ON THE CAUCHY PROBLEM FOR THE FARADAY TENSOR ON GLOBALLY HYPERBOLIC MANIFOLDS WITH TIMELIKE BOUNDARY

by

Nicoló Drago[1], **Nicolas Ginoux** [2] **and Simone Murro** [3]

[1] *Dipartimento di Matematica, Università di Trento & INFN-TIFPA, Italy*

[2] *Université de Lorraine & CNRS & IECL, F-57000 Metz, France*

[3] *Dipartimento di Matematica, Università di Genova & INdAM & INFN, Italy*

email: `nicolo.drago@unitn.it; nicolas.ginoux@univ-lorraine.fr; murro@dima.unige.it`

Abstract

We study the well-posedness of the Cauchy problem for the Faraday tensor on globally hyperbolic manifolds with timelike boundary. The existence of Green operators for the operator $\mathrm{d}+\delta$ and a suitable pre-symplectic structure on the space of solutions are discussed.

Keywords: Overdetermined initial-boundary value problem, Maxwell's equations, Faraday tensor, Cauchy problem, globally hyperbolic manifolds with timelike boundary.

MSC 2020: Primary 35Q61, 35N30; Secondary 58J45, 53C50.

1 Introduction

Electromagnetic interactions play a key role in the history of physics since they are related to the first successful example of unification of two apparently different fields, the electric and the magnetic one, into a single body, the Faraday tensor. The latter tensor contains all the physical information both at a classical and at a quantum level. Indeed, as noted for example in [18], in all idealized and real experiments of the Aharonov-Bohm kind, the true observable is actually the flux of the magnetic component of the Faraday tensor which is present inside an impenetrable region, typically a cylinder. It is far from the scope of this paper to discuss the details of this procedure, but it is sufficient to say that, on Minkowski background and in absence of sources, the result is pretty much satisfactory. Yet the situation starts to complicate itself as soon as it is assumed that a spacetime M has a non-trivial geometry.

In this paper we will be interested in the Cauchy problem for Maxwell's equations (for k-forms) $\delta F = j$ and $\mathrm{d}F = 0$ on a globally hyperbolic manifold M with timelike boundary [1]. Within this setting, boundary conditions have to be imposed to ensure the well-posedness of the resulting Cauchy problem: For the case at end we will impose the vanishing of the normal component of the Faraday tensor F at the boundary.

The well-posedness of the Cauchy problem allows to introduce advanced/retarded propagators for the operator $D = \mathrm{d}+\delta$. This opens to the possibility of applying a standard quantization scheme [10, Chap. 3] which is well-established for the case of the Faraday tensor on globally hyperbolic manifolds without boundaries [12, 13], for $U(1)$-gauge theories [5–7] and for gauge theories on globally hyperbolic manifolds with timelike boundary [8, 11].

Statement of the problem and main results. Through this paper, (M, g) denotes a globally hyperbolic manifold with timelike boundary $\partial\mathsf{M}$ as defined in [1, Definition 2.14], see also e.g. [17, Definition 2.1]. In more details, (M, g) is a connected, oriented smooth Lorentzian n-dimensional manifold M with boundary $\partial\mathsf{M}$ such that $(\partial\mathsf{M}, g|_{\partial\mathsf{M}})$ is a Lorentzian manifold and there exists a smooth Cauchy temporal function $t : \mathsf{M} \to \mathbb{R}$ such that

$$\mathsf{M} = \mathbb{R} \times \Sigma \qquad g = -\beta^2 dt^2 + h_t \,,$$

where $\beta : \mathbb{R} \times \Sigma \to \mathbb{R}$ is a smooth positive function, h_t is a Riemannian metric on each slice $\Sigma_t := \{t\} \times \Sigma$ varying smoothly with t, and these slices are spacelike Cauchy hypersurfaces with boundary $\partial\Sigma_t := \{t\} \times \partial\Sigma$, namely achronal sets intersected exactly once by every inextensible timelike curve.

The (sourceless) Maxwell's equations for the Faraday tensor $F \in \Omega^k(\mathsf{M})$ are given by satisfying

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \mathrm{d} *_g F = 0 \,.$$

Clearly, if the boundary of $\partial\mathsf{M}$ is not empty, then the uniqueness of a solution to the Cauchy problem for F can be expected only if a boundary condition is imposed. To this end, we shall consider the boundary condition

$$\mathtt{n} \lrcorner F = 0$$

where the vector field $\mathtt{n}$ is the outward-pointing unit normal vector field along $\partial\mathsf{M}$. If we fix $\nu \in \Gamma(T^*\mathsf{M}_{|\partial\mathsf{M}})$ to be a 1-form such that

$$\ker \nu_p = T_p\partial\mathsf{M} \,, \qquad \nu_p(\mathtt{n}_p) > 0 \,, \qquad \text{and} \qquad \mathcal{L}_{\partial_t}\nu = 0 \,,$$

for all $p \in \partial\mathsf{M}$, then there exists a positive smooth function c_t on $\partial\mathsf{M}$ such that

$$\mathtt{n}_p = c_t \nu^{\sharp_t}$$

for all $p \in \partial\mathsf{M}$, where $\sharp_t \colon T^*\Sigma \to T\Sigma$ denotes the musical isomorphism associated with h_t. For later convenience we set

$$\begin{aligned}\Omega^k_{c,\mathtt{n}}(\mathsf{M}) &:= \{F \in \Omega^k_c(\mathsf{M}) \,|\, \mathtt{n} \lrcorner F = 0\}\,,\\ \Omega^\bullet_{c,\mathtt{n},\delta}(\mathsf{M}) &:= \{\alpha \in \Omega^\bullet_{c,\mathtt{n}}(\mathsf{M}) \,|\, \delta\alpha = 0\}\,,\\ \Omega^\bullet_{c,\mathtt{n},\mathrm{d}}(\mathsf{M}) &:= \{\alpha \in \Omega^\bullet_{c,\mathtt{n}}(\mathsf{M}) \,|\, \mathrm{d}\alpha = 0\}\,.\end{aligned}$$

Within this setting, the main result of the paper is the following:

Theorem 1.1. *Let (M, g) be a globally hyperbolic manifold with timelike boundary and let be $j \in \Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})$, $\zeta \in \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})$ and $F_0 \in \Omega^k_c(\mathsf{M})$ such that*

$$(\mathrm{supp}(\zeta) \cup \mathrm{supp}(j)) \cap \Sigma_0 = \varnothing\,, \quad \mathrm{supp}(F_0) \cap \partial\mathsf{M} = \varnothing\,, \quad \mathrm{d}_{\Sigma_0} \iota^*_{\Sigma_0} F_0 = 0\,, \quad \mathrm{d}_{\Sigma_0} \iota^*_{\Sigma_0} *_g F_0 = 0\,.$$

Then the Cauchy problem for the Faraday tensor

$$\mathrm{d}F = \zeta \tag{1.1a}$$

$$\delta F = j \tag{1.1b}$$

$$\mathtt{n} \lrcorner F = 0 \tag{1.1c}$$

$$F|_{\Sigma_0} = F_0 \tag{1.1d}$$

has a unique solution $F \in \Omega^k_{sc,\mathtt{n}}(\mathsf{M})$. Moreover,

$$\mathrm{supp}(F) \subseteq J\left[\mathrm{supp}(F_0) \cup \mathrm{supp}(j) \cup \mathrm{supp}(\zeta)\right]\,, \tag{1.2}$$

where $J(A)$ denotes the causal development of A.

Remarks 1.2.

1. It is worth pointing out that Theorem 1.1 proves that any closed compactly supported form $\zeta \in \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})$ is necessarily exact, $\zeta = \mathrm{d}F$, for a spacelike form $F \in \Omega^k_{sc}(\mathsf{M})$. (A similar argument applies for the coexactness of j in Equation (1.1b).) Actually, the inclusion $\Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M}) \subset \mathrm{d}\Omega^k(\mathsf{M})$ can be proved by cohomological arguments[1] and is based on the fact that M is homeomorphic to $\mathbb{R} \times \Sigma$. Indeed, let $f \in C^\infty_c(\mathbb{R})$ be such that $\int_{\mathbb{R}} f(t)\mathrm{d}t = 1$ and consider the following maps between chain complexes

$$\Omega^{\bullet-1}_c(\Sigma) \xrightarrow{f\mathrm{d}t\wedge\cdot} \Omega^\bullet_c(\mathsf{M}) \xrightarrow{\mathrm{Id}} \Omega^\bullet(\mathsf{M}) \xrightarrow{\iota^*_\Sigma} \Omega^\bullet(\Sigma)\,,$$

where the Id is the identity map while ι^*_Σ is the pull-back to Σ. All these maps induces (de Rham) cohomology maps —denoted by $[f\mathrm{d}t \wedge \cdot]$, [Id], $[\iota^*_\Sigma]$— and by [9, Prop. 4.7] we have $H^{\bullet-1}_c(\Sigma) \simeq H^\bullet_c(\mathsf{M})$ while [9, Prop. 4.1] proves that $H^\bullet(\mathsf{M}) \simeq H^\bullet(\Sigma)$. Let now $[\omega]_c \in H^\bullet_c(\mathsf{M})$. Since $H^\bullet_c(\mathsf{M}) \simeq H^{\bullet-1}_c(\mathsf{M})$ there exists $[\alpha]_c \in H^{\bullet-1}_c(\Sigma)$ such that

[1] We are grateful to M. Benini for this observation.

$[\omega]_c = [f\mathrm{d}t \wedge \alpha]_c$. Considering the equivalence class $[\mathrm{Id}][f\mathrm{d}t \wedge \alpha]_c = [f\mathrm{d}t \wedge \alpha] \in H^\bullet(\mathsf{M})$ and the isomorphism $H^\bullet(\mathsf{M}) \simeq H^\bullet(\Sigma)$ we then find

$$[f\mathrm{d}t \wedge \alpha] = [\iota_\Sigma^*]^{-1}[\iota_\Sigma^*][f\mathrm{d}t \wedge \alpha] = [\iota_\Sigma^*]^{-1}[\iota_\Sigma^*(f\mathrm{d}t \wedge \alpha)] = [0]\,,$$

where in the last line we used $\pi_\Sigma \circ f\mathrm{d}t \wedge \cdot = 0$. This proves that $[\mathrm{Id}]$ is the zero map, hence the claim.

2. Our analysis extends straightforwardly to the Cauchy problem for a Faraday tensor coupled with the boundary condition $\mathtt{n}\lrcorner *_g F_0 = 0$.

3. Theorem 1.1 can be generalized by dropping the assumption $\mathrm{supp}(F_0) \cap \partial\mathsf{M} = \varnothing$ and $(\mathrm{supp}(\zeta) \cup \mathrm{supp}(j)) \cap \Sigma_0 = \varnothing$. This requires introducing suitable "compatibility conditions" between F_0 and j as described in [17]. We will refrain from discussing this case as the hypotheses of Theorem 1.1 are sufficient for the application we have in mind, *cf.* Proposition 5.1.

4. The boundary condition (1.1c) can be derived with the following variational argument —*cf.* [11, Rmk. 27]. In this setting one introduces the formal action

$$I(A) = \frac{1}{2}(\mathrm{d}A, \mathrm{d}A)_\mathsf{M} = \frac{1}{2}\int_\mathsf{M} \mathrm{d}A \wedge *_g \mathrm{d}A\,,$$

where the convergence of the integral is not discussed. The homogeneous Maxwell's equations $\delta\mathrm{d}A = 0$ are recovered by requiring A to be a critical point of the formal action I, namely

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(A + \varepsilon\alpha)\Big|_{\varepsilon=0} = 0 \qquad \forall \alpha \in \Omega_c^{k-1}(\mathsf{M})\,,$$

where $\alpha \in \Omega_c^{k-1}(\mathsf{M})$ is an arbitrarily chosen compactly supported smooth $(k-1)$-form. Notably, although $I(A)$ may be ill-defined, the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(A + \varepsilon\alpha)\Big|_{\varepsilon=0}$ is always well-defined and it can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(A + \varepsilon\alpha)\Big|_{\varepsilon=0} = (\mathrm{d}A, \mathrm{d}\alpha)_\mathsf{M} = (\delta\mathrm{d}A, \alpha)_\mathsf{M} + (\mathtt{n}\lrcorner\mathrm{d}A, \iota_{\partial\mathsf{M}}^*\alpha)_{\partial\mathsf{M}}\,,$$

where $(\cdot\,,\cdot)_{\partial\mathsf{M}}$ is the canonical pairing between forms on $\partial\mathsf{M}$. Because α can be chosen arbitrarily, this leads to $\delta\mathrm{d}A = 0$ and $\mathtt{n}\lrcorner\mathrm{d}A = 0$.

5. The well-posedness of the Cauchy problem will guarantee the existence of Green operators (*cf.* Proposition 5.1) which play a pivotal role in the algebraic approach to linear quantum field theory, see e.g. [4, 10, 16] for textbooks and [3, 10, 15] for recent reviews.

Plan of the proof. As a preliminary, in Section 2 we will decompose the Faraday tensor $F \in \Omega^k(\mathsf{M})$ into its electric and magnetic components F_E, F_B, *cf.* Equation (2.1). The equations of motion (1.1a)-(1.1b) are then written in terms of F_E, F_B leading to the standard formulation of Maxwell's equations in terms of electric and magnetic "fields". Within this setting the system made by (1.1a)-(1.1b) decouples in a system of 2 dynamical equations, which determine F_E, F_B once initial data and boundary conditions are provided, and 2 constraint equations, which must be fulfilled along the motion and in particular by the initial data. Similarly, the initial condition (1.1d) leads to initial conditions for F_E, F_B; moreover, the same applies for the boundary condition (1.1c) which leads to 2 boundary conditions for F_E and F_B. As we will see more in details, the boundary conditions we obtain are somehow redundant: The first one can be used to determine F_E, F_B uniquely —together with the initial data and the dynamical equations of motion— whereas the latter plays the role of a constraint. Summing up, the initial-value problem with boundary conditions (1.1) for F will be turned into an initial-value problem with boundary conditions and constraints for F_E, F_B.

In Section 3, we will solve the initial-boundary value problem for F_E, F_B relying on the results of [17]. Henceforth, in Section 4 we will prove that the constraints are fulfilled once they are fulfilled by the initial data. We conclude our paper with Section 5, devoted to prove the existence of Green operators for the Faraday tensor. This leads to a pre-symplectic form on the space of solutions to the Cauchy problem for the Faraday tensor: To this avail, however, one has to consider Faraday of all degrees in a unified non-trivial fashion.

Acknowledgment. We are grateful to Marco Benini for helpful discussions related to the topic of this paper. This work was produced within the activities of the INdAM-GNFM. N.D. acknowledges the support of the GNFM-INdAM Progetto Giovani *Non-linear sigma models and the Lorentzian Wetterich equation.* S.M. acknowledges the support of the GNFM-INdAM Progetto Giovani *Feynman propagator for Dirac fields: a microlocal analytic approach.*

2 Reformulation of the Cauchy problem

Let $\pi_2 \colon \mathsf{M} \to \Sigma$ be the projection on the second factor in the Cartesian product $\mathsf{M} = \mathbb{R} \times \Sigma$ and let $\mathsf{V}^\bullet := \pi_2^*(\Lambda^\bullet T^*\Sigma) \to \mathsf{M}$ be the pull-back over M of the exterior bundle of Σ. The **electric and the magnetic components of a given** $F \in \Omega^k(\mathsf{M})$ are the forms $F_B \in \Gamma(\mathsf{V}^k)$ and $F_E \in \Gamma(\mathsf{V}^{n-k})$ defined by

$$F = \mathrm{d}t \wedge *_{h_t} F_E + F_B \,, \tag{2.1}$$

where $*_{h_t}$ denotes the Hodge dual with respect to the metric h_t. More explicitly we have

$$*_{h_t} F_E := \partial_t \lrcorner F \,, \qquad F_B = F - \mathrm{d}t \wedge *_{h_t} F_E \,,$$

where $\partial_t \lrcorner$ denotes the interior product with ∂_t. Clearly F_E, F_B determines F uniquely and viceversa.

Remark 2.1. For later convenience we shall recollect here some useful identities concerning the differential, codifferential, Hodge operators, pull-backs and interior products. Let (M, g) be an m-dimensional pseudo-Riemannian manifold with possibly non-empty boundary ∂M; in most applications below, M^m will be either the spacetime M, its boundary $\partial\mathsf{M}$ together with its induced Lorentzian metric or the Cauchy hypersurface Σ with Riemannian metric h_t. We denote by σ_M the index of M. The orientation of M will be chosen such that, for any oriented pointwise basis $(e_1^*, \ldots, e_{n-1}^*)$ of $T^*\Sigma$, the n-tuple $(\mathrm{d}t, e_1^*, \ldots, e_{n-1}^*)$ is an oriented basis of $T^*\mathsf{M}$. We denote by $\mathrm{d}_\bullet \colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ the differential on M while $*_\bullet \colon \Omega^\bullet(M) \to \Omega^{m-\bullet}(M)$ denotes the Hodge dual of (M, g). To emphasize the difference between operators on M and on $\partial\mathsf{M}$, the differential and Hodge dual of ∂M will be denoted by $\mathrm{d}_\bullet^{\partial M}$ and $*_\bullet^{\partial M}$ respectively —we will suppress the superscript when the latter is clear from the context. We then have

$$\begin{aligned}
*_{m-k}*_k &= (-1)^{k(m-k)+\sigma_M} \quad (\Rightarrow *_k^{-1} = (-1)^{k(m-k)+\sigma_M}*_{m-k}) \\
\delta_k &= \mathrm{d}_k^* = (-1)^k(*_{k-1})^{-1}\mathrm{d}_{m-k}*_k \\
*_{k-1}\delta_k &= (-1)^k\mathrm{d}_{m-k} *_k \qquad \delta_{m-k}*_k = (-1)^{k+1} *_{k+1} \mathrm{d}_k \\
*_{k-1}^{\partial M}\mathtt{n}\lrcorner &= \iota_{\partial M}^* *_k \\
\mathtt{n}\lrcorner *_{m-k} &= (-1)^{m-k+\sigma_M+\sigma_{\partial M}} *_{m-k}^{\partial M} \iota_{\partial M}^* \\
X^\flat \wedge *_g\omega &= (-1)^{k+1} *_g (X\lrcorner\,\omega) \\
*_g(X^\flat \wedge \omega) &= (-1)^k X\lrcorner *_g \omega \\
\delta_{m-k-1}^{\partial M}\mathtt{n}\lrcorner|_{\Omega^{m-k}(\mathsf{M})} &= -\mathtt{n}\lrcorner\delta_{m-k}\,,
\end{aligned}$$

for all $\omega \in \Lambda^k T^*M$ and $X \in TM$. Moreover, defining the pointwise nondegenerate inner product $\langle\cdot\,,\cdot\rangle$ on $\Lambda^k T^*M$ via

$$\langle \omega, \omega' \rangle := (-1)^{\sigma_M} \cdot *_g(\omega \wedge *_g\omega'),$$

we have, for all $X \in TM$, $\omega \in \Lambda^k T^*M$ and $\omega' \in \Lambda^{k+1}T^*M$,

$$\begin{aligned}
\langle X^\flat \wedge \omega, \omega' \rangle &= (-1)^{\sigma_M} \cdot *_g(X^\flat \wedge \omega \wedge *_g\omega') \\
&= (-1)^{\sigma_M} \cdot (-1)^k \cdot *_g(\omega \wedge X^\flat \wedge *_g\omega') \\
&= (-1)^{\sigma_M} \cdot (-1)^k \cdot (-1)^k *_g (\omega \wedge *_g(X\lrcorner\,\omega')) \\
&= \langle \omega, X\lrcorner\,\omega' \rangle.
\end{aligned}$$

Moreover, for all $\omega \in \Lambda^k T^*M$ and $\omega' \in \Lambda^{m-k}T^*M$,

$$\begin{aligned}
\langle *_g\omega, \omega' \rangle &= (-1)^{k(m-k)+\sigma_M} \cdot \langle *_g\omega, *_g^2\omega' \rangle \\
&= (-1)^{k(m-k)+2\sigma_M} \cdot \langle \omega, *_g\omega' \rangle \\
&= (-1)^{k(m-k)} \cdot \langle \omega, *_g\omega' \rangle.
\end{aligned}$$

The next lemma converts equations (1.1a)-(1.1b) into dynamical and constraint equations for F_E, F_B.

Lemma 2.2. *A k-form $F \in \Omega^k(\mathsf{M})$ solves* (1.1a)-(1.1b) *if and only if its electric and magnetic components F_E, F_B solve*

$$\beta^{-1}\mathcal{L}_{\partial_t}(\beta^{-1}F_E) + (-1)^{(n-k+1)(k+1)+1}\beta^{-1}\mathrm{d}_\Sigma(*_{h_t}\beta F_B) = (-1)^{(n-k)(k+1)} *_{h_t} j_B\,, \tag{2.2a}$$

$$\mathcal{L}_{\partial_t}F_B - \mathrm{d}_\Sigma *_{h_t} F_E = *_{h_t}\zeta_E\,, \tag{2.2b}$$

$$\mathrm{d}_\Sigma(\beta^{-1}F_E) = (-1)^{n-k}\beta^{-1}j_E\,, \tag{2.2c}$$

$$\mathrm{d}_\Sigma F_B = \zeta_B\,, \tag{2.2d}$$

where d_Σ denotes the differential on Σ, while $j_E \in \Gamma(\mathsf{V}^{n+1-k})$ and $j_B \in \Gamma(\mathsf{V}^{k-1})$ are the electric and magnetic components of $j \in \Omega^{k-1}(\mathsf{M})$.

Proof. We recall that the differential d on M and the differential d_Σ on Σ are related by

$$\mathrm{d}\omega = \mathrm{d}t \wedge \partial_t \lrcorner \mathrm{d}\omega + \mathrm{d}_\Sigma \iota_\Sigma^* \omega\,,$$

for all $\omega \in \Omega^k(\mathsf{M})$. By direct inspection we have

$$\begin{aligned}
\zeta = \mathrm{d}F &= -\mathrm{d}t \wedge \mathrm{d}_\Sigma *_{h_t} F_E + \mathrm{d}t \wedge \partial_t \lrcorner \mathrm{d}F_B + \mathrm{d}_\Sigma F_B && \text{Eq. (2.1)}\\
&= \mathrm{d}t \wedge [\mathcal{L}_{\partial_t}F_B - \mathrm{d}_\Sigma *_{h_t} F_E] + \mathrm{d}_\Sigma F_B && \partial_t \lrcorner F_B = 0\,,
\end{aligned}$$

which leads to Equations (2.2b) and (2.2d) once we consider the decomposition $\zeta = \mathrm{d}t \wedge *_{h_t}\zeta_E + \zeta_B$.

For what concerns Equations (2.2a) and (2.2c) we consider the Hodge dual of Equation (1.1b):

$$*_g j = *_g \delta F = (-1)^k \mathrm{d} *_g F\,.$$

Moreover, for all $\omega \in \Gamma(\mathsf{V}^k)$ we have $\beta dt \wedge *_{h_t}\omega = (-1)^k *_g \omega$, which implies

$$\begin{aligned}
*_g F &= \beta^{-1} *_g (\beta \mathrm{d}t \wedge *_{h_t} F_E) + *_g F_B\\
&= (-1)^{n-k}\beta^{-1} *_g^2 F_E + *_g F_B\\
&= (-1)^{(n-k)(k+1)+\sigma_{\mathsf{M}}}\beta^{-1}F_E + (-1)^k \beta \mathrm{d}t \wedge *_{h_t} F_B\\
&= (-1)^{(n-k)(k+1)+1}\beta^{-1}F_E + (-1)^k \beta \mathrm{d}t \wedge *_{h_t} F_B
\end{aligned}$$

and similarly

$$\begin{aligned}
*_g j &= *_g(\mathrm{d}t \wedge *_{h_t} j_E + j_B)\\
&= \beta^{-1} *_g (\beta \mathrm{d}t \wedge *_{h_t} j_E) + *_g j_B\\
&= (-1)^{n-k+1}\beta^{-1} *_g^2 j_E + (-1)^{k-1}\beta \mathrm{d}t \wedge *_{h_t} j_B\\
&= (-1)^{n-k+1+(n-k+1)(k-1)+\sigma_{\mathsf{M}}}\beta^{-1} j_E + (-1)^{k-1}\beta \mathrm{d}t \wedge *_{h_t} j_B\\
&= (-1)^{k(n-k+1)+1}\beta^{-1} j_E + (-1)^{k-1}\beta \mathrm{d}t \wedge *_{h_t} j_B\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}*_g F &= (-1)^{(n-k)(k+1)+1}\mathrm{d}(\beta^{-1}F_E) + (-1)^k \mathrm{d}(\mathrm{d}t \wedge *_{h_t}\beta F_B) \\
&= (-1)^{(n-k)(k+1)+1}\mathrm{d}t \wedge \partial_t \lrcorner\, \mathrm{d}(\beta^{-1}F_E) + (-1)^{(n-k)(k+1)+1}\mathrm{d}_\Sigma(\beta^{-1}F_E) \\
&\quad +(-1)^{k+1}\mathrm{d}t \wedge \mathrm{d}_\Sigma(*_{h_t}\beta F_B) \\
&= \mathrm{d}t \wedge \Big((-1)^{(n-k)(k+1)+1}\mathcal{L}_{\partial_t}(\beta^{-1}F_E) + (-1)^{k+1}\mathrm{d}_\Sigma(*_{h_t}\beta F_B)\Big) \\
&\quad +(-1)^{(n-k)(k+1)+1}\mathrm{d}_\Sigma(\beta^{-1}F_E).
\end{aligned}$$

It can be deduced that $\mathrm{d}*_g F = (-1)^k *_g j$ if and only if

$$\begin{cases} (-1)^{(n-k)(k+1)+1}\mathcal{L}_{\partial_t}(\beta^{-1}F_E) + (-1)^{k+1}\mathrm{d}_\Sigma(*_{h_t}\beta F_B) = -\beta *_{h_t} j_B \\ (-1)^{(n-k)(k+1)+1}\mathrm{d}_\Sigma(\beta^{-1}F_E) = (-1)^{k(n-k)+1}\beta^{-1}j_E \end{cases},$$

that is

$$\begin{cases} \beta^{-1}\mathcal{L}_{\partial_t}(\beta^{-1}F_E) + (-1)^{(n-k+1)(k+1)+1}\beta^{-1}\mathrm{d}_\Sigma(*_{h_t}\beta F_B) = (-1)^{(n-k)(k+1)} *_{h_t} j_B \\ \mathrm{d}_\Sigma(\beta^{-1}F_E) = (-1)^{n-k}\beta^{-1}j_E \end{cases}.$$

This leads to Equations (2.2a) and (2.2c). □

Remark 2.3. The constraint $\delta j = 0$ on the current $j \in \Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})$ assumed in Theorem 1.1 reduces to the standard continuity equation in terms of j_E, j_B:

$$\mathcal{L}_{\partial_t}[\beta^{-1}j_E] + (-1)^{k(n-k)+1}\mathrm{d}_\Sigma[\beta *_{h_t} j_B] = 0\,, \qquad \mathrm{d}_\Sigma[\beta^{-1}j_E] = 0\,. \tag{2.3}$$

Similarly $\zeta \in \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})$ has to be closed, therefore,

$$\mathcal{L}_{\partial_t}\zeta_B - \mathrm{d}_\Sigma *_{h_t} \zeta_E = 0\,, \qquad \mathrm{d}_\Sigma\zeta_B = 0\,. \tag{2.4}$$

Thus, Equations (1.1a)-(1.1b) can be recast into Equations (2.2). Notice that the latter consists of two dynamical equations (2.2a)-(2.2b) and two constraint equations (2.2c)-(2.2d). In the next section we will prove that Equations (2.2a)-(2.2b) define a symmetric hyperbolic system [17, Def. 2.4-2.5]. Before that, we observe that the boundary condition (1.1c) can be equivalently written in terms of the electric and magnetic components F_E, F_B as

$$\mathtt{n}\lrcorner *_{h_t} F_E = 0 \qquad (\Leftrightarrow \iota^*_{\partial\Sigma_t}F_E = 0) \tag{2.5}$$

$$\mathtt{n}\lrcorner F_B = 0 \qquad (\Leftrightarrow \iota^*_{\partial\Sigma_t} *_{h_t} F_B = 0)\,. \tag{2.6}$$

As we will see, in order to apply the results of [17] only one among (2.5)-(2.6) is needed —in the following we will choose (2.5). The remaining boundary condition is redundant, in fact, it plays the role of an additional constrained boundary condition.

3 Maxwell's equations as a constrained symmetric hyperbolic system

We now recast Equations (2.2a)-(2.2b) into a symmetric hyperbolic system. Following [17, Def. 2.4-2.5] we recall that a differential operator $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ on a Riemannian vector bundle $\mathsf{E} \to \mathsf{M}$, is called **symmetric hyperbolic system** over M if

(S) The principal symbol $\sigma_{\mathsf{S}}(\xi)\colon \mathsf{E}_p \to \mathsf{E}_p$ is pointwise self-adjoint resp. symmetric with respect to $\prec \cdot \,|\, \cdot \succ_p$ for every $\xi \in \mathsf{T}^*_p\mathsf{M}$ and for every $p \in \mathsf{M}$ —here $\prec \cdot \,|\, \cdot \succ_p$ denotes the Riemannian resp. symmetric fiber pairing at E_p;

(H) For every future-directed timelike covector $\tau \in \mathsf{T}^*_p\mathsf{M}$, the bilinear form $\prec \sigma_{\mathsf{S}}(\tau) \cdot \,|\, \cdot \succ_p$ is positive definite on E_p for every $p \in \mathsf{M}$.

A symmetric hyperbolic system S is said **of constant characteristic** if $\dim\ker \sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ is constant, where $\sigma_{\mathsf{S}}(\mathtt{n}^\flat) \in \mathrm{End}\,(T^*\mathsf{M}|_{\partial\mathsf{M}})$. In particular, if $\sigma_{\mathsf{S}}(\mathtt{n}^\flat)$ has maximal rank at each point of $\partial\mathsf{M}$ we say that S is **nowhere characteristic**.

Concerning boundary conditions for a symmetric hyperbolic system S with constant characteristic we quote from [17, Definition 2.13]. A smooth subbundle B of $\mathsf{E}_{|\partial\mathsf{M}}$ is called a **self-adjoint admissible boundary condition** for S if

(i) the quadratic form $\Psi \mapsto \prec \sigma_{\mathsf{S}}(\nu)\Psi \,|\, \Psi \succ_p$ vanishes on B —here $\nu \in \Omega^1(\mathsf{M})$ is any form such that $\ker\nu_x = T_x\partial\mathsf{M}$ for all $x \in \partial\mathsf{M}$;

(ii) the rank of B is equal to the number of pointwise non-negative eigenvalues of $\sigma_{\mathsf{S}}(\nu)$ counting multiplicity;

(iii) the identity $\mathsf{B} = \mathsf{B}^\dagger$ holds, where $\mathsf{B}^\dagger := [\sigma_{\mathsf{S}}(\mathtt{n}^\flat)\mathsf{B}]^\perp$ and the symbol $(\cdot)^\perp$ denotes the pointwise orthogonal complement with respect to $\prec \cdot \,|\, \cdot \succ$.

The next Proposition shows that Equations (2.2a)-(2.2b) can be interpreted as a symmetric hyperbolic system of constant characteristic. Moreover, the boundary condition (2.5) is a self-adjoint boundary condition for that symmetric hyperbolic system.

Proposition 3.1. *Let $\mathsf{E} = \mathsf{V}^{n-k} \oplus \mathsf{V}^k \to \mathsf{M}$ be the vector bundle over M with the standard positive-definite fiber metric $\prec \cdot \,|\, \cdot \succ$ between forms. Actually for $F_B, F'_B \in \Gamma(\mathsf{V}^k)$ we have*

$$\prec F_B \,|\, F'_B \succ := *_{h_t}[F_B \wedge *_{h_t} F'_B] = - *_g [F_B \wedge *_g F'_B]\,.$$

Then:

1. *The first-order differential operator $\mathsf{S}\colon \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ defined by*

$$\mathsf{S}\begin{bmatrix} F_E \\ F_B \end{bmatrix} = \begin{pmatrix} \beta^{-1}\mathcal{L}_{\partial_t} \circ \beta^{-1} & (-1)^{(n-k+1)(k+1)+1}\beta^{-1}\mathrm{d}_\Sigma *_{h_t} \beta \\ -\mathrm{d}_\Sigma *_{h_t} & \mathcal{L}_{\partial_t} \end{pmatrix} \begin{bmatrix} F_E \\ F_B \end{bmatrix}, \tag{3.1}$$

is a symmetric hyperbolic system of constant characteristic.

2. The subbundle $\mathsf{B} \subset \mathsf{E}|_{\partial\mathsf{M}}$ *defined by*

$$\mathsf{B} := \{(F_E, F_B) \in \mathsf{E}|_{\partial\mathsf{M}} \,|\, \mathtt{n} \lrcorner F_B = 0\} := \{(F_E, F_B) \in \mathsf{E}|_{\partial\mathsf{M}} \,|\, \nu \wedge *_{h_t} F_B = 0\}\,, \tag{3.2}$$

defines a self-adjoint admissible boundary condition for S.

Proof.

$\boxed{1}$ The principal symbol of S at $\xi \in T^*_p\mathsf{M}$, $p \in \Sigma_t$, is given by

$$\sigma_{\mathsf{S}}(\xi) = \begin{pmatrix} \beta^{-2}\xi(\partial_t)\,\mathrm{Id}_{\mathsf{V}^{n-k}|_p} & (-1)^{(n-k+1)(k+1)+1}\xi_{\Sigma_t} \wedge *_{h_t} \\ -\xi_{\Sigma_t} \wedge *_{h_t} & \xi(\partial_t)\,\mathrm{Id}_{\mathsf{V}^k|_p} \end{pmatrix}\,,$$

where $\xi_{\Sigma_t} := \iota^*_{\Sigma_t}\xi$ being $\iota_{\Sigma_t} \colon \Sigma_t \to \mathsf{M}$. By direct inspection we have, for all $F_E \in \mathsf{V}^{n-k}_p$, $F_B \in \mathsf{V}^k_p$, and $\xi \in T^*_p\mathsf{M}$,

$$\begin{aligned} \prec -\xi_{\Sigma_t} \wedge *_{h_t} F_E \,|\, F_B \succ &= - \prec *_{h_t} F_E \,|\, \xi^{\sharp_t}_{\Sigma_t} \lrcorner F_B \succ \\ &= -(-1)^{(n-k)(k-1)} \prec F_E \,|\, *_{h_t} (\xi^{\sharp_t}_{\Sigma_t} \lrcorner F_B) \succ \\ &= -(-1)^{(n-k)(k-1)+k+1} \prec F_E \,|\, \xi_{\Sigma_t} \wedge *_{h_t} F_B \succ \\ &= (-1)^{(n-k+1)(k+1)+1} \prec F_E \,|\, \xi_{\Sigma_t} \wedge *_{h_t} F_B \succ\,, \end{aligned}$$

which shows $\sigma_{\mathsf{S}}(\xi)^\dagger = \sigma_{\mathsf{S}}(\xi)$ and therefore that condition (S) holds.
Next we prove condition (H). Let $\xi = \xi(\partial_t)dt + \xi_{\Sigma_t} \in T^*_p\mathsf{M}$ be any future-directed timelike covector that is, $\|\xi_{\Sigma_t}\|^2_{h_t} < \beta^{-2}\xi(\partial_t)^2$ and $\xi(\partial_t) > 0$. For any $F_E \in \mathsf{V}^k_p$ and $F_B \in \mathsf{V}^{n-k}_p$ we have

$$\begin{aligned} \prec \sigma_{\mathsf{S}}(\xi)(F_E, F_B) \,|\, (F_E, F_B) \succ &= \beta^{-2}\xi(\partial_t) \prec F_E \,|\, F_E \succ + \xi(\partial_t) \prec F_B \,|\, F_B \succ \\ &\quad - 2 \prec \xi_{\Sigma_t} \wedge *_{h_t} F_E \,|\, F_B \succ \\ &\geq \beta^{-2}\xi(\partial_t) \prec F_E \,|\, F_E \succ + \xi(\partial_t) \prec F_B \,|\, F_B \succ \\ &\quad - 2\|\xi_{\Sigma_t}\|_{h_t} \prec F_E \,|\, F_E \succ^{1/2} \prec F_B \,|\, F_B \succ^{1/2} \\ &\geq \beta^{-2}\xi(\partial_t) \prec F_E \,|\, F_E \succ + \xi(\partial_t) \prec F_B \,|\, F_B \succ \\ &\quad - 2\beta^{-1}\xi(\partial_t) \prec F_E \,|\, F_E \succ^{1/2} \prec F_B \,|\, F_B \succ^{1/2} \\ &= \xi(\partial_t) \Big[\beta^{-1} \prec F_E \,|\, F_E \succ^{1/2} - \prec F_B \,|\, F_B \succ^{1/2}\Big]^2 \geq 0\,. \end{aligned}$$

Moreover, if $\prec \sigma_{\mathsf{S}}(\xi)(F_E, F_B) \,|\, (F_E, F_B) \succ = 0$ then the above inequalities implies

$$\|\xi_{\Sigma_t}\|_{h_t} \prec F_E \,|\, F_E \succ \prec F_B \,|\, F_B \succ = \beta^{-2}\xi(\partial_t)^2 \prec F_E \,|\, F_E \succ \prec F_B \,|\, F_B \succ\,,$$

which forces $F_E = 0$ and $F_B = 0$ due to the condition $\|\xi_{\Sigma_t}\|^2_{h_t} < \xi(\partial_t)^2\beta^{-2}$. This proves that $\sigma_{\mathsf{S}}(\xi)$ is positive definite and therefore condition (H) holds.

Finally, since $\sigma_{\mathsf{S}}(\nu)$ is given by

$$\sigma_{\mathsf{S}}(\nu) = \begin{pmatrix} 0 & (-1)^{k(n-k)}\nu \wedge *_{h_t} \\ -\nu \wedge *_{h_t} & 0 \end{pmatrix},$$

it follows that

$$\begin{aligned} \ker \sigma_{\mathsf{S}}(\nu) &= \{(F_E, F_B) \in \mathsf{V}^{n-k} \oplus \mathsf{V}^k \,|\, \mathtt{n}\lrcorner F_E = 0 = \mathtt{n}\lrcorner F_B\} \\ &= \pi_2^* \Lambda^{n-k} T^*\partial\Sigma \oplus \pi_2^*\Lambda^k T^*\partial\Sigma\,, \end{aligned}$$

which proves that S is of constant characteristic.

$\boxed{2}$ We now prove that the subbundle B introduced in Equation (3.2) identifies a future admissible boundary condition for S. By direct inspection we have

$$\mathsf{E}_{|\partial\mathsf{M}} = \ker \sigma_{\mathsf{S}}(\nu) \oplus \ker[\sigma_{\mathsf{S}}(\nu) + 1] \oplus \ker[\sigma_{\mathsf{S}}(\nu) - 1]\,,$$

where

$$\begin{aligned} \ker \sigma_{\mathsf{S}}(\nu) &\simeq \pi_2^*\Lambda^{n-k}T^*\partial\Sigma \oplus \pi_2^*\Lambda^k T^*\partial\Sigma\,, \\ \ker[\sigma_{\mathsf{S}}(\nu) - \varepsilon] &= \{(F_E, -\varepsilon\nu \wedge *_{h_t} F_E) \in \mathsf{E}_{|\partial\mathsf{M}} \,|\; *_{h_t} F_E \in \pi_2^*\Lambda^{k-1}T^*\partial\Sigma\}\,, \qquad \varepsilon \in \{1, -1\}\,. \end{aligned}$$

Notice $\dim\ker\sigma_{\mathsf{S}}(\nu) = \binom{n-2}{n-k} + \binom{n-2}{k}$, moreover, each eigenspace associated to $\varepsilon \in \{\pm 1\}$ has pointwise rank $\binom{n-2}{k-1}$. Thus, an admissible boundary condition must have rank $\binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{n-k}$ because of condition (ii). But this is exactly the case for B, whose dimension is $\binom{n-1}{k-1} + \binom{n-2}{k}$ so that condition (ii) is fulfilled. Moreover, for all $(F_E, F_B) \in \mathsf{B}$ it holds

$$\prec \sigma_{\mathsf{S}}(\nu)(F_E, F_B) \,|\, (F_E, F_B) \succ = -2 \prec F_E \,|\, \nu \wedge *_{h_t} F_B \succ = 0\,.$$

The latter equality implies condition (i). Finally, since $\mathsf{B} = \mathsf{V}^{n-k} \oplus \pi_2^*\Lambda^k T^*\partial\Sigma$ and $\sigma_{\mathsf{S}}(\nu)(\mathsf{B}) = \{(0, -\nu \wedge *_{h_t} F_E) \,|\, F_E \in \mathsf{V}^{n-k}\}$ we have that $\mathsf{B}^\dagger = \mathsf{V}^{n-k} \oplus \pi_2^*\Lambda^k T^*\partial\Sigma = \mathsf{B}$ *i.e.* condition (iii) is fulfilled.

This concludes our proof. □

4 The Cauchy problem for the Faraday tensor

We have finally all the ingredients to prove our main theorem.

Proof of Theorem 1.1. On account of Lemma 2.2 we may reduce our problem to the initial-value problem

$$\mathsf{S}(F_E, F_B) = ((-1)^{(n-k)(k+1)} *_{h_t} j_B, *_{h_t}\zeta_E)\,, \tag{4.1a}$$

$$(F_E, F_B)|_{\Sigma_0} = (F_{0,E}, F_{0,B}) \tag{4.1b}$$

$$(F_E, F_B)|_{\partial\mathsf{M}} \in \mathsf{B} \tag{4.1c}$$

subjected to the constraint equations

$$\mathrm{d}_\Sigma[\beta^{-1}F_E] = (-1)^{n-k}\beta^{-1}j_E\,, \qquad \mathrm{d}_\Sigma F_B = \zeta_B\,, \qquad \iota^*_{\partial\Sigma_t}F_E = 0\,. \tag{4.2}$$

Here $F_{0,E,}, F_{0,B}$ denote the electric and magnetic component of the initial datum $F_0 \in \Omega^k(\mathsf{M})$. Notice that the assumptions on the initial data F_0 implies

$$(F_{0,E}, F_{0,B}) \in \mathsf{B}\,, \qquad \mathrm{d}_\Sigma[\beta^{-1}F_{0,E}] = 0\,, \qquad \mathrm{d}_\Sigma F_{0,B} = 0\,,$$

Since S is symmetric hyperbolic and B is an admissible self-adjoint boundary condition for S, we may apply [17, Thm. 1.2]. Notice that the compatibility conditions mentioned therein —*cf.* [17, Eq. (4.3)]— are automatically fulfilled on account of our assumption that $\mathrm{supp}(F_0) \cap \partial\mathsf{M} = \varnothing$ and $(\mathrm{supp}(\zeta) \cup \mathrm{supp}(j)) \cap \Sigma_0 = \varnothing$.

Then [17, Thm. 1.2] guarantees the existence of a unique solution $(F_E, F_B) \in \Gamma(\mathsf{V}^{n-k} \oplus \mathsf{V}^k)$ to (4.1). Moreover [17, Prop. 3.3] entails (1.2) and thus $F \in \Omega^k_{sc}(\mathsf{M})$, where $F = \mathrm{d}t \wedge *_{h_t} F_E + F_B$.

It remains to prove that (4.2) holds —notice that this would also prove that $F \in \Omega^k_{c,\mathtt{n}}(\mathsf{M})$. In fact by direct inspection we find

$$\begin{aligned}
\mathcal{L}_{\partial_t}\mathrm{d}_\Sigma[\beta^{-1}F_E] &= \mathrm{d}_\Sigma\mathcal{L}_{\partial_t}[\beta^{-1}F_E] \\
&= (-1)^{(n-k+1)(k+1)}\cancel{\mathrm{d}^2_\Sigma[*_{h_t}\beta F_B]} + (-1)^{(n-k)(k+1)}\mathrm{d}_\Sigma[\beta *_{h_t} j_B] && \text{Eq. (2.2a)} \\
&= (-1)^{n-k}\mathcal{L}_{\partial_t}[\beta^{-1}j_E] && \text{Eq. (2.3)} \\
\mathcal{L}_{\partial_t}\mathrm{d}_\Sigma F_B &= \mathrm{d}_\Sigma\mathcal{L}_{\partial_t}F_B = \cancel{\mathrm{d}^2_\Sigma[*_{h_t}F_E]} + \mathrm{d}_\Sigma[*_{h_t}\zeta_E] = \mathcal{L}_{\partial_t}\zeta_B && \text{Eq. (2.2b)} \\
\mathcal{L}_{\partial_t}\iota^*_{\partial\Sigma}\beta^{-1}F_E &= \iota^*_{\partial\Sigma}\mathcal{L}_{\partial_t}[\beta^{-1}F_E] = 0 && \text{Eq. (2.2a)} - (2.5)\,,
\end{aligned}$$

where in the last equality we also used that $\mathtt{n}\lrcorner j = 0$ is equivalent to $\iota^*_{\partial\Sigma}[*_{h_t}j_B] = 0$. The latter equations proves that (4.2) is fulfilled once is fulfilled by the initial datum F_0: This is the case by assumption. □

5 Existence of Green operators and pre-symplectic structures

In this section we establish the existence of the Green operators for the differential operator $D = \delta + \mathrm{d}$ acting on k-forms and with boundary conditions (1.1c). To this end, we will profit from [3, 11, 17]. For later convenience we recall that $\Omega^k_{sfc}(\mathsf{M})$ (*resp.* $\Omega^k_{spc}(\mathsf{M})$) denotes the space of strictly future- (*resp.* past-) compactly supported k-forms that is, of all $F \in \Omega^k(\mathsf{M})$ such that $\mathrm{supp}(F) \subset J^-(K)$ (*resp.* $\mathrm{supp}(F) \subset J^+(K)$) for a suitable compact subset $K \subset \mathsf{M}$. We also set $\Omega^k_{sc}(\mathsf{M}) := \Omega^k_{sfc}(\mathsf{M}) \cup \Omega^k_{spc}(\mathsf{M})$. Similarly $\Omega^k_{fc}(\mathsf{M})$ (*resp.* $\Omega^k_{pc}(\mathsf{M})$) denotes the space of future- (*resp.* past-) compactly supported k-forms that is, of all $F \in \Omega^k(\mathsf{M})$ such that $\mathrm{supp}(F) \cap J^+(x)$ (*resp.* $\mathrm{supp}(F) \cap J^-(x)$) is compact for all $x \in \mathsf{M}$. We set $\Omega^k_{tc}(\mathsf{M}) := \Omega^k_{fc}(\mathsf{M}) \cup \Omega^k_{pc}(\mathsf{M})$.

Proposition 5.1. *Let $k \in \{0, \dots, n\}$ and let $D\colon \Omega^k(\mathsf{M}) \to \Omega^{k-1}(\mathsf{M}) \oplus \Omega^{k+1}(\mathsf{M})$ be the differential operator $D\omega := \delta\omega + \mathrm{d}\omega$. There exists linear operators*

$$G^+_k\colon \Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M}) \to \Omega^k_{spc,\mathtt{n}}(\mathsf{M})\,, \qquad G^-_k\colon \Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M}) \to \Omega^k_{sfc,\mathtt{n}}(\mathsf{M})\,,$$

which fulfil the following properties:

$$\mathrm{d}G_k^{\pm}(\alpha_{k-1}\oplus\zeta_{k+1})=\zeta_{k+1} \tag{5.1}$$

$$\delta G_k^{\pm}(\alpha_{k-1}\oplus\zeta_{k+1})=\alpha_{k-1} \tag{5.2}$$

$$G_k^{\pm}(\delta\omega_k\oplus\mathrm{d}\omega_k)=\omega_k \qquad \forall\omega_k\in\Omega^k_{c,\mathtt{n}}(\mathsf{M}) \tag{5.3}$$

$$\operatorname{supp} G_k^{\pm}(\alpha_{k-1}\oplus\zeta_{k+1})\subseteq J^{\pm}[\operatorname{supp}(\alpha_{k-1})\cup\operatorname{supp}(\zeta_{k+1})]\,. \tag{5.4}$$

Moreover, the $G_k^{\pm}$ can be extended to

$$G_k^+\colon\Omega^{k-1}_{spc,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{spc,\mathrm{d}}(\mathsf{M})\to\Omega^k_{spc,\mathtt{n}}(\mathsf{M})\,,\quad G_k^-\colon\Omega^{k-1}_{sfc,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{sfc,\mathrm{d}}(\mathsf{M})\to\Omega^k_{sfc,\mathtt{n}}(\mathsf{M})\,, \tag{5.5}$$

still preserving properties (5.1)-(5.4)*.*

Finally, if $G_k:=G_k^+-G_k^-$, then there exists a short exact sequence

$$\{0\}\to\Omega^k_{c,\mathtt{n}}(\mathsf{M})\xrightarrow{D}\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})\xrightarrow{G_k}\Omega^k_{sc,\mathtt{n}}(\mathsf{M})\xrightarrow{D}\delta\Omega^k_{sc,\mathtt{n}}(\mathsf{M})\oplus\mathrm{d}\Omega^k_{sc}(\mathsf{M})\to\{0\}\,. \tag{5.6}$$

Proof. Let $k\in\{0,\ldots,n\}$. Following [3, 11–13, 17], we define $G_k^+\colon\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})\to\Omega^k_{spc,\mathtt{n}}(\mathsf{M})$ so that $G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})$ is the unique solution $\omega_k\in\Omega^k(\mathsf{M})$ to the initial-value problem with boundary conditions

$$\mathrm{d}\omega_k=\zeta_{k+1}\,,\qquad\delta\omega_k=\alpha_{k-1}\,,\qquad\mathtt{n}\lrcorner\omega_k=0\,,\qquad\omega_k|_\Sigma=0\,, \tag{5.7}$$

where Σ is an arbitrary but fixed Cauchy surface such that $J^-(\Sigma)\cap[\operatorname{supp}(\alpha_{k-1})\cup\operatorname{supp}(\zeta_{k+1})]=\varnothing$. Existence and uniqueness of $G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})$ follows from Theorem 1.1, moreover, G_k^+ is easily shown to be linear and independent on the chosen Σ. The map G_k^- is similarly defined by assigning vanishing Cauchy data on a Cauchy surface Σ so that $J^+(\Sigma)\cap[\operatorname{supp}(\alpha_{k-1})\cup\operatorname{supp}(\zeta_{k+1})]=\varnothing$.

Equations (5.1)-(5.2) follow from the definition of $G_k^{\pm}\alpha$ while the inclusion (5.4) is a consequence of (1.2). Finally, Equation (5.3) follows from the uniqueness of (5.7) together with the condition $\mathtt{n}\lrcorner\omega_k=0$. Notice that the latter condition is necessary for (5.3) as the latter equation implies $\mathtt{n}\lrcorner\omega_k=\mathtt{n}\lrcorner G_k^{\pm}(\delta\omega_k\oplus\mathrm{d}\omega_k)=0$.

The extension (5.5) is obtained by using property (5.4), *cf.* [2, Thm. 3.8] whose proof we mimic for the sake of self-containedness of the article. To wit, let $\alpha_{k-1}\in\Omega^{k-1}_{spc,\mathtt{n},\delta}(\mathsf{M})$ and $\zeta_{k+1}\in\Omega^{k+1}_{spc,\mathrm{d}}(\mathsf{M})$. We define $G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})$ as follows —a similar argument goes for G_k^-. For fixed $x\in\mathsf{M}$, let $K_x:=J^-(x)\cap[\operatorname{supp}(\alpha_{k-1}\oplus\zeta_{k+1})]$. Then K_x is compact and we may choose $\chi\in C^\infty_c(\mathsf{M})$ such that $\chi|_{K_x}=1$. For any such χ we set

$$G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})|_x:=G_k^+(\chi\alpha_{k-1}\oplus\chi\zeta_{k+1})|_x\,. \tag{5.8}$$

Note that $\operatorname{supp}(\chi)$ being compact ensures that $\chi\alpha_{k-1}$ and $\chi\zeta_{k+1}$ are compactly supported. Moreover, $\operatorname{supp}(\mathrm{d}[\chi\zeta_{k+1}])\cap J^-(x)=\varnothing$ and similarly $\operatorname{supp}(\delta[\chi\alpha_{k-1}])\cap J^-(x)=\varnothing$. On account of property (5.4) this entails that $G_k^+(\chi\alpha_{k-1}\oplus\chi\zeta_{k+1})|_x$ is well-posed and defines the wanted extension.

The resulting map G_k^+ is independent on the particular choice of χ. Indeed, any pair of functions χ, χ' with the above properties fulfil $\mathrm{supp}[(\chi-\chi')(\alpha_{k-1}\oplus\zeta_{k+1})]\cap J^-(x)=\varnothing$, therefore, $G_k^+[\chi\alpha_{k-1}\oplus\chi\zeta_{k+1}]|_x = G_k^+[\chi'\alpha_{k-1}\oplus\chi'\zeta_{k+1}]|_x$.

The χ-independence implies linearity of the resulting map G_k^+. Indeed, if $\alpha_{k-1}\oplus\zeta_{k+1}$, $\alpha'_{k-1}\oplus\zeta'_{k+1}$ are in $\Omega^{k-1}_{spc,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{spc,\mathrm{d}}(\mathsf{M})$, then for all $x\in\mathsf{M}$ we may choose $\chi\in C^\infty(\mathsf{M})$ so that $\chi=1$ on $J^-(x)\cap[\mathrm{supp}(\alpha_{k-1}\oplus\zeta_{k+1})\cup\mathrm{supp}(\alpha'_{k-1}\oplus\zeta'_{k+1})]$, thus

$$\begin{aligned}G_k^+[(\alpha_{k-1}+\alpha'_{k-1})\oplus(\zeta_{k+1}+\zeta'_{k+1})]|_x &= G_k^+[(\chi\alpha_{k-1}+\chi\alpha'_{k-1})\oplus(\chi\zeta_{k+1}+\chi\zeta'_{k+1})]|_x\\ &= G_k^+[\chi\alpha_{k-1}\oplus\chi\zeta_{k+1}]|_x + G_k^+[\chi\alpha'_{k-1}\oplus\chi\zeta'_{k+1}]|_x\\ &= G_k^+[\alpha_{k-1}\oplus\zeta_{k+1}]|_x + G_k^+[\alpha'_{k-1}\oplus\zeta'_{k+1}]|_x\,.\end{aligned}$$

Property (5.4) follows from Equation (5.8). The same holds for properties (5.2)-(5.1). Note also that, because it is of vanishing order, the boundary condition $\mathtt{n}\lrcorner G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})=0$ is also a straightforward consequence of the definition of G_k^+. For what concerns (5.3) we observe that, for all $\omega_k\in\Omega^k_{spc,\mathtt{n}}(\mathsf{M})$ it holds

$$G_k^+(\delta\omega_k\oplus\mathrm{d}\omega_k)|_x = G_k^+(\chi\delta\omega_k\oplus\chi\mathrm{d}\omega_k)|_x = G_k^+(\delta\chi\omega_k\oplus\mathrm{d}\chi\omega_k)|_x = \chi\omega_k|_x = \omega_k|_x\,,$$

where we used $\mathrm{supp}(\mathrm{d}\chi)\cap\mathrm{supp}(\alpha_{k-1}\oplus\zeta_{k+1})\cap J^-(x)=\varnothing$.

We now prove the exactness of (5.6). To begin with, notice that if $\alpha_k\in\Omega^k_{c,\mathtt{n}}(\mathsf{M})$ is such that $D\alpha_k=0$ —*i.e.* $\delta\alpha_k=0$ and $\mathrm{d}\alpha_k=0$— then we have $\alpha_k=G_k^+(\delta\alpha_k,\mathrm{d}\alpha_k)=0$: This shows exactness in the first arrow of (5.6).

If $\alpha_k\in\Omega^k_{c,\mathtt{n}}(\mathsf{M})$ then $G_kD\alpha_k=G_k^+(\delta\alpha_k,\mathrm{d}\alpha_k)-G_k^-(\delta\alpha_k,\mathrm{d}\alpha_k)=\alpha_k-\alpha_k=0$, proving that $D\Omega^k_{c,\mathtt{n}}(\mathsf{M})\subset\ker G_k$. Conversely, if $\alpha_{k-1}\oplus\zeta_{k+1}\in\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^k_{c,\mathrm{d}}(\mathsf{M})$ is such that $G_k(\alpha_{k-1}\oplus\zeta_{k+1})=0$ then $G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})=G_k^-(\alpha_{k-1}\oplus\zeta_{k+1})\in\Omega^k_{c,\mathtt{n}}(\mathsf{M})$ is such that

$$DG_k^+(\alpha_{k-1}\oplus\zeta_{k+1})=\delta G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})+\mathrm{d}G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})=\alpha_{k-1}\oplus\zeta_{k+1}\,.$$

This proves exactness of (5.6) in the second arrow.

Let $\alpha_{k-1}\oplus\zeta_{k+1}\in\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})$: Then $\delta G_k(\alpha_{k-1}\oplus\zeta_{k+1})=\delta G_k^+(\alpha_{k-1}\oplus\zeta_{k+1})-\delta G_k^-(\alpha_{k-1}\oplus\zeta_{k+1})=\alpha_{k-1}-\alpha_{k-1}=0$, and similarly $\mathrm{d}G_k(\alpha_{k-1}\oplus\zeta_{k+1})=0$. This shows that $DG_k(\alpha_{k-1}\oplus\zeta_{k+1})=0$ and thus $G_k[\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M})\oplus\Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})]\subset\ker D$. Moreover, let $\omega_k\in\Omega^k_{sc,\mathtt{n}}(\mathsf{M})$ be such that $D\omega_k=0$. Consider a function $\chi\in C^\infty(\mathsf{M})$ such that $\mathrm{d}\chi\in\mathrm{span}\,\mathrm{d}t$ and such that $\chi(t)=1$ for $t\geq t_0$, $t_0\in\mathbb{R}$ being arbitrary, and $\chi(t)=0$ for $t\leq -t_0$. Let $\omega_k^+:=\chi\omega_k$ and $\omega_k^-:=(1-\chi)\omega_k$. Then $\omega_k^+\in\Omega^k_{spc,\mathtt{n}}(\mathsf{M})$ and $\omega_k^-\in\Omega^k_{sfc,\mathtt{n}}(\mathsf{M})$. Moreover, $\delta\omega_k^+=-\delta\omega_k^-\in\Omega^{k-1}_{c,\mathtt{n}}(\mathsf{M})$ and similarly $\mathrm{d}\omega_k^\pm\in\Omega^{k+1}_c(\mathsf{M})$. Finally

$$\begin{aligned}G_k(\delta\omega_k^+\oplus\mathrm{d}\omega_k^+) &= G_k^+(\delta\omega_k^+\oplus\mathrm{d}\omega_k^+)-G_k^-(\delta\omega_k^+\oplus\mathrm{d}\omega_k^+)\\ &= G_k^+(\delta\omega_k^+\oplus\mathrm{d}\omega_k^+)+G_k^-(\delta\omega_k^-\oplus\mathrm{d}\omega_k^-)\\ &= \omega_k^++\omega_k^-\\ &= \omega_k\,,\end{aligned}$$

where we used the extension (5.5). This shows exactness in the third arrow of (5.6).

Finally, let $\alpha_k \in \Omega^k_{sc,\mathtt{n}}(\mathsf{M})$ and $\beta_k \in \Omega^k_{sc}(\mathsf{M})$. We wish to prove the existence of $\omega_k \in \Omega^k_{sc,\mathtt{n}}(\mathsf{M})$ such that $D\omega_k = \delta\alpha_k \oplus \mathrm{d}\beta_k$, that is, $\delta\omega_k = \delta\alpha_k$ and $\mathrm{d}\omega_k = \mathrm{d}\beta_k$. To this avail, we consider $\chi \in C^\infty(\mathsf{M})$ as above and let $\alpha_k = \alpha_k^+ + \alpha_k^-$, where $\alpha_k^+ := \chi\alpha_{k-1}$ and $\alpha_k^- := (1-\chi)\alpha_k^-$ and similarly $\beta_k = \beta_k^+ + \beta_k^-$. Notice that, per construction $\alpha_k^+ \in \Omega^+_{spc,\mathtt{n}}(\mathsf{M})$, $\alpha_k^- \in \Omega^k_{sfc,\mathtt{n}}(\mathsf{M})$ and similarly $\beta_k^+ \in \Omega^k_{spc}(\mathsf{M})$ and $\beta_k^- \in \Omega^k_{sfc}(\mathsf{M})$. We then set $\omega_k := G_k^+(\delta\alpha_k^+ \oplus \mathrm{d}\beta_k^+) + G_k^-(\delta\alpha_k^- \oplus \mathrm{d}\beta_k^-)$. Per definition $\omega_k \in \Omega^k_{sc,\mathtt{n}}(\mathsf{M})$, moreover, $D\omega_k = \delta\alpha_k^+ \oplus \mathrm{d}\beta_k^+ + \delta\alpha_k^- \oplus \mathrm{d}\beta_k^- = \delta\alpha_k \oplus \mathrm{d}\beta_k$, where we used the extension (5.5). This shows exactness of (5.6) in the fourth and last arrow. □

Remark 5.2. From (5.5) it follows that the causal propagator G_k extends to a linear map $G_k \colon \Omega^{k-1}_{tc,\mathrm{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{tc,\mathrm{d}}(\mathsf{M}) \to \Omega^k_{\mathtt{n}}(\mathsf{M})$, *cf.* [2, Thm. 3.8]. Furthermore, one may generalize the exact sequence (5.6) by relaxing the compactness support assumption to timelike compactness, while dropping the spacelike compactness condition:

$$\{0\} \to \Omega^k_{tc,\mathtt{n}}(\mathsf{M}) \xrightarrow{D} \Omega^{k-1}_{tc,\mathrm{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{tc,\mathrm{d}}(\mathsf{M}) \xrightarrow{G_k} \Omega^k_{\mathtt{n}}(\mathsf{M}) \xrightarrow{D} \delta\Omega^k_{\mathtt{n}}(\mathsf{M}) \oplus \mathrm{d}\Omega^k(\mathsf{M}) \to \{0\}\,. \tag{5.9}$$

The exactness of (5.6) leads to the following isomorphism, which provides a complete description of the solution space to Maxwell's equations by generalizing the well-known situation on a globally hyperbolic spacetime without boundary:

$$\mathrm{Sol}^k_{sc,\mathtt{n}}(\mathsf{M}) := \{F_k \in \Omega^k_{sc}(\mathsf{M}) \,|\, \delta F_k = 0\,,\, \mathrm{d}F_k = 0\,,\, \mathtt{n}\lrcorner F_k = 0\} \tag{5.10}$$

$$\simeq G_k[\Omega^{k-1}_{c,\mathrm{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})] \simeq \frac{\Omega^{k-1}_{c,\mathrm{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})}{D\Omega^k_{c,\mathtt{n}}(\mathsf{M})}\,. \tag{5.11}$$

5.1 Causal propagator and the pre-symplectic structure

We conclude the paper by endowing the space of homogeneous solutions to the Faraday Cauchy problem with a pre-symplectic form. The latter is constructed out of the causal propagators $\{G_k\}_{k=1}^n$ introduced in Proposition 5.1. The resulting pre-symplectic structure requires to consider all k-forms at once in a non-trivial fashion. To this avail we set $\Omega^\oplus(\mathsf{M}) := \oplus_{k=0}^n \Omega^k(\mathsf{M})$: An element of this latter space will be denoted by $\underline{F} = \sum_{k=0}^n F_k$, $F_k \in \Omega^k(\mathsf{M})$. The natural pairing $\Omega^\oplus(\mathsf{M})^2 \to \mathbb{R}$ inherited from the pairings on $\Omega^k(\mathsf{M})$ is denoted by $(\cdot\,,\cdot)_\oplus$. Let

$$\mathcal{S} := \{\underline{F} \in \Omega^\oplus_{sc,\mathtt{n}}(\mathsf{M}) \,|\, D\underline{F} = 0\} \tag{5.12}$$

$$= \{\underline{F} \in \Omega^\oplus_{sc}(\mathsf{M}) \,|\, F_k \in \Omega^k_{sc,\mathtt{n}}(\mathsf{M})\,,\, \mathrm{d}F_k = 0\,,\, \delta F_k = 0\,, \forall k \in \{0,\ldots,n\}\}\,. \tag{5.13}$$

Notice that $F_0 = 0$, moreover,

$$(D\underline{F}^{(1)}, \underline{F}^{(2)})_\oplus = (\underline{F}^{(1)}, D\underline{F}^{(2)})_\oplus \quad \forall \underline{F}^{(1)}, \underline{F}^{(2)} \in \Omega^\oplus_{\mathtt{n}}(\mathsf{M})\,, \mathrm{supp}(\underline{F}^{(1)}) \cap \mathrm{supp}(\underline{F}^{(2)}) \text{ compact}\,. \tag{5.14}$$

A direct application of Proposition 5.1 leads to the following isomorphism of vector spaces:

$$\bigoplus_{k=1}^n \frac{\Omega^{k-1}_{c,\mathrm{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})}{D\Omega^k_{c,\mathtt{n}}(\mathsf{M})} \simeq \bigoplus_{k=1}^n \mathrm{Sol}^k_{sc,\mathtt{n}}(\mathsf{M}) = \mathcal{S} \qquad \underline{\alpha} \oplus \underline{\zeta} \mapsto \underline{G}\,(\underline{\alpha} \oplus \underline{\zeta})\,, \tag{5.15}$$

where $\underline{G} := \oplus_{k=1}^{n} G_k$.

Proposition 5.3. *With the notation introduced above, let $\sigma_{\mathcal{S}} \colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be defined by*

$$\sigma_{\mathcal{S}}(\underline{F}^{(1)}, \underline{F}^{(2)}) := (D\underline{F}^{(1),+}, \underline{F}^{(2)})_{\oplus}\,, \tag{5.16}$$

where $\underline{F}^{(1)} = \underline{F}^{(1),+} + \underline{F}^{(1),-}$, $\underline{F}^{(1),+} \in \Omega^{\oplus}_{sfc,\mathtt{n}}(\mathsf{M})$, $\underline{F}^{(1),-} \in \Omega^{\oplus}_{spc,\mathtt{n}}(\mathsf{M})$, is an arbitrary decomposition of $\underline{F}^{(1)}$ in strictly future/past compactly supported forms.

Then $\sigma_{\mathcal{S}}$ is a well-defined pre-symplectic structure on $\mathcal{S}$. Moreover, if M admits a finite good cover [9, 19] it holds

$$\sigma_{\mathcal{S}}(\cdot, \underline{F}) = 0 \quad \Longleftrightarrow \quad \underline{F} = \mathrm{d}\underline{A} = \delta\underline{B}\,, \tag{5.17}$$

where $\underline{A} \in \Omega^{\oplus}_{sc}(\mathsf{M})$ and $\underline{B} \in \Omega^{\oplus}_{sc,\mathtt{n}}(\mathsf{M})$ —in particular $\underline{A} \in \Omega^{\oplus}_{sc}(\mathsf{M})$ is such that $\delta\mathrm{d}\underline{A} = 0$ and $\mathtt{n}\lrcorner\mathrm{d}\underline{A} = 0$.

Proof. We adapt the arguments of [5, 11] to the current case. To begin with, we observe that a decomposition of the form $\underline{F} = \underline{F}^{+} + \underline{F}^{-}$ can always be realized by multiplying $\underline{F}$ by a suitable time-dependent function $\chi \in C^{\infty}(\mathsf{M})$: Notice that this also preserves the boundary conditions. Moreover, if $D\underline{F} = 0$ then $D\underline{F}^{+} = -D\underline{F}^{-}$, therefore, $D\underline{F}^{+} \in \Omega^{\oplus}_{c}(\mathsf{M})$. This implies that the pairing $(D\underline{F}^{(1),+}, \underline{F}^{(2)})_{\oplus}$ is well-defined for all $\underline{F}^{(1)}, \underline{F}^{(2)} \in \mathcal{S}$.

Next we observe that $\underline{F}^{(1)}, \underline{F}^{(2)} \mapsto (D\underline{F}^{(1),+}, \underline{F}^{(2)})_{\oplus}$ is in fact independent on the splitting $\underline{F}^{(1)} = \underline{F}^{(1),+} + \underline{F}^{(1),-}$. Indeed, if $\underline{F}^{(1)} = \underline{F}^{(1),+\prime} + \underline{F}^{(1),-\prime}$ is another such splitting we have $\underline{F}^{(1),+\prime} - \underline{F}^{(1),+} = \underline{F}^{(1),-} - \underline{F}^{(1),-\prime}$ which ensures that $\underline{F}^{(1),+\prime} - \underline{F}^{(1),+} \in \Omega^{k}_{c,\mathtt{n}}(\mathsf{M})$. This implies that

$$\begin{aligned}
(D\underline{F}^{(1),+\prime}, \underline{F}^{(2)})_{\oplus} - (D\underline{F}^{(1),+}, \underline{F}^{(2)})_{\oplus} &= (D(\underline{F}^{(1),+\prime} - \underline{F}^{(1),+}), \underline{F}^{(2)})_{\oplus} \\
&= (\underline{F}^{(1),+\prime} - \underline{F}^{(1),+}, D\underline{F}^{(2)})_{\oplus} = 0\,,
\end{aligned}$$

where we applied Equation (5.14).

Thus, the map $\sigma_{\mathcal{S}} \colon \mathcal{S}^2 \to \mathbb{R}$ is well-defined and readily bilinear. We now prove that it is skew-symmetric, therefore, it provides a pre-symplectic structure on $\mathcal{S}$. To this avail let $\underline{F}^{(1)}, \underline{F}^{(2)} \in \mathcal{S}$ and consider two decompositions $\underline{F}^{(j)} = \underline{F}^{(j),+} + \underline{F}^{(j),-}$, $j \in \{1, 2\}$, as above. Then repeatedly using Equation (5.14) we have

$$\begin{aligned}
\sigma_{\mathcal{S}}(\underline{F}^{(1)}, \underline{F}^{(2)}) &= (D\underline{F}^{(1),+}, \underline{F}^{(2)})_{\oplus} \\
&= (D\underline{F}^{(1),+}, \underline{F}^{(2),+})_{\oplus} + (D\underline{F}^{(1),+}, \underline{F}^{(2),-})_{\oplus} \\
&= -(D\underline{F}^{(1),-}, \underline{F}^{(2),+})_{\oplus} + (\underline{F}^{(1),+}, D\underline{F}^{(2),-})_{\oplus} \\
&= -(\underline{F}^{(1),-}, D\underline{F}^{(2),+})_{\oplus} - (\underline{F}^{(1),+}, D\underline{F}^{(2),+})_{\oplus} \\
&= -(\underline{F}^{(1)}, D\underline{F}^{(2),+})_{\oplus} = -\sigma_{\mathcal{S}}(\underline{F}^{(1)}, \underline{F}^{(2)})\,.
\end{aligned}$$

Finally, let assume that M has a finite cover and let $\underline{F} \in \mathcal{S}$ be such that $\sigma_{\mathcal{S}}(\underline{F}', \underline{F}) = 0$. We observe that each component F_k of $\underline{F} \in \mathcal{S}$ induces an element, still denoted by F_k, of the dual space $H_{k,c,\mathtt{n}}(\mathsf{M})^*$ where

$$H_{k,c,\mathtt{n}}(\mathsf{M}) := \frac{\{\alpha_k \in \Omega^k_{c,\mathtt{n}}(\mathsf{M}) \,|\, \delta\alpha_k = 0\}}{\delta\Omega^{k+1}_{c,\mathtt{n}}(\mathsf{M})} \,.$$

Indeed $F_k([\alpha_k]) := (\alpha_k, F_k)$ is well-defined for all $[\alpha] \in H_{k,c,\mathtt{n}}(\mathsf{M})$ on account of the identity $(\delta\beta_{k+1}, F_k) = (\beta_{k+1}, \mathrm{d}F_k) = 0$ for all $\beta_{k+1} \in \Omega^{k+1}_{c,\mathtt{n}}(\mathsf{M})$. Notice that, since M has a good cover, $H_{k,c,\mathtt{n}}(\mathsf{M})^* \simeq H^k(\mathsf{M})$, where $H^k(\mathsf{M})$ is the standard k-th de Rham cohomology group, *cf.* [9,19] and [11, App. C]. A similar argument shows that the assignment $\alpha_k \mapsto F_k([\zeta_k]) := (\zeta_k, F_k)$ defines an element in $H^k_c(\mathsf{M})^* \simeq H_{k,\mathtt{n}}(\mathsf{M})$.

On account of (5.15) we have

$$\underline{F}' = \underline{G}(\underline{\alpha} \oplus \underline{\zeta}) \,, \qquad \underline{\alpha} \oplus \underline{\zeta} \in \bigoplus_{k=1}^{n} \frac{\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})}{D\Omega^k_{c,\mathtt{n}}(\mathsf{M})} \,.$$

Thus, we may set $\underline{F}'^{,+} := \underline{G}^+(\underline{\alpha} \oplus \underline{\zeta})$ which leads to

$$\sigma_{\mathcal{S}}(\underline{F}', \underline{F}) = (D\underline{G}^+(\underline{\alpha} \oplus \underline{\zeta}), \underline{F})_\oplus = (\underline{\alpha} \oplus \underline{\zeta}, \underline{F})_\oplus \,.$$

The condition $\sigma_{\mathcal{S}}(\underline{F}', \underline{F}) = 0$ and the arbitrariness of $\underline{\alpha}$ implies in particular that $(\alpha_k, F_k) = 0$ for all $\alpha_k \in \Omega^k_{c,\mathtt{n},\delta}(\mathsf{M})$ and $k \in \{0, \ldots, n\}$. This entails that $F_k = 0 \in H_{k,c,\mathtt{n}}(\mathsf{M})^* \simeq H^k(\mathsf{M})$, that is, $F_k = \mathrm{d}A_{k-1}$: Thus, $\underline{F} = \mathrm{d}\underline{A}$. With a similar argument, the arbitrariness of $\underline{\zeta}$ leads to $(\zeta_k, F_k) = 0$ for all $\zeta_k \in \Omega^k_{c,\mathrm{d}}(\mathsf{M})$ which implies $F_k = 0 \in H^k_c(\mathsf{M})^* \simeq H_{k,\mathtt{n}}(\mathsf{M})$, therefore $F_k = \delta B_{k+1}$ for $B_{k+1} \in \Omega^{k+1}_{\mathtt{n}}(\mathsf{M})$.

Conversely, by direct inspection any element $\underline{F} \in \mathcal{S}$ such that $\underline{F} = \mathrm{d}\underline{A} = \delta\underline{B}$ for $\underline{B} \in \Omega^{\oplus}_{\mathtt{n}}(\mathsf{M})$ fulfils $\sigma_{\mathcal{S}}(\ , \underline{F}) = 0$. □

Remarks 5.4.

1. The pre-symplectic form $\sigma_{\mathcal{S}}$ involves forms of different degrees in a non-trivial fashion. In particular, this spoils the possibility of inducing a pre-symplectic form on a single component of $\underline{F} \in \mathcal{S}$. At its core, this difficulty is due to the different degrees in the domain and codomain of the operators $G^\pm_k$, *cf.* Proposition 5.1. Moreover, the degeneracy space of $\sigma_{\mathcal{S}}$ coincides with the space of spacelike solutions to Maxwell's equation for the electromagnetic potential [11, Def. 28]. These two facts do not allow a clear physical interpretation of the resulting structure.

   For the purpose of quantizing the solution space $\mathrm{Sol}^k_{sc,\mathtt{n}}(\mathsf{M})$ for fixed k it is likely more appropriate to proceed as in [12], which is based on the connection with the solution space to the wave operator $\Box$. For the case at hand, such connection would require the identification of appropriate boundary conditions which guarantee formal self-adjointness of $\Box$. The latter can be easily determined by observing that any $F \in \mathrm{Sol}^k_{sc,\mathtt{n}}(\mathsf{M})$ fulfils

$\mathtt{n}\lrcorner F = 0$ as well as $\mathtt{n}\lrcorner \mathrm{d}F = 0$. Moreover, $(\Box\alpha_k, \beta_k) = (\alpha_k, \Box\beta_k)$ if $\alpha_k, \beta_k \in \Omega^k(\mathsf{M})$ are such that $\mathrm{supp}(\alpha_k) \cap \mathrm{supp}(\beta_k)$ is compact and $\mathtt{n}\lrcorner\alpha_k = \mathtt{n}\lrcorner\beta_k = 0$ as well as $\mathtt{n}\lrcorner\mathrm{d}\alpha_k = \mathtt{n}\lrcorner\mathrm{d}\beta_k = 0$. Forms abiding by these boundary conditions were investigated in [11], which deals with the quantization of the electromagnetic vector potential in the framework of gauge theories.

2. Similarly to [5,11] one may promote the isomorphism of vector spaces (5.15) to an isomorphism of pre-symplectic vector spaces. This requires to define a pre-symplectic form

$$\varsigma_{\mathcal{S}} \colon \left(\bigoplus_{k=1}^{n} \frac{\Omega^{k-1}_{c,\mathtt{n},\delta}(\mathsf{M}) \oplus \Omega^{k+1}_{c,\mathrm{d}}(\mathsf{M})}{D\Omega^k_{c,\mathtt{n}}(\mathsf{M})}\right)^2 \to \mathbb{R}$$
$$\varsigma_{\mathcal{S}}(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}, \underline{\alpha}^{(2)} \oplus \underline{\zeta}^{(2)}) := (\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}, \underline{G}(\underline{\alpha}^{(2)} \oplus \underline{\zeta}^{(2)}))_{\oplus}\,,$$

from which $\sigma_{\mathcal{S}}(\underline{G}(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}), \underline{G}(\underline{\alpha}^{(2)} \oplus \underline{\zeta}^{(2)})) = \varsigma_{\mathcal{S}}(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}, \underline{\alpha}^{(2)} \oplus \underline{\zeta}^{(2)})$ follows by decomposing $\underline{G}(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}) = \underline{G}^+(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}) - \underline{G}^-(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)})$ together with the observation that $D\underline{G}^+(\underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}) = \underline{\alpha}^{(1)} \oplus \underline{\zeta}^{(1)}$.

References

[1] L. Aké, J. L. Flores and M. Sánchez, *Structure of globally hyperbolic spacetimes with timelike boundary.* Rev. Mat. Iberoam. 37 (2021), 45–94.

[2] C. Bär, *Green-hyperbolic operators on globally hyperbolic spacetimes,* Comm. Math. Phys. **333** (2015), 1585.

[3] C. Bär and N. Ginoux, *Classical and Quantum Fields on Lorentzian Manifolds*, Global Differential Geometry, Springer Proceedings in Math. 17, Springer-Verlag (2011) 359–400.

[4] C. Bär, N. Ginoux and F. Pfäffle, *Wave equations on Lorentzian Manifolds and Quantization.* ESI Lectures in Mathematics and Physics (2007).

[5] M. Benini, *Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies*, J. Math. Phys. **57** (2016), 053502.

[6] M. Benini, C. Dappiaggi, T.-P. Hack and A. Schenkel, *A C^*-algebra for quantized principal $U(1)$-connections on globally hyperbolic Lorentzian manifolds*, Commun. Math. Phys. 332, 477 (2014)

[7] M. Benini, C. Dappiaggi and A. Schenkel, *Quantized Abelian principal connections on Lorentzian manifolds*, Commun. Math. Phys. 330, 123 (2014).

[8] M. Benini, C. Dappiaggi and A. Schenkel, *Algebraic quantum field theory on spacetimes with timelike boundary*, Annales Henri Poincaré, 19(8), 2401-2433 (2018).

[9] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Springer, Berlin (1982).

[10] R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason, *Advances in algebraic quantum field theory.* Springer (2015)

[11] C. Dappiaggi, N. Drago and R. Longhi, *On Maxwell's equations on globally hyperbolic spacetimes with timelike boundary*, Ann. Henri Poincaré, **21** (2020), 2367–2409.

[12] C. Dappiaggi and B. Lang, *Quantization of Maxwell's equations on curved backgrounds and general local covariance.* Lett. Math. Phys. **101** (2012), 265–287.

[13] J. Dimock, *Quantized Electromagnetic field on a manifold*, Rev. Math. Phys. **04** (1992), 223–233.

[14] N. Drago, N. Ginoux and S. Murro, *Møller operators and Hadamard states for Dirac fields with MIT boundary conditions,* Doc. Math. **27** (2022), 1643–1687.

[15] K. Fredenhagen and K. Rejzner, *Quantum field theory on curved spacetimes: Axiomatic framework and examples,* J. Math. Phys. **57** (2016), 031101 .

[16] C. Gérard, *Microlocal Analysis of Quantum Fields on Curved Spacetimes.* ESI Lectures in Mathematics and Physics (2019).

[17] N. Ginoux and S. Murro, *On the Cauchy problem for Friedrichs systems on globally hyperbolic manifolds with timelike boundary*, Ad. Differential Equations **27** (2022), 497–542.

[18] J.J. Sakurai, *Modern Quantum Mechanics.* Addison-Wesley Publishing (1994).

[19] G. Schwarz, *Hodge Decomposition - A Method for Solving Boundary Value Problems*, Springer, Berlin (1995).