CW-structure of real Grassmannians

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Abstract: We describe the standard CW-structure of the Grassmannians $G_n(\mathbb{R}^{n+k})$ and $G_n(\mathbb{R}^\infty)$. We stick to [1, App. pp. 519-523] for basics on CW-complexes and to [2, Sec. 1.2 pp. 27-34] for the CW-structure itself.

1 CW-complexes

Definition 1.1 (inductive definition) *A* CW-complex *is a Hausdorff topological space X which can be written as $X = \bigcup_{n\in\mathbb{N}} X_n$, where:*

i) for $n = 0$ the subset X_0 is a discrete set (collection of points with the discrete topology);

ii) for each $n \geq 1$ the subset X_n arises as $X_n = X_{n-1} \bigcup_{f^n} \coprod_{\alpha\in I_n} D^n_\alpha$, where I_n is an arbitrary set, $D^n_\alpha := D^n := \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$ is the usual closed n-dimensional ball and $f^n : \coprod_{\alpha\in I_n} \partial D^n_\alpha \longrightarrow X_{n-1}$ is a continuous map; the topology of X_n is the quotient topology induced by f^n, the standard topology on D^n_α and that of X_{n-1};

iii) the space X has the topology induced by the direct limit of the non-decreasing family $(X_n)_n$ (with inclusions as maps), that is, a subset Ω of X is open in X if and only if $\Omega \cap X_n$ is open in X_n for all n.

Recall that, for two topological spaces X, Y and a map $f : A \longrightarrow X$ defined on a subset A of Y, the space $X \bigcup_f Y$ is the quotient set ${X \coprod Y}/{a \sim f(a)}$ endowed with the quotient topology.

The subspace X_n is called *n-dimensional skeleton* of X. An n-dimensional (open) *cell* of X is the homeomorphic image under the quotient map of $\mathrm{int}(D^n_\alpha) := D^n_\alpha \setminus \partial D^n_\alpha$ for some $\alpha \in I_n$. We shall denote that cell by e^n_α. Each cell e^n_α has a so-called *characteristic map* $\phi^n_\alpha : D^n_\alpha \longrightarrow X$, which is the composition $D^n_\alpha \stackrel{\text{incl.}}{\to} \coprod_{\beta \in I_n} D^n_\beta \stackrel{\text{incl.}}{\to} X_{n-1} \coprod \coprod_{\beta \in I_n} D^n_\beta \stackrel{\text{proj.}}{\to} X_n \stackrel{\text{incl.}}{\to} X$. By construction, ϕ^n_α is continuous and maps $\mathrm{int}(D^n_\alpha)$ homeomorphically onto the open cell e^n_α.

By definition, each CW-complex can be written as the disjoint union of its open cells (of different dimensions). Note that this decomposition into cells need not be unique; e.g. a circle can be written as a CW-complex with one 0- and one 1-cell or with two 0- and two 1-cells. A *finite* CW-complex is a CW-complex having only a finite number of cells (in that case, $X = X_n$ for some $n \in \mathbb{N}$). A *subcomplex* of a CW-complex X is a closed subset $A \subset X$ which the union of cells of X.

Standard examples of CW-complexes include spheres, (real or complex) projective spaces and... Grassmannians, see Section 2.

A CW-complex can be completely described by its cells and the corresponding characteristic maps:

Proposition 1.2 (direct definition) *Let X be a Hausdorff topological space and $\phi^n_\alpha : D^n_\alpha \longrightarrow X$ be a family of maps, where $\alpha \in I_n$ for some (possibly empty) set I_n and n runs over $\mathbb{N}$[1]. Then the $(\phi^n_\alpha)_{\alpha,n}$ are the characteristic maps of a CW-structure on X if and only if the following conditions are fulfilled:*

1. *each ϕ^n_α is continuous and maps $\mathrm{int}(D^n_\alpha)$ homeomorphically onto its image, which we denote e^n_α;*
2. *the e^n_α's are disjoint from each other and their union is X;*
3. *for all α, n the subset $\phi^n_\alpha(\partial D^n_\alpha)$ lies in a finite union of e^k_β's, where $k \le n-1$;*
4. *a subset $A \subset X$ is closed in X if and only if $A \cap \overline{e^n_\alpha}$ is closed for all α, n.*

The proof uses the fact that a subset $A \subset X$ is closed in X if and only if $(\phi^n_\alpha)^{-1}(A)$ is closed in D^n_α for all α, n. We refer to [1, App. pp. 519-523] for

[1] Note this $\mathbb{N}_0$ if you prefer.

a proof of Proposition 1.2 and further basic or less basic remarks on the topology of CW-complexes (e.g. that CW-complexes are always paracompact and locally contractible).

Further on in this seminar we shall make use of the following

Theorem 1.3 (Whitehead) *Let $f : X \longrightarrow Y$ be a continuous map between connected CW-complexes. Assume $\pi_n(f) : \pi_n(X) \longrightarrow \pi_n(Y)$ to be a group-isomorphism for all $n \in \mathbb{N}$. Then the map f is a homotopy equivalence.*

The reverse statement ("*f homotopy equivalence $\Longrightarrow$ all $\pi_n(f) : \pi_n(X) \longrightarrow \pi_n(Y)$ are group-isomorphisms*") is, of course, trivial. For the introduction of higher homotopy groups and the proof of Theorem 1.3, we refer to [1, Sec. 4.1].

2 Real Grassmannians as CW-complexes

Recall that the (real) Grassmannian of n-dimensional vector subspaces in $\mathbb{R}^{n+k}$ (where $n, k \in \mathbb{N}$) is defined as

$$G_n(\mathbb{R}^{n+k}) := \{n\text{-dimensional vector subspaces of } \mathbb{R}^{n+k}\}.$$

It is a closed manifold which is homeomorphic to the nk-dimensional homogeneous space ${}^{\mathrm{O}(n+k)}/_{\mathrm{O}(n)\times\mathrm{O}(k)}$. In just the same way one can define the Grassmannian $G_n(\mathbb{R}^\infty)$ as the collection of all n-dimensional vector subspaces of $\mathbb{R}^\infty := \bigoplus_{l\in\mathbb{N}} \mathbb{R}$. Notice that it can be written as the direct limit $G_n(\mathbb{R}^\infty) = \bigcup_{k\in\mathbb{N}} G_n(\mathbb{R}^{n+k})$ (with inclusions as maps). We endow $G_n(\mathbb{R}^\infty)$ with the topology induced by that direct limit.

We begin by fixing n, k and look for a CW-structure on $G_n(\mathbb{R}^{n+k})$. This is done with the help of the so-called *Schubert symbols* of a matrix. Given $V \in G_n(\mathbb{R}^{n+k})$, choose a basis of V. W.r.t. the canonical coordinates of $\mathbb{R}^{n+k}$, this basis defines a matrix $A \in \mathbb{M}_{(n+k)\times n}(\mathbb{R})$ of rank n. Elementary operations on the columns of A (see below) allow for A to admit the following "echelon" form: there exists an n-tuple $(\sigma_1, \ldots, \sigma_n) \in \{1, \ldots, n+k\}^n$ with $1 \leq \sigma_1 < \ldots < \sigma_n \leq n+k$ such that $A_{\sigma_j j} = 1$, $A_{ij} = 0$ for all $i > \sigma_j$ and $A_{\sigma_j k} = 0$ for all $k \neq j$ [figure]. Namely the following algorithm can be implemented, where the only allowed operations on the columns are permutations between them or replacing a column by a "non-trivial" linear combination of them (non-trivial so as to keep the rank maximal):

- For each $j \in \{1, \ldots, n\}$ let $m_j := \max(\{i \,|\, A_{ij} \neq 0\})$ and j_0 with $m_{j_0} = \max\limits_{1 \leq j \leq n}(m_j)$ (this j_0 need not be unique of course). Up to permuting the columns we can assume that $j_0 = n$ and, up to rescaling the n^{th} column by a non-zero scalar, we can assume that $A_{m_{j_0}n} = 1$. We set $\sigma_n := m_{j_0}$ and note that, by definition of σ_n, we have $A_{ij} = 0$ for all $i > \sigma_n$ and all j. Replacing[2] the column $A_{\cdot j}$ by $A_{\cdot j} - A_{\sigma_n j} A_{\cdot n}$ for all $j < n$ (and this is an invertible transformation since in the basis $(A_{\cdot j})_j$ its matrix is upper triangular with only 1's on the diagonal) we can achieve $A_{\sigma_n j} = 0$ for all $j < n$.

- In the same way, we proceed inductively on the remaining columns $(A_{\cdot j})_{1 \leq j \leq n-1}$. Note that, at each step, necessarily $\sigma_j < \sigma_{j+1}$ holds because of $A_{\sigma_{j+1} k} = 0$ for all $k < j + 1$. One should also pay attention to the elimination of the coefficients A_{ij} for $i \neq \sigma_j$: namely for $i < \sigma_j$ it is the operation analogous to the one above, for $i > \sigma_j$ we can still do the same since the coefficients A_{kl} remain unaffected whenever $k > \sigma_j$ and $l > j$ (because of $A_{kj} = 0$ for all $k > \sigma_j$) [figure].

- Obviously the number of σ_j's we obtain at the end is the rank of the matrix, which has not changed in the process, therefore we obtain the desired tuple $1 \leq \sigma_1 < \ldots < \sigma_n \leq n + k$.

By the choice of operations on the columns of the original matrix, the columns of the new echelon matrix still span V. However, it could be *a priori* possible for different echelon forms of A to exist. This is not the case. First, the tuple $(\sigma_1, \ldots, \sigma_n)$ has to be independent of the echelon form. To see this, consider the family of canonical projections $p_j : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k-j}$ onto the last $n+k-j$ coordinates, $0 \leq j \leq n+k$. Then the map $\{0, \ldots, n+k\} \to \mathbb{N}$, $j \mapsto d_j := \dim(p_j(V))$, is obviously non-increasing with $d_0 = n$ and $d_{n+k} = 0$ and the j's with $d_{j-1} > d_j$ are exactly the σ_j's. In particular those are uniquely determined by V and not by the choice of echelon basis. Moreover, two echelon bases $(v_1, \ldots, v_n)$ and $(\tilde{v}_1, \ldots, \tilde{v}_n)$ have to coincide: for each j decompose $\tilde{v}_j$ in the basis $(v_1, \ldots, v_n)$, then necessarily $\tilde{v}_j \in \mathrm{Span}(v_1, \ldots, v_j)$ because of $(\tilde{v}_j)_i = 0$ for all $i > \sigma_j$; moreover $\tilde{v}_j$ has to be proportional to v_j because of $(\tilde{v}_j)_{\sigma_i} = 0$ for all $i < j$; finally $\tilde{v}_j = v_j$ because of $(v_j)_{\sigma_j} = (\tilde{v}_j)_{\sigma_j} = 1$.

On the whole, we obtain the

Lemma 2.1 *Each $V \in G_n(\mathbb{R}^{n+k})$ has a unique echelon basis and in particular provides a unique $(\sigma_1, \ldots, \sigma_n)$ as above.*

[2] We thank Nikolai Nowaczyk for correcting a mistake here.

The tuple $(\sigma_1, \ldots, \sigma_n)$ is called the *Schubert symbol* of the vector subspace V. Now fix such a tuple $\sigma := (\sigma_1, \ldots, \sigma_n)$ and consider the set $e(\sigma)$ of those $V \in G_n(\mathbb{R}^{n+k})$ having Schubert-symbol σ. Then $e(\sigma)$ canonically stands in one-to-one correspondence with $\mathbb{R}^l$, where l is the number of free entries in the echelon basis, thus $l = \sum_{j=1}^n \sigma_j - j = \sum_{j=1}^n \sigma_j - \frac{n(n+1)}{2}$. Using the definition of the quotient topology on $G_n(\mathbb{R}^{n+k})$, the subset $e(\sigma)$ equipped with the induced topology is even homeomorphic to $\mathbb{R}^l$, hence to $\mathrm{int}(D^l)$. The set $e(\sigma)$ is called *Schubert cell* associated to σ. It is not called "cell"by chance:

Proposition 2.2 *The Schubert cells are the cells of a CW-structure on the Grassmannian $G_n(\mathbb{R}^{n+k})$. In particular $G_n(\mathbb{R}^{n+k})$ is a finite CW-complex with $\begin{pmatrix} n+k \\ n \end{pmatrix}$ cells.*

Proof: By Lemma 2.1, cells corresponding to different Schubert symbols are disjoint. Obviously, the union of all cells is $G_n(\mathbb{R}^{n+k})$. To describe the gluing maps, we introduce a slightly different echelon form as the one above. More precisely, given $V \in G_n(\mathbb{R}^{n+k})$, we claim the existence of a unique *orthonormal* basis of V (w.r.t. the standard inner product on $\mathbb{R}^{n+k}$) and a tuple $1 \leq \sigma_1 < \ldots < \sigma_n \leq n+k$ such that, for the matrix A of that basis in the canonical coordinates, we have $A_{ij} = 0$ for all $i > \sigma_j$ and $A_{\sigma_j j} > 0$. For instance, the Gram-Schmidt process applied to the echelon basis of Lemma 2.1 provides such a basis. Notice in particular that the row indices where the "steps" appear coincide with the components of the Schubert symbol of V. Again, there is actually only one such orthonormal echelon basis for V: if $(w_1, \ldots, w_n)$ and $(\tilde{w}_1, \ldots, \tilde{w}_n)$ are two such bases, then argueing inductively on j one shows that first $\tilde{w}_j \in \mathrm{Span}(\{w_k,\, 1 \leq k \leq j\})$ using the echelon form, second $\tilde{w}_j \in \mathbb{R} \cdot w_j$ using the induction step, third $\tilde{w}_j = \pm w_j$ using $|\tilde{w}_j| = |w_j| = 1$ and finally $\tilde{w}_j = w_j$ because of $(\tilde{w}_j)_{\sigma_j}, (w_j)_{\sigma_j} > 0$.
Now, given a Schubert symbol σ, consider

$$E(\sigma) := \{(w_1, \ldots, w_n) \in \mathbb{R}^{(n+k)n} \,|\, \langle w_i, w_j \rangle = \delta_{ij} \text{ and } w_j \in \mathbb{S}_+^{\sigma_j - 1} \text{ for all } i, j\},$$

where $\mathbb{S}_+^l := \{x \in \mathbb{R}^{l+1} \,|\, |x| = 1 \text{ and } x_{l+1} \geq 0\}$ is the l-dimensional *closed* upper half-sphere. Here and as usual, we identify $\mathbb{R}^l \cong \mathbb{R}^l \times \{0_{n+k-l}\} \subset \mathbb{R}^{n+k}$. The set $E(\sigma)$ can be seen as the set of orthonormal echelon bases of all $V \in G_n(\mathbb{R}^{n+k})$ having Schubert symbol σ, enlarged by those having Schubert symbol σ' with $\sigma'_j \leq \sigma_j$ for all j and $\sigma'_{j_0} < \sigma_{j_0}$ for at least one j_0 (for in $E(\sigma)$ we allow for the component $(w_j)_{\sigma_j}$ to vanish).
Claim: *The set $E(\sigma)$ - endowed with the topology induced from $\mathbb{R}^{(n+k)n}$ - is homeomorphic to the closed ball D^q, where $q = \sum_{j=1}^n \sigma_j - j$.*

Proof of the claim: We proceed inductively on n. For $n=1$ the result is trivial since then $E(\sigma)=\mathbb{S}_+^{\sigma_1-1}\cong D^{\sigma_1-1}$ (this remains true if $\sigma_1=1$). For the step from $n-1$ to n we consider the projection onto the first factor $\pi : E(\sigma)\longrightarrow \mathbb{S}_+^{\sigma_1-1}$ and show that it is isomorphic (hence homeomorphic) to the trivial fibre bundle $\mathbb{S}_+^{\sigma_1-1}\times E(\sigma')\longrightarrow \mathbb{S}_+^{\sigma_1-1}$, where $\sigma' := (\sigma_2-1,\dots,\sigma_n-1)$; the induction assumption then shows that $E(\sigma')\cong D^{l'}$ with $l'=\sum_{j=2}^n \sigma_j-1-(j-1)=\sum_{j=2}^n \sigma_j - j$, hence $E(\sigma)\cong D^q$ with $q=\sigma_1-1+l'=\sum_{j=1}^n\sigma_j-j$, which is the result. To prove that π is isomorphic to the above trivial fibre bundle, we construct as in [2] a continuous map $p : E(\sigma)\longrightarrow \pi^{-1}(\{n_{\sigma_1}\})$ which maps each fibre $\pi^{-1}(\{x\})$ homeomorphically onto $\pi^{-1}(\{n_{\sigma_1}\})$ and such that $p_{|_{\pi^{-1}(\{n_{\sigma_1}\})}}=\mathrm{id}$. Here $n_{\sigma_1}:=(0,\dots,0,1)\in\mathbb{S}^{\sigma_1-1}$ denotes the North pole of $\mathbb{S}^{\sigma_1-1}$. Note that

$$\begin{aligned}
\pi^{-1}(\{n_{\sigma_1}\}) &= \{(n_{\sigma_1},w_2,\dots,w_n)\in E(\sigma)\}\\
&\cong \{(w_2,\dots,w_n)\in \prod_{j=2}^n \mathbb{S}_+^{\sigma_j-1}\,|\,(w_j)_{\sigma_1}=0 \text{ and } \langle w_i,w_j\rangle=\delta_{ij}\}\\
&\cong \{(\hat{w}_2,\dots,\hat{w}_n)\in \prod_{j=2}^n \mathbb{S}_+^{\sigma_j-2}\,|\,\langle \hat{w}_i,\hat{w}_j\rangle=\delta_{ij}\}\\
&\cong E((\sigma_2-1,\dots,\sigma_n-1)).
\end{aligned}$$

The map p can be defined as follows. Given $v\in\mathbb{S}^{\sigma_1-1}$, let $r_v\in \mathrm{O}(n+k)$ denote an[3] orthogonal map sending v on n_{σ_1} and restricting to the identity on $\{v,n_{\sigma_1}\}^\perp$ (for example the direct sum of a rotation in the plane $\mathrm{Span}(v,n_{\sigma_1})$ with $\mathrm{id}_{\{v,n_{\sigma_1}\}^\perp}$ does the job). In case $v=n_{\sigma_1}$ just take $r_v=\mathrm{id}$. Then set $p(v_1,\dots,v_n):=(n_{\sigma_1},r_{v_1}(v_2),\dots,r_{v_1}(v_n))$ for all $(v_1,\dots,v_n)\in E(\sigma)$. Note that $p(v_1,\dots,v_n)$ is orthonormal because of $r_{v_1}\in\mathrm{O}(n+k)$ and that $r_{v_1}(\mathbb{S}_+^{\sigma_j-1})\subset\mathbb{S}_+^{\sigma_j-1}$ for all $j\geq 2$ because of $n_j\in\{v_1,n_{\sigma_1}\}^\perp$ for all $j>\sigma_1$; in particular $p(v_1,\dots,v_n)\in\pi^{-1}(\{n_{\sigma_1}\})$. The map $p: E(\sigma)\longrightarrow\pi^{-1}(\{n_{\sigma_1}\})$ can be shown to be continuous and fibrewise bijective (since r_{v_1} is bijective $\mathbb{S}_+^{\sigma_j-1}\to\mathbb{S}_+^{\sigma_j-1}$ for each $j\geq 2$). Since all fibres of π are compact and Hausdorff, the map p induces a homeomorphism $\pi^{-1}(\{v_1\})\to\pi^{-1}(\{n_{\sigma_1}\})$ for each $v_1\in\mathbb{S}_+^{\sigma_1-1}$. Now $\pi\times p : E(\sigma)\longrightarrow \mathbb{S}_+^{\sigma_1-1}\times\pi^{-1}(\{n_{\sigma_1}\})$ is continuous, bijective (since p is fibrewise bijective) and because of $E(\sigma)$ being compact (it is a closed subset of a compact set) the map $\pi\times p$ is actually a homeomorphism. This concludes the proof of the claim. $\surd$

The map $\phi_\sigma : E(\sigma)\longrightarrow G_n(\mathbb{R}^{n+k})$, $(v_1,\dots,v_n)\longmapsto \mathrm{Span}(v_1,\dots,v_n)$, is continuous by definition of the quotient topology on $G_n(\mathbb{R}^{n+k})$. Its restriction to $\mathrm{int}(E(\sigma))$ induces a bijective map $\mathrm{int}(E(\sigma))\to e(\sigma)$ by the existence and

[3] Actually the map r_v can be constructed so as to *continuously* depend on v.

uniqueness of orthonormal echelon bases and the coincidence of the row indices where the "steps" appear. By Brouwer's theorem on the invariance of the domain, $\phi_{\sigma}|_{\mathrm{int}(E(\sigma))}$ must be a homeomorphism $\mathrm{int}(E(\sigma)) \to e(\sigma)$. Moreover, $\phi_{\sigma}(\partial E(\sigma))$ is contained in a finite union of $e(\sigma')$ with $\sigma'_j \leq \sigma_j$ for all j and $\sigma'_{j_0} < \sigma_{j_0}$ for at least one j_0, see definition of $E(\sigma)$ above. On the whole, $G_n(\mathbb{R}^{n+k})$ can be constructed inductively (as in Definition 1.1) by beginning with a point (take $\sigma = (1, \ldots, n)$) and by attaching at each step finitely many cells of the next possible dimension using the ϕ_{σ} above. This concludes the proof of Proposition 2.2. $\square$

Since $G_n(\mathbb{R}^{\infty})$ can be obtained as the (topological) direct limit of the CW-complexes $G_n(\mathbb{R}^{n+k})$ and each inclusion $G_n(\mathbb{R}^{n+k}) \subset G_n(\mathbb{R}^{n+k+1})$ obviously maps $G_n(\mathbb{R}^{n+k})$ onto a subcomplex of $G_n(\mathbb{R}^{n+k+1})$, the Grassmannian $G_n(\mathbb{R}^{\infty})$ has a unique CW-structure such that all inclusions $G_n(\mathbb{R}^{n+k}) \subset G_n(\mathbb{R}^{\infty})$ are again inclusions of subcomplexes.

References

[1] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[2] A. Hatcher, *Vector bundles and K-theory*, lecture notes, available at `http://www.math.cornell.edu/~hatcher`.