up to now only studied the eigenvalue corresponding to one particular representation.

3. In the same way the eigenvalue computed in Corollary 2.9 is not necessarily the smallest one in absolute value. Choose for example the ε_0-spin structure, $a_2 = a_3 < 0$ and $a_1 \in]-\frac{a_2}{8}, -\frac{a_2}{2}[$. Then $\frac{4a_1 a_2 - a_2^2}{2a_1}$ and $-\frac{8a_1 a_2 + a_2^2}{2a_1}$ are eigenvalues of the Dirac operator of M, the first one corresponding to $n = 1$ (i.e., to the one computed in Corollary 2.9) and the second one to $n = 3$, see Corollary 0.2. However one has from the assumptions on a_1, a_2, a_3 that $|-\frac{8a_1 a_2 + a_2^2}{2a_1}| < |\frac{4a_1 a_2 - a_2^2}{2a_1}|$.

We end this section with an important remark which actually constitutes the main motivation for this work. The manifold M can be seen as hypersurface of the 4-dimensional round sphere S^4 (with sectional curvature 1): consider the manifold $\{A \in \mathrm{M}_{3\times 3}(\mathbb{R}),\ {}^tA = A,\ \mathrm{tr}(A) = 0 \text{ and } \mathrm{tr}(A^2) = 2\} \cong S^4$ with metric $(A, B) \longmapsto \langle A, B\rangle := \frac{1}{2}\mathrm{tr}(AB)$. Let $B := \mathrm{diag}(\lambda, -\lambda-\mu, \mu) \in S^4$ where $\lambda, \mu \in \mathbb{R}$ satisfy $\lambda + 2\mu \neq 0$, $\lambda \neq \mu$, $\mu + 2\lambda \neq 0$ and $\lambda^2 + (\lambda + \mu)^2 + \mu^2 = 2$. Set

$$N := \{\pi(P) \cdot B \cdot \pi(P)^{-1},\ P \in \mathrm{SU}_2\} \subset S^4,$$

where $\mathrm{SU}_2 \xrightarrow{\pi} \mathrm{SO}_3$ is the universal 2-fold covering map. Then it is an elementary exercise to show that N is a hypersurface of S^4 which is diffeomorphic to $\mathrm{SU}_2/\mathrm{Q}_8$, that the homogeneous metric induced by the inclusion map $N \subset S^4$

is given by $a_1 := -\frac{1}{2(\lambda+2\mu)}, a_2 := \frac{1}{2(\mu-\lambda)}, a_3 := \frac{1}{2(\mu+2\lambda)}$ and that choosing $\nu_B := \frac{1}{\sqrt{3}}\mathrm{diag}(2\mu+\lambda, \lambda-\mu, -2\lambda-\mu) \in T_B S^4$ as unit normal vector field the induced spin structure on N is the ε_0-one. Here beware that the metrics obtained form a one-parameter strict subfamily of that of all homogeneous metrics on M. Furthermore, the Weingarten endomorphism-field of N w.r.t. ν_B - seen as endomorphism of $\mathfrak{su}(2)$ - is given in the basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ by

$$\mathrm{Mat}(\mathcal{A}) = \sqrt{3}\cdot\mathrm{diag}(\frac{\lambda}{2\mu+\lambda}, \frac{\mu+\lambda}{\mu-\lambda}, -\frac{\mu}{2\lambda+\mu}).$$

In particular, the mean curvature $\mathcal{H} := \frac{1}{3}\mathrm{tr}(\mathcal{A})$ of N in S^4 w.r.t. ν_B is

$$\mathcal{H} = \frac{3\sqrt{3}\cdot\lambda\mu(\lambda+\mu)}{(2\mu+\lambda)(\mu-\lambda)(2\lambda+\mu)}.$$

Corollary 2.11 *Under the hypotheses of* Theorem 0.1 *assume furthermore that M sits in S^4, i.e., that $a_1 = -\frac{1}{2(\lambda+2\mu)}, a_2 = \frac{1}{2(\mu-\lambda)}, a_3 = \frac{1}{2(\mu+2\lambda)}$ for some $\lambda, \mu \in \mathbb{R}$ satisfying $\lambda + 2\mu \neq 0$, $\lambda \neq \mu$, $\mu + 2\lambda \neq 0$ and $\lambda^2 + (\lambda+\mu)^2 + \mu^2 = 2$. Then $\frac{9}{4}(\mathcal{H}^2+1)$ is an eigenvalue of the Dirac Laplacian of M for the induced (ε_0-)spin structure.*

Proof: The result follows straightforward from Corollary 2.9 in the case $j = 0$ and from an elementary computation giving

$$\begin{aligned}\frac{9}{4}(\mathcal{H}^2+1) &= \frac{9}{(\lambda+2\mu)^2(\mu-\lambda)^2(\mu+2\lambda)^2} \\ &= \left(\frac{-(a_2-a_3)^2a_1^2 + 2a_2a_3(a_2+a_3)a_1 - a_2^2a_3^2}{2a_1a_2a_3}\right)^2.\end{aligned}$$

□

Corollary 2.11 confirms what had been already noticed since Christian Bär's work [5] on extrinsic upper eigenvalue bounds for the lower part of the Dirac spectrum: for any compact orientable hypersurface $\overline{M}^m$ with constant mean curvature $\mathcal{H}$ (and carrying the induced metric and spin structure) in the $(m+1)$-dimensional round sphere the number $\frac{m^2}{4}(\mathcal{H}^2+1)$ is an eigenvalue of its Dirac Laplacian. However the question still remains open whether this eigenvalue should be the smallest one or not.

3 Computation of the spectrum of the Dirac operator on M for particular metrics

Although the matrices representing the Dirac operator D of M have a "simple" shape (they are tridiagonal, see Theorem 0.1), their spectrum is still hard to compute explicitly since there does not exist any general formula giving the eigenvalues of such matrices. It is however possible to compute them for particular values of the parameters $a_1, a_2, a_3 \in \mathbb{R}^*$, i.e., for particular metrics on M. In Corollary 0.2 we do it for the so-called Berger metrics on M (compare with [2, p.71] where the author chooses $a_2 = 1 = -a_3$ and $a_1 = -\frac{1}{T}$ with $T > 0$). Namely, if we assume that $a_2 = a_3$ then the identities for $D'_n(F_k \pm F_{n-k})$ and $D'_n(G_k \pm G_{n-k})$ become

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\ &\quad +(k+1)(1+(-1)^k)a_2(F_{k+1} \pm F_{n-k-1}) \\ &\quad +(n-k+1)(1-(-1)^k)a_2(F_{k-1} \pm F_{n-k+1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\ &\quad +(k+1)(1-(-1)^k)a_2(G_{k+1} \pm G_{n-k-1}) \\ &\quad +(n-k+1)(1+(-1)^k)a_2(G_{k-1} \pm G_{n-k+1}) \end{aligned}$$

for every $k \in \{0, \ldots, \frac{n-1}{2}\}$. In particular, if k is even, then

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= a_1(n-2k)(F_k \pm F_{n-k}) \\ &\quad +2(k+1)a_2(F_{k+1} \pm F_{n-k-1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= -a_1(n-2k)(G_k \pm G_{n-k}) \\ &\quad +2(n-k+1)a_2(G_{k-1} \pm G_{n-k+1}). \end{aligned}$$

If k is odd then

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= -a_1(n-2k)(F_k \pm F_{n-k}) \\ &\quad +2(n-k+1)a_2(F_{k-1} \pm F_{n-k+1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= a_1(n-2k)(G_k \pm G_{n-k}) \\ &\quad +2(k+1)a_2(G_{k+1} \pm G_{n-k-1}). \end{aligned}$$

We now consider each case separately. Remember that from Theorem 2.1 the Dirac operator D restricted to $V_n \otimes \mathrm{Hom}_{Q_8,\varepsilon_j}(V_n, \mathbb{C}^2)$ is given by $\mathrm{Id} \otimes D_n$ where $D_n = D'_n - (\frac{a_1^2a_2^2+a_1^2a_3^2+a_2^2a_3^2}{2a_1a_2a_3})\mathrm{Id}$. In particular the multiplicity of each eigenvalue of D_n should be counted $n+1$ times for the spectrum of D.

- *Case* $j = 0$:
  - * If $n \equiv 1$ (4): It follows from the identities just above and from Lemma 2.6 that the matrix of D'_n consists of $\frac{n-1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{0, \ldots, \frac{n-5}{2}\}$ is even and of the isolated eigenvalue $a_1 + (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$). The eigenvalues of each such 2×2-matrix are simple and given by

$$a_1 \pm \sqrt{((n-2k)(n-2(k+1))+1)a_1^2 + 4(n-k)(k+1)a_2^2}$$

with $((n-2k)(n-2(k+1))+1) = (n-2k-1)^2$.

* If $n \equiv 3$ (4): It follows from the identities just above and from Lemma 2.6 that the matrix of D'_n consists of $\frac{n-3}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{1, \dots, \frac{n-5}{2}\}$ is odd and of the isolated eigenvalues $-na_1$ (corresponding to $k = 0$) and $a_1 - (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$).

This shows 0.

• *Case* $j = 1$:

* If $n \equiv 1$ (4): It follows from the identities just above and from Lemma 2.6 that the matrix of D'_n consists of $\frac{n-1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{0, \dots, \frac{n-5}{2}\}$ is even and of the isolated eigenvalue $a_1-(n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$). The eigenvalues of each such 2×2-matrix have already been computed in the case $j = 0$ above.

* If $n \equiv 3$ (4): It follows from the identities just above and from Lemma 2.6 that the matrix of D'_n consists of $\frac{n-3}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{1, \dots, \frac{n-5}{2}\}$ is odd and of the isolated eigenvalues $-na_1$ (corresponding to $k = 0$) and $a_1 + (n+1)a_2$ (corresponding to $k = \frac{n-1}{2}$).

This shows 1.

• *Case* $j = 2$ *or* $j = 3$: Since $a_2 = a_3$ the Dirac spectra for the ε_2- and ε_3- spin structures coincide, see Examples 2.4.2 with $\sigma = (2\ 3)$.

* If $n \equiv 1$ (4): It follows from the identities just above and from Lemma 2.6 that the matrix of D'_n consists of $\frac{n-1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{1, \dots, \frac{n-3}{2}\}$ is odd and of the isolated eigenvalue $-na_1$ (corresponding to $k = 0$).

* If $n \equiv 3$ (4): It follows from the identities just above and from Lemma 2.6 that the matrix of D'_n consists of $\frac{n+1}{4}$ blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where $k \in \{0, \dots, \frac{n-3}{2}\}$ is even.

This shows 2. and concludes the proof of Corollary 0.2.

Note 3.1 Of course one should understand each upper bound (e.g. $\frac{n-5}{2}$) for the possible values of k in Corollary 0.2 as follows: if for a given n it is negative then the corresponding eigenvalues do not appear. For example if M carries the ε_0-spin structure and $n = 1$ then $D_n + \frac{2a_1^2+a_2^2}{2a_1}\mathrm{Id}$ has only one eigenvalue, namely $a_1 + 2a_2$ (with multiplicity 2). Similarly, if $j = 2, 3$ and $n = 1$, then only $-a_1$ appears with multiplicity 2.

One could in a similar way compute the spectrum of the Dirac operator for $a_2 = -a_3$, in which case the spectra would coincide for the ε_0- and the ε_1-spin structure on M (use Examples 2.4).

We end this section with deriving from Corollary 0.2 the spectrum of the Dirac operator on M for any of the 4 spin structures and the following metrics: for one of the metrics with constant sectional curvature and for one of the 6 metrics induced by minimal isometric embeddings into S^4 (i.e., for $(\lambda = 0, \mu = \pm 1)$, $(\lambda = \pm 1, \mu = 0)$ or $(\lambda, \mu) = \pm(1, -1)$, see end of Section 2). In the first case the spectrum has already been computed by Christian Bär in [3, Thm. 2] and it can be easily checked that his results coincide with ours.

Corollary 3.2 *Under the hypotheses of* Theorem 0.1*, assume furthermore that*

i) $a_1 = a_2 = a_3 = 1$. *Then the spectrum of the Dirac operator of M w.r.t. the ε_0-spin structure consists of the family*

$$\left|\begin{array}{ll} \frac{3}{2} + 4k & \textit{with multiplicity } 2(k+1)(2k+1) \\ \frac{3}{2} + 4k + 2 & \textit{with multiplicity } 4k(k+1) \\ -\frac{3}{2} - 4k - 1 & \textit{with multiplicity } 2k(2k+1) \\ -\frac{3}{2} - 4k - 3 & \textit{with multiplicity } 4(k+1)(k+2) \end{array}\right.$$

where k runs over $\mathbb{N}$ and w.r.t. any of the other spin structures ε_j of the family

$$\left|\begin{array}{ll} \frac{3}{2} + 4k & \textit{with multiplicity } 2k(2k+1) \\ \frac{3}{2} + 4k + 2 & \textit{with multiplicity } 4(k+1)^2 \\ -\frac{3}{2} - 4k - 1 & \textit{with multiplicity } 2(k+1)(2k+1) \\ -\frac{3}{2} - 4k - 3 & \textit{with multiplicity } 4(k+1)^2 \end{array}\right.$$

where k runs over $\mathbb{N}$.

ii) $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$. *Then the spectrum of the Dirac operator of M*

* *w.r.t. the ε_0-spin structure is given by*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$\,|\, k\in\{0,\ldots,\frac{n-5}{2}\}\text{ even}, \frac{n}{2}+1\Big\}$$
$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$\,|\, k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, -\frac{n}{2}, \frac{n+3}{4}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

* *w.r.t. the ε_1-spin structure is given by*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$\,|\, k\in\{0,\ldots,\frac{n-5}{2}\}\text{ even}, -\frac{n}{2}\Big\}$$
$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$\,|\, k\in\{1,\ldots,\frac{n-5}{2}\}\text{ odd}, \frac{n}{2}+1, \frac{n+3}{4}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

* *w.r.t. the ε_2- or ε_3-spin structure is given by*

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 1\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$\,|\, k\in\{1,\ldots,\frac{n-3}{2}\}\text{ odd}, \frac{n+3}{4}\Big\}$$
$$\bigcup \quad \bigcup_{\substack{n\in\mathbb{N}\\ n\equiv 3\,(4)}} \Big\{\frac{1}{2}\pm\frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$$
$$\,|\, k\in\{0,\ldots,\frac{n-3}{2}\}\text{ even}\Big\},$$

each eigenvalue having multiplicity $n+1$ for the corresponding n.

Proof: In case $a_1=a_2=a_3=1$ one has on the one hand

$$(n-2k-1)^2a_1^2+4(n-k)(k+1)a_2^2=(n+1)^2$$

for every possible k and on the other hand $\frac{2a_1^2+a_2^2}{2a_1} = \frac{3}{2}$. The result in i) straightforward follows using Corollary 0.2 and Examples 2.4.
Assuming now $a_1 = -\frac{1}{4}$ and $a_2 = a_3 = \frac{1}{2}$, one has

$$\begin{aligned} & a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \\ & = -\frac{1}{4} \pm \frac{\sqrt{(n-2k-1)^2 + 16(n-k)(k+1)}}{4} \end{aligned}$$

and $\frac{2a_1^2+a_2^2}{2a_1} = -\frac{3}{4}$. This concludes the proof.

□

One can deduce from Corollary 3.2 and Examples 2.4 the spectrum of the Dirac operator of M for any spin structure and any metric induced by (a_1, a_1, a_1) with $a_1 \in \mathbb{R}^*$ or any metric induced by a minimal embedding into S^4: in the first case rescale by a_1, in the second one exchange the roles of a_1, a_2, a_3 and possibly multiply all of them by -1.
For the next corollary recall that, for a given $\beta \in \mathbb{C}$, a β-Killing spinor on a spin manifold N is a smooth section ψ of the spinor bundle of N such that $\nabla_X \psi = \beta X \cdot \psi$ for every $X \in TN$.

Corollary 3.3 *Under the hypotheses of* Theorem 0.1 *the following holds:*

i) If $a_1 = a_2 = a_3 = 1$ then the ε_0-spin structure is the only one for which M admits a non-zero space of Killing spinors, which is then 2*-dimensional and associated to the constant $\beta = -\frac{1}{2}$. In particular $\frac{3}{2}$ is in absolute value the smallest eigenvalue of the Dirac operator of M for the ε_0-spin structure.*

ii) If $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$ and M carries the ε_0-spin structure then $\frac{3}{2}$ is in absolute value the smallest eigenvalue of the Dirac operator of M. In particular inequality (1) *is an equality on M for the induced metric and spin structure.*

Proof: If $a_1 = a_2 = a_3 = 1$ then on the one hand the metric induced on M has constant sectional curvature 1; on the other hand Corollary 3.2 i) implies that the smallest eigenvalue in absolute value of the Dirac operator of M is $\frac{3}{2}$ with multiplicity 2 w.r.t. the ε_0-spin structure and $-\frac{5}{2}$ with multiplicity 2 w.r.t. any of the other spin structures (both obtained for $n = 1$, i.e., they are the eigenvalues computed in Corollary 2.9). Now M carries a non-trivial Killing spinor if and only if the smallest eigenvalue of its Dirac Laplacian coincides with T. Friedrich's lower bound $\frac{3}{4(3-1)} \inf_M(\mathrm{Scal}_M)$ in terms of the scalar curvature of M, see [7]. Here $\frac{3}{4(3-1)}\mathrm{Scal}_M = \frac{9}{4}$ so that M carries a 2-dimensional space of non-zero Killing spinors only for the ε_0-spin structure; in that case the corresponding constant β should obviously be $-\frac{1}{2}$. This shows i)
If $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$ and M carries the ε_0-spin structure then from Corollary 3.2 ii) the eigenvalues corresponding to $n = 1$ and $n = 3$ are $\frac{3}{2}$ and $-\frac{3}{2}, \frac{3}{2}$ with

multiplicities 2, 4 and 4 respectively. Next we show that all eigenvalues corresponding to $n \geq 5$ are greater than $\frac{3}{2}$ in absolute value. Since this is obviously the case for $\frac{n}{2}+1$, $-\frac{n}{2}$ and $\frac{n+3}{4}$ we just have to deal with the eigenvalues $\frac{1}{2} \pm \frac{1}{4}\sqrt{(n-2k-1)^2+16(n-k)(k+1)}$, of which absolute value is greater than $\frac{3}{2}$ if and only if

$$(n-2k-1)^2 + 16(n-k)(k+1) - 64 > 0 \tag{7}$$

for every $k \in \{0, \ldots, \frac{n-5}{2}\}$. The l.h.s. of (7) is a trinomial in k with negative dominant coefficient and of which roots are given by $\frac{n-1}{2} \pm \sqrt{\frac{(n-3)(n+5)}{3}}$. If $n \geq 5$ then $\frac{n-1}{2} - \sqrt{\frac{(n-3)(n+5)}{3}} < 0 < \frac{n-1}{2} < \frac{n-1}{2} + \sqrt{\frac{(n-3)(n+5)}{3}}$, which shows that (7) is satisfied. Hence $\frac{3}{2}$ is in absolute value the smallest eigenvalue of the Dirac operator. Apply Corollary 2.11 to the case $\lambda = 0$ and $\mu = 1$ to conclude.

□

That M admits a 2-dimensional space of Killing spinors w.r.t. its ε_0-spin structure and any normal metric is also not a surprise, see [1, Cor. 5.2.5 (1b)]. Moreover, following the symmetry arguments already used above (see Examples 2.4) Corollary 3.3 *ii*) actually holds for any of the metrics induced by a minimal embedding into S^4. This proves Corollary 0.3.

Corollary 0.3 provides a further example (after geodesic spheres [5] and generalized Clifford tori [8]) of homogeneous hypersurface of the round sphere for which Christian Bär's inequality [5, Cor. 4.3] is an equality for the smallest Dirac eigenvalue. Here it should furthermore be noticed that, still under the assumptions of Corollary 0.3, the multiplicity of the smallest eigenvalue of the Dirac Laplacian on M is greater than the corresponding one on the 3-dimensional round sphere. This shows an analogy with the generalized Clifford tori tested in [8], on which the multiplicity of the smallest eigenvalue of the Dirac Laplacian is also greater than or equal to the corresponding one on the round sphere of same dimension.

We conjecture that the inequality in [5, Cor. 4.3] for the smallest Dirac eigenvalue is an equality for every homogeneous hypersurface in the round sphere. We refer to [9] for further work in this direction.

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