International Journal of Mathematics, Vol. 14, No. 1 (2002) 1–16

ON EIGENVALUE ESTIMATES FOR THE SUBMANIFOLD DIRAC OPERATOR

NICOLAS GINOUX* and BERTRAND MOREL†

Institut Élie Cartan, Université Henri Poincaré
Nancy I, B.P. 239, 54506 Vandœuvre-Lès-Nancy Cedex, France
**ginoux@iecn.u-nancy.fr*
†morel@iecn.u-nancy.fr

Received 2 October 2001

We give lower bounds for the eigenvalues of the submanifold Dirac operator in terms of intrinsic and extrinsic curvature expressions. We also show that the limiting cases give rise to a class of spinor fields generalizing that of Killing spinors. We conclude by translating these results in terms of intrinsic twisted Dirac operators.

Keywords and phrases: Dirac operator; conformal geometry; spectrum; submanifolds; energy-momentum tensor.

Mathematics Subject Classification 2000: 53C27, 53C40, 53C80, 58G25

1. Introduction

It is well known that limiting cases in classical estimates for the eigenvalues of the fundamental Dirac operator on a compact manifold without boundary [5, 8] give rise to special geometries. Indeed, these limiting cases are characterized by the existence of special spinor fields, such as Killing spinor fields which imply severe restrictions on the holonomy [1]. Considering hypersurfaces bounding a domain, the hypersurface Dirac operator has been introduced by E. Witten to prove the positive mass theorem [15]. The spinorial background that has been developed to extend the classical estimates to hypersurfaces has now become a powerful tool to investigate extrinsic geometry and manifolds with boundary problems (see e.g. [10, 11]).

In this direction, the spectrum of the submanifold Dirac operator has been studied in [9], where some estimates are obtained for odd codimensions. In this paper, we first give new lower bounds for the eigenvalues of the submanifold Dirac operator (Theorems 3.5 and 3.6) and discuss their limiting cases.

We start by restricting the spinor bundle of a Riemannian spin manifold to a spin submanifold endowed with the induced metric. We then relate this bundle to the twisted spinor bundle on the submanifold. For further study of the limiting cases, we have to adapt the algebraic identifications of the spinor spaces and Clifford multiplications given in [2].

Defining appropriate Dirac operators and relating them with the help of the spinorial Gauss formula, the submanifold Dirac operator D_H is the natural generalization of the hypersurface Dirac operator (see for example [14,16]). We then get lower bounds for the eigenvalues of D_H in terms of the norm of the mean curvature vector, the Energy-Momentum tensor associated with an eigenspinor, and an adapted conformal change of the metric.

Lower bounds also involve the scalar curvature of the submanifold as well as a normal curvature term which only appears in codimension greater than one.

As a consequence of our definitions, the established estimates hold *for all codimensions* (compare with [9]).

Our identifications allow to discuss the limiting cases in terms of special sections of the spinor bundle. These particular spinor fields generalize the notion of Killing spinors to the spinor bundle of the submanifold twisted with the normal spinor bundle.

The main point of this paper is that such estimates (see also [9, 16, 17]) can always be discussed in an intrinsic way by considering any auxiliary vector bundle attached to a manifold instead of the normal bundle of a submanifold (see Theorems 4.1–4.4).

2. Dirac Operators on Submanifolds

2.1. *Algebraic preliminaries*

In this section, we adapt algebraic material developped by C. Bär in [2]. Basic facts concerning spinor representations can be found in classical books (see [3, 6, 12] or [4]).

Let m and n be two integers, we start by constructing an irreducible representation of the complex Clifford algebra $\mathbb{C}l_{m+n}$ from irreducible representations ρ_n and ρ_m of $\mathbb{C}l_n$ and $\mathbb{C}l_m$ respectively. Let Σ_p be the space of complex spinors for the representation ρ_p. Recall that if p is even, ρ_p is unique up to an isomorphism, and if p is odd, there are two inequivalent irreducible representations of $\mathbb{C}l_p$; in this case (ρ_p^j, Σ_p^j), $j = 0, 1$, denotes the representation which sends the complex volume form to $(-1)^j \operatorname{Id}_{\Sigma_p^j}$. So we have to consider four cases according to the parity of m and n.

First case: Assume that n and m are even. Define

$$\begin{aligned}\gamma : \mathbb{R}^m \oplus \mathbb{R}^n &\longrightarrow \operatorname{End}_{\mathbb{C}}(\Sigma_m \otimes \Sigma_n)\\ (v, w) &\longmapsto \rho_m(v) \otimes (\operatorname{Id}_{\Sigma_n^+} - \operatorname{Id}_{\Sigma_n^-}) + \operatorname{Id}_{\Sigma_m} \otimes \rho_n(w)\,,\end{aligned}$$

where $\Sigma_n^\pm$ is the ± 1-eigenspace for the action of the complex volume form ω_n of $\mathbb{C}l_n$. Recall that $\omega_n = i^{[\frac{n+1}{2}]} e_1 \cdot \ldots \cdot e_n$, where $(e_1, \ldots, e_n)$ stands for any positively oriented orthonormal basis of $\mathbb{R}^n$ and '$\cdot$' denotes the Clifford multiplication in $\mathbb{C}l_n$.

Then, for $\sigma \in \Sigma_m$, $\theta \in \Sigma_n$, for any vectors $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$, we have:

$$\begin{aligned}\gamma(v+w)^2(\sigma\otimes\theta) &= \rho_m(v)^2\sigma\otimes(\theta^+ + \theta^-) + \rho_m(v)\sigma\otimes(\rho_n(w)\theta^- - \rho_n(w)\theta^+)\\ &\quad + \rho_m(v)\sigma\otimes(\rho_n(w)\theta^+ - \rho_n(w)\theta^-) + \sigma\otimes\rho_n(w)^2\theta\\ &= -(|v|^2+|w|^2)\sigma\otimes\theta\,.\end{aligned}$$

Therefore, since $\gamma(v+w)^2 = -|v+w|^2\,\mathrm{Id}$, the map γ induces a non-trivial complex representation of $\mathbb{C}l_{m+n}$ of dimension $2^{\frac{m+n}{2}}$ and so γ is equivalent to ρ_{m+n}.

With respect to the inclusions of $\mathbb{C}l_m$ and $\mathbb{C}l_n$ in $\mathbb{C}l_{m+n}$ corresponding to

$$\begin{aligned}\mathbb{R}^m &\longrightarrow \mathbb{R}^{m+n} = \mathbb{R}^m\oplus\mathbb{R}^n &\quad\text{and}\quad \mathbb{R}^n &\longrightarrow \mathbb{R}^{m+n} = \mathbb{R}^m\oplus\mathbb{R}^n\\ v &\longmapsto (v,0) & w &\longmapsto (0,w)\,,\end{aligned}$$

we can write

$$\begin{aligned}\omega_{m+n} &= i^{[\frac{m+n+1}{2}]}e_1\cdot\ldots\cdot e_{m+n}\\ &= i^{[\frac{m+1}{2}]}i^{[\frac{n+1}{2}]}e_1\cdot\ldots\cdot e_m\cdot e_{m+1}\cdot\ldots\cdot e_{m+n}\\ &= \omega_m\cdot\omega_n\,.\end{aligned}\tag{2.1}$$

On the other hand, if $\sigma \in \Sigma_m$ and $\theta \in \Sigma_n$, then for all $v \in \mathbb{R}^m$,

$$\gamma(v\cdot\omega_n)(\sigma\otimes\theta) = \rho_m(v)\sigma\otimes\theta\,.\tag{2.2}$$

Therefore, since m is even, we have

$$\gamma(\omega_{m+n})(\sigma\otimes\theta) = \rho_m(\omega_m)\sigma\otimes\rho_n(\omega_n)\theta\,,$$

so that

$$\begin{aligned}\Sigma^+_{m+n} &= \Sigma^+_m\otimes\Sigma^+_n \oplus \Sigma^-_m\otimes\Sigma^-_n\,,\\ \Sigma^-_{m+n} &= \Sigma^+_m\otimes\Sigma^-_n \oplus \Sigma^-_m\otimes\Sigma^+_n\,.\end{aligned}$$

We can then define

$$\Sigma := \Sigma_m\otimes\Sigma_n = \Sigma^+_{m+n}\oplus\Sigma^-_{m+n}\,.$$

Second case: Assume that m is odd and n is even. For $j = 0, 1$, set

$$\begin{aligned}\gamma^j : \mathbb{R}^m\oplus\mathbb{R}^n &\longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma^j_m\otimes\Sigma_n)\\ (v,w) &\longmapsto \rho^j_m(v)\otimes(\mathrm{Id}_{\Sigma^+_n} - \mathrm{Id}_{\Sigma^-_n}) + \mathrm{Id}_{\Sigma^j_m}\otimes\rho_n(w)\,.\end{aligned}$$

As before, the map γ^j induces a non-trivial complex representation of $\mathbb{C}l_{m+n}$ of dimension $2^{[\frac{m+n}{2}]}$. Since $\omega_{m+n} = \omega_m\cdot\omega_n$ as in (2.1), we have $\gamma^j(\omega_{m+n}) = (-1)^j\,\mathrm{Id}$, and therefore the representations γ^j and ρ^j_{m+n} are equivalent. Note that

$$\gamma^j(v\cdot\omega_n) = \rho^j_m(v)\otimes\mathrm{Id}_{\Sigma_n}\,,\quad \forall\, v\in\mathbb{R}^m\,.\tag{2.3}$$

Third case: Assume that m is even and n is odd. For $j = 0, 1$, set

$$\gamma^j : \mathbb{R}^m \oplus \mathbb{R}^n \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_m \otimes \Sigma_n^j)$$

$$(v, 0) \longmapsto i \begin{pmatrix} 0 & -\rho_m(v) \\ \rho_m(v) & 0 \end{pmatrix} \otimes \mathrm{Id}_{\Sigma_n^j}$$

$$(0, w) \longmapsto \begin{pmatrix} \mathrm{Id}_{\Sigma_m^+} & 0 \\ 0 & -\mathrm{Id}_{\Sigma_m^-} \end{pmatrix} \otimes \rho_n^j(w)\,,$$

where the matrices are given with respect to the decomposition $\Sigma_m = \Sigma_m^+ \oplus \Sigma_m^-$. Once again, γ^j is an irreducible complex representation of $\mathbb{C}l_{m+n}$. As in the previous case, $\omega_{m+n} = \omega_m \cdot \omega_n$ and we see that $\gamma^j(\omega_{m+n}) = (-1)^j\,\mathrm{Id}$.

So we proved that γ^j is equivalent to ρ^j_{m+n} and

$$\gamma^j(v \cdot \omega_n) = (-1)^j\, i\, \rho_m(v) \otimes \mathrm{Id}_{\Sigma_n^j}\,, \quad \forall\, v \in \mathbb{R}^m\,. \tag{2.4}$$

Fourth case: Assume that m and n are odd. Define

$$\Sigma^+ := \Sigma_m^0 \otimes \Sigma_n^0\,,$$

$$\Sigma^- := \Sigma_m^0 \otimes \Sigma_n^1\,,$$

$$\Sigma := \Sigma^+ \oplus \Sigma^-\,,$$

and

$$\gamma : \mathbb{R}^m \oplus \mathbb{R}^n \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma)$$

$$(v, 0) \longmapsto i \begin{pmatrix} 0 & \rho_m^0(v) \otimes \tau^{-1} \\ -\rho_m^0(v) \otimes \tau & 0 \end{pmatrix}$$

$$(0, w) \longmapsto \begin{pmatrix} 0 & -\mathrm{Id}_{\Sigma_m^0} \otimes \tau^{-1} \circ \rho_n^1(w) \\ \mathrm{Id}_{\Sigma_m^0} \otimes \tau \circ \rho_n^0(w) & 0 \end{pmatrix},$$

where τ is an isomorphism from Σ_n^0 to Σ_n^1 satisfying

$$\tau \circ \rho_n^0(w) \circ \tau^{-1} = -\rho_n^1(w)\,, \quad \forall\, w \in \mathbb{R}^n\,.$$

Now, as in previous cases, we have $\gamma(v+w)^2 = -(|v|^2 + |w|^2)\,\mathrm{Id}_\Sigma$ for all $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$. Moreover, since in the case where m and n are odd, $\omega_{m+n} = -i\,\omega_m \cdot \omega_n$, we can show that

$$\gamma(\omega_{m+n}) = \begin{pmatrix} \mathrm{Id}_{\Sigma^+} & 0 \\ 0 & -\mathrm{Id}_{\Sigma^-} \end{pmatrix}.$$

Therefore we conclude that γ is equivalent to ρ_{m+n} and $\Sigma^\pm_{m+n} \cong \Sigma^\pm$.

Besides, we have the relation

$$\gamma(v \cdot \omega_n) = i \begin{pmatrix} \rho_m^0(v) \otimes \mathrm{Id}_{\Sigma_n^0} & 0 \\ 0 & -\rho_m^0(v) \otimes \mathrm{Id}_{\Sigma_n^1} \end{pmatrix}, \quad \forall\, v \in \mathbb{R}^m\,. \tag{2.5}$$

2.2. *Restriction of spinors to a submanifold*

Let $(\tilde{M}^{m+n}, g)$ be a Riemannian spin manifold and let M^m be an immersed oriented submanifold in $\tilde{M}$ with the induced Riemannian structure. Assume that $(M^m, g_{|M})$ is spin. If NM is the normal vector bundle of M in $\tilde{M}$, then there exists a spin structure on NM, denoted by $\mathrm{Spin}N$. Let $\mathrm{Spin}M \times_M \mathrm{Spin}N$ be the pull-back of the product fibre bundle $\mathrm{Spin}M \times \mathrm{Spin}N$ over $M \times M$ by the diagonal map. There exists a principal bundle morphism $\Phi : \mathrm{Spin}M \times_M \mathrm{Spin}N \to \mathrm{Spin}\tilde{M}_{|M}$, with

$$\Phi((s_M, s_N)(a, a')) = \Phi((s_M, s_N))(a \cdot a') \tag{2.6}$$

for all (s_M, s_N) in $\mathrm{Spin}M \times_M \mathrm{Spin}N$ and for all (a, a') in $\mathrm{Spin}(m) \times \mathrm{Spin}(n)$, such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Spin}M \times_M \mathrm{Spin}N & \xrightarrow{\Phi} & \mathrm{Spin}\tilde{M}_{|M} & & \\
\Big\downarrow & & \Big\downarrow & \searrow & \\
 & & & & M \\
 & & & \nearrow & \\
\mathrm{SO}M \times_M \mathrm{SO}N & \longrightarrow & \mathrm{SO}\tilde{M}_{|M} & &
\end{array}$$

where the lower horizontal arrow is just given by juxtaposition of bases (see [13]).

Now, let $\mathbb{S} := \Sigma\tilde{M}_{|M}$, where $\Sigma\tilde{M}$ is the spinor bundle of $\tilde{M}$ and

$$\Sigma := \begin{cases} \Sigma M \otimes \Sigma N & \text{if } n \text{ or } m \text{ is even}\,, \\ \Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N & \text{otherwise}\,. \end{cases}$$

Recall that there exists a hermitian inner product on $\mathbb{S}$, denoted by $\langle .\,,.\rangle$, such that Clifford multiplication by a vector of $T\tilde{M}_{|M}$ is skew-symmetric. In the following, we write $(.\,,.) = \Re e(\langle .\,,.\rangle)$.

2.3. *Identification of the restricted spinor bundle*

From the preceding considerations, it is now possible to identify $\mathbb{S}$ with Σ. For example, if m and n are even, we have the following isomorphism:

$$\Sigma M \otimes \Sigma N \longrightarrow \mathbb{S}$$

$$([s_M, \sigma], [s_N, \eta]) \longmapsto [\Phi(s_M, s_N), \sigma \otimes \eta]$$

where the last equivalence class is given, for all $(a, a') \in \mathrm{Spin}(m) \times \mathrm{Spin}(n)$, by

$$(\Phi((s_M, s_N)(a, a')), \sigma \otimes \eta) \sim (\Phi(s_M, s_N), \gamma(a \cdot a')(\sigma \otimes \eta))\,,$$

with respect to (2.6). From now on, the inverse of this isomorphism will be denoted by

$$\psi \in \Gamma(\mathbb{S}) \mapsto \psi^\star \in \Gamma(\Sigma)\,. \tag{2.7}$$

With respect to $\langle .\,,.\rangle$ and the naturally induced hermitian inner product on Σ, this isomorphism is unitary. This is why both inner products will be denoted by the same symbol when using this identification.

Let $\omega_\perp = \omega_n$ if n is even, and $\omega_\perp = -i\omega_n$ if n is odd. Recall that in both cases $\omega_\perp^2 = (-1)^n$ (compare with the definition of $\omega_\perp$ in [9] and note that it keeps the same properties). From (2.2)–(2.5), it is easy to see that, with respect to the representation γ defined in Sec. 2.1, Clifford multiplication by a vector field X tangent to M satisfies

$$\forall\,\psi \in \Gamma(\mathbb{S})\,, \quad X \underset{M}{\cdot} \psi^\star = (X \cdot \omega_\perp \cdot \psi)^\star\,. \tag{2.8}$$

2.4. *The Gauss formula and the submanifold Dirac operator*

Fix $p \in M$ and denote by $(e_1, \ldots, e_m, \nu_1, \ldots, \nu_n)$ a positively oriented local orthonormal basis of $T\tilde{M}_{|M}$ such that $(e_1, \ldots, e_m)$ (respectively $(\nu_1, \ldots, \nu_n)$) is a positively oriented local orthonormal basis of TM (respectively NM). If $\tilde{\nabla}$ denotes the Levi–Civita connection of $(\tilde{M}, g)$, then for all $X \in \Gamma(TM)$, for all $Y \in \Gamma(NM)$ and for $i = 1, \ldots, m$, the Gauss formula can be written as

$$\tilde{\nabla}_i(X + Y) = \nabla_i(X + Y) + h(e_i, X) - h^*(e_i, Y)\,, \tag{2.9}$$

where $\nabla_i(X + Y) = \nabla_i^M X + \nabla_i^N Y$, and $h^*(e_i, .)$ is the transpose of the second fundamental form h viewed as a linear map from TM to NM. Here $\tilde{\nabla}_i$ stands for $\tilde{\nabla}_{e_i}$.

Denote also by $\tilde{\nabla}$ and ∇ the induced spinorial covariant derivatives on $\Gamma(\mathbb{S})$. Therefore, on $\Gamma(\mathbb{S})$, $\nabla = \nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N}$ except for n and m odd where $\nabla = (\nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N}) \oplus (\nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N})$. For $\psi \in \Gamma(\mathbb{S})$, the covariant derivative $\nabla\psi$ is understood via the relation $(\nabla\psi)^\star = \nabla\psi^\star$.

As in [2], one can deduce from (2.9) the spinorial Gauss formula:

$$\forall\,\psi \in \Gamma(\mathbb{S})\,, \quad \tilde{\nabla}_i\psi = \nabla_i\psi + \frac{1}{2}\sum_{j=1}^{m} e_j \cdot h_{ij} \cdot \psi\,. \tag{2.10}$$

Now, define the following Dirac operators

$$\tilde{D} = \sum_{i=1}^{m} e_i \cdot \tilde{\nabla}_i\,, \quad D = \sum_{i=1}^{m} e_i \cdot \nabla_i\,,$$

and, $H = \sum_{i=1}^m h(e_i, e_i)$ denoting the mean curvature vector field,

$$D_H := (-1)^n \omega_\perp \cdot \tilde{D} = (-1)^n \omega_\perp \cdot D + \frac{1}{2} H \cdot \omega_\perp \cdot \psi \tag{2.11}$$

since $H \cdot \omega_\perp \cdot = (-1)^{n-1} \omega_\perp \cdot H \cdot$ and $\tilde{D} = D - \frac{1}{2} H \cdot$ by (2.10).

Remark 2.1. Another Dirac operator can be defined by using intrinsic Clifford multiplication and twisting the Dirac operator on the submanifold with the normal spinor bundle. This has been done by C. Bär in [2] by setting

$$D_M^{\Sigma N} : \Gamma(\Sigma) \longrightarrow \Gamma(\Sigma)$$

$$\varphi \longmapsto \begin{cases} \sum_i e_i \underset{M}{\cdot} \nabla_i \varphi & \text{if } m \text{ or } n \text{ is even}\,, \\ \sum_i e_i \underset{M}{\cdot} \nabla_i \varphi \oplus - \sum_i e_i \cdot_M \nabla_i \varphi & \text{if } m \text{ and } n \text{ are odd}\,. \end{cases}$$

In fact, with the help of (2.8) and (2.11), we can relate D_H and $D_M^{\Sigma N}$ by

$$(D_H\psi)^\star = \left((-1)^n\omega_\perp \cdot D\psi + \frac{1}{2}H \cdot \omega_\perp \cdot \psi\right)^\star$$
$$= D_M^{\Sigma N}\psi^\star + \frac{1}{2}(H \cdot \omega_\perp \cdot \psi)^\star\,, \quad \forall\, \psi \in \Gamma(\mathbb{S})\,. \tag{2.12}$$

It is known that D_H is formally self-adjoint and that $D_H^2 = \tilde{D}^*\tilde{D}$, where $\tilde{D}^*$ is the formal adjoint of $\tilde{D}$ w.r.t. $\int_M(.\,,.)v_g$ (see [9]).

3. Estimates for the Eigenvalues of the Submanifold Dirac Operator

3.1. *Basic estimates*

First, for any spinor field $\psi \in \Gamma(\mathbb{S})$, define the function

$$R_\psi^N := 2\sum_{i,j=1}^{m}(e_i \cdot e_j \cdot \mathrm{Id} \otimes \mathrm{R}^N_{e_i,e_j}\psi, \psi/|\psi|^2) \tag{3.1}$$

on $M_\psi := \{x \in M : \psi(x) \neq 0\}$, where $\mathrm{R}^N_{e_i,e_j}$ stands for spinorial normal curvature tensor. We start by giving a proof of the following result (see [9]):

Theorem 3.1 (Hijazi–Zhang). *Let $M^m \subset \tilde{M}^{m+n}$ be a compact spin submanifold of a Riemannian spin manifold $(\tilde{M}, g)$. Consider a non-trivial spinor field $\psi \in \Gamma(\mathbb{S})$ such that $D_H\psi = \lambda\psi$. Assume that $m \geq 2$ and*

$$m(R + R_\psi^N) > (m-1)\|H\|^2 > 0$$

on M_ψ, where R is the scalar curvature of $(M^m, g_{|M})$ and R_ψ^N is given by (3.1). *Then one has*

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\psi}\left(\sqrt{\frac{m}{m-1}(R + R_\psi^N)} - \|H\|\right)^2. \tag{3.2}$$

Proof. For any function q, nowhere equal to $\frac{1}{m}$, define the modified connection,

$$\nabla_i^\lambda = \nabla_i + \frac{1-q}{2(1-mq)}\, e_i \cdot H \cdot + q\lambda e_i \cdot \omega_\perp \cdot\ .$$

Using the Lichnerowicz–Schrödinger formula (see [12]), we have

$$(D^2\psi, \psi) = (\nabla^*\nabla\psi, \psi) + \frac{1}{4}(R + R_\psi^N)|\psi|^2\,,$$

and one can easily compute

$$\int_M |\nabla^\lambda\psi|^2 v_g = \int_M (1 + mq^2 - 2q)$$
$$\times\left[\lambda^2 - \frac{1}{4}\left(\frac{R + R_\psi^N}{(1 + mq^2 - 2q)} - \frac{(m-1)\|H\|^2}{(1-mq)^2}\right)\right]|\psi|^2 v_g\,. \tag{3.3}$$

Then, assuming $m(R+R_\psi^N) > (m-1)\|H\|^2 > 0$ on M_ψ, we can choose q so that

$$(1-mq)^2 = \frac{(m-1)\|H\|}{\sqrt{\frac{m}{m-1}(R+R_\psi^N)} - \|H\|} \quad \text{on } M_\psi\,. \tag{3.4}$$

Inserting Eq. (3.4) in (3.3), and since the complement of M_ψ in M is of zero-measure, we conclude by observing that the left member of (3.3) is nonnegative. □

Let κ_1 be the lowest eigenvalue of the self-adjoint operator $\mathcal{R}^N$ defined by

$$\mathcal{R}^N : \Gamma(\mathbb{S}) \longrightarrow \Gamma(\mathbb{S}) \tag{3.5}$$

$$\varphi \longmapsto 2\sum_{i,j=1}^{m} e_i \cdot e_j \cdot \mathrm{Id} \otimes \mathrm{R}^N_{e_i,e_j}\varphi\,. \tag{3.6}$$

The hypothesis $m(R+R_\psi^N) > (m-1)\|H\|^2 > 0$ in Theorem 3.1 can be strengthened to give

Corollary 3.2. *Under the same hypotheses as in Theorem* 3.1, *assume that* $m \geq 2$ *and*

$$m(R+\kappa_1) > (m-1)\|H\|^2 > 0$$

on M, *then*

$$\lambda^2 \geq \frac{1}{4}\inf_M \left(\sqrt{\frac{m}{m-1}(R+\kappa_1)} - \|H\|\right)^2\,. \tag{3.7}$$

Recall that in the case of hypersurfaces, limiting cases are characterized by the existence of a real Killing spinor on M and the fact that the mean curvature H is constant (see [14] and [16]). A non-zero section ψ of $\mathbb{S}$ satisfying

$$\forall\, X \in \Gamma(TM)\,, \quad \nabla_X\psi^\star = -\frac{\mu}{m} X \underset{M}{\cdot} \psi^\star$$

for a given real constant μ will be called a twisted (real) Killing spinor.

Proposition 3.3. *If equality holds in* (3.7), *then* $(M^m, g_{|M})$ *admits a twisted Killing spinor and* $\|H\|$ *is constant.*

Proof. Suppose the limiting case holds in (3.7), then the right hand side has to be constant on M, and

$$\lambda^2 = \frac{1}{4}\left(\sqrt{\frac{m}{m-1}(R+\kappa_1)} - \|H\|\right)^2\,, \quad \nabla^\lambda\psi = 0\,, \quad \text{on } M\,. \tag{3.8}$$

Note that equality holds in (3.2) which yields $R_\psi^N = \kappa_1$. Hence ψ is an eigenspinor for the operator $\mathcal{R}^N$ with eigenvalue κ_1. Using (3.8), we can show that $|\psi|$ must be constant on M (therefore, $M_\psi = M$) and compute

$$D\psi = -\sum_{i=1}^{m} e_i \cdot \left(\frac{1-q}{2(1-mq)}\, e_i \cdot H \cdot \psi + q\lambda e_i \cdot \omega_\perp \cdot \psi\right)$$

$$= \frac{m(1-q)}{2(1-mq)} H \cdot \psi + mq\lambda\omega_\perp \cdot \psi\,.$$

Then, by (2.11) and the fact that $H \cdot \omega_\perp \cdot = (-1)^{n-1}\omega_\perp \cdot H\cdot$,

$$0 = \lambda\omega_\perp \cdot \psi + \frac{H}{2} \cdot \psi - \frac{m(1-q)}{2(1-mq)} H \cdot \psi - mq\lambda\omega_\perp \cdot \psi$$

$$0 = (1-mq)^2\lambda\omega_\perp \cdot \psi - (m-1)\frac{H}{2} \cdot \psi\,.$$

Since in the equality case, $\sqrt{\frac{m}{m-1}(R+\kappa_1)} - \|H\| = 2|\lambda|$, we can deduce the relation:

$$\omega_\perp \cdot \psi = \mathrm{sgn}(\lambda)\frac{H}{\|H\|} \cdot \psi\,.$$

With respect to the isomorphism "$\star$", we can rewrite (3.8) as an intrinsic equation on $\Gamma(\Sigma)$:

$$\forall\, X \in \Gamma(TM)\,, \quad \nabla_X \psi^\star = -\frac{\mu}{m} X \underset{M}{\cdot} \psi^\star$$

$$\text{with } \mu = \frac{\mathrm{sgn}(\lambda)}{2}\sqrt{\frac{m}{m-1}(R+\kappa_1)}\,.$$

Note that if there exists two smooth real functions f and κ on M and a non-zero section ψ of $\mathbb{S}$ satisfying for all vector field X on M

$$\nabla_X \psi^\star = -\frac{f}{m} X \underset{M}{\cdot} \psi^\star \quad \text{and} \quad \mathcal{R}^N \psi = \kappa\psi\,,$$

then, by computing the action of the curvature tensor on $\psi^\star$, we see that necessarily

$$\frac{1}{2}\mathrm{Ric}(X) \underset{M}{\cdot} \psi^\star - \sum_{i=1}^{m} (e_i \cdot \mathrm{Id} \otimes \mathrm{R}^N_{X,e_i})\psi^\star$$

$$= -\frac{1}{m}\mathrm{d}f \underset{M}{\cdot} X \underset{M}{\cdot} \psi^\star - \mathrm{d}f(X)\psi^\star + 2\frac{m-1}{m^2} f^2 X \underset{M}{\cdot} \psi^\star$$

which implies

$$f^2 = \frac{m}{4(m-1)}(R+\kappa) = \text{constant}\,.$$

Moreover, in the equality case, the fact that f is constant implies that $\|H\|$ is constant. □

Remark 3.4. If the normal curvature tensor is zero, then μ has to be constant and the manifold M must be Einstein with mean curvature vector being of constant length. Besides, the equality case corresponds to that of Friedrich's inequality. Therefore μ is the first eigenvalue of the Dirac operator $D_M^{\Sigma N}$.

3.2. *Estimate involving the Energy-Momentum tensor*

If $\psi \in \Gamma(\mathbb{S})$ is a spinor field, we define the Energy-Momentum tensor Q^ψ associated with ψ on M_ψ by

$$Q^\psi_{ij} = \frac{1}{2}(e_i \cdot \omega_\perp \cdot \nabla_j \psi + e_j \cdot \omega_\perp \cdot \nabla_i \psi, \psi/|\psi|^2)\,.$$

Note that

$$Q^\psi_{ij} = \frac{1}{2}(e_i \underset{M}{\cdot} \nabla_j \psi^\star + e_j \underset{M}{\cdot} \nabla_i \psi^\star, \psi^\star/|\psi^\star|^2)\,.$$

Therefore, Q^ψ is the intrinsic Energy-Momentum tensor associated with $\psi^\star$. Observe that this intrinsic Energy-Momentum tensor is the one that appears in the Einstein–Dirac equation (see [7]). We prove the following (compare with [14])

Theorem 3.5. *Let $M^m \subset \tilde{M}^{m+n}$ be a compact spin submanifold of a Riemannian spin manifold $(\tilde{M}, g)$. Consider a non-trivial spinor field $\psi \in \Gamma(\mathbb{S})$ such that $D_H\psi = \lambda\psi$. Assume that*

$$R + \kappa_1 + 4|Q^\psi|^2 > \|H\|^2 > 0$$

on M_ψ. Then one has

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{R + \kappa_1 + 4|Q^\psi|^2} - \|H\|\right)^2. \tag{3.9}$$

Proof. For any real function q that never vanishes, consider the modified covariant derivative defined on $\Gamma(\mathbb{S})$ by

$$\nabla^Q_i = \nabla_i - \frac{1}{2mq} e_i \cdot H \cdot + (-1)^{n+1} q\lambda e_i \cdot \omega_\perp \cdot + \sum_j Q^\psi_{ij} e_j \cdot \omega_\perp \cdot\,.$$

As in the proof of Theorem 3.1, we compute

$$\begin{aligned}\int_M |\nabla^Q \psi|^2 v_g = {} & \int_M (1 + mq^2)\left[\lambda^2 - \frac{1}{4}\left(\frac{R + R^N_\psi + 4|Q^\psi|^2}{(1 + mq^2)} - \frac{\|H\|^2}{mq^2}\right)\right] |\psi|^2 v_g \\ & - \frac{1}{4}\int_M (1 + mq^2)\left[\frac{2}{mq(1 + mq^2)}\left(\|H\|^2 - \frac{\langle H \cdot \psi, \omega_\perp \cdot \psi\rangle^2}{|\psi|^4}\right)\right] |\psi|^2 v_g\,.\end{aligned} \tag{3.10}$$

To finish the proof of Theorem 3.5, if $R + \kappa_1 + 4|Q^\psi|^2 > \|H\|^2 > 0$, we take

$$q = \sqrt{\frac{\|H\|}{m(\sqrt{R + \kappa_1 + 4|Q^\psi|^2} - \|H\|)}}\,,$$

and then observe that by the Cauchy–Schwarz inequality, we have

$$\|H\|^2 - \frac{\langle H \cdot \psi, \omega_\perp \cdot \psi\rangle^2}{|\psi|^4} \geq 0\,. \tag{3.11}$$

□

Suppose now that equality holds in (3.9). Then

$$\nabla^Q \psi = 0\,, \quad |\lambda| = \frac{1}{2}\left(\sqrt{R+\kappa_1+4|Q^\psi|^2} - \|H\|\right) \quad \text{and} \quad \mathcal{R}^N\psi = \kappa_1\psi\,.$$

Moreover,

$$\|H\|^2 - \frac{\langle H\cdot\psi, \omega_\perp\cdot\psi\rangle^2}{|\psi|^4} = 0\,,$$

so that, by the equality case in the Cauchy–Schwarz inequality,

$$\omega_\perp \cdot \psi = fH\cdot\psi\,,$$

for some real function f on M. As in the preceding section, and taking into account the identification (2.7), we deduce that $f = \frac{\mathrm{sgn}(\lambda)}{\|H\|}$, and that the section ψ satisfies

$$\nabla_i \psi^\star = -\sum_j Q^\psi_{ij}\, e_j \underset{M}{\cdot} \psi^\star\,. \tag{3.12}$$

Hence, we can say that ψ is a kind of Energy-Momentum spinor (see [14]). We will call such a section a twisted EM-spinor. One can give an integrability condition for the existence of twisted EM-spinors, by computing the action of the curvature tensor on $\Gamma(\mathbb{S})$:

$$(\mathrm{tr}(Q^\psi))^2 = \frac{1}{4}(R + R^N_\psi + 4|Q^\psi|^2)\,.$$

This implies, with Eq. (3.12), that the section $\psi^\star$ is an "eigenspinor" for $D^{\Sigma N}_M$ associated with the function $\pm\frac{1}{2}\sqrt{R+\kappa_1+4|Q^\psi|^2}$. Note that this function is constant if and only if $\|H\|$ is constant.

3.3. *Conformal lower bounds*

Consider a conformal change of the metric $\bar{g} = e^{2u}g$ for a real function u on $\tilde{M}$. Let

$$\begin{aligned} \mathbb{S} &\longrightarrow \bar{\mathbb{S}} \\ \psi &\longmapsto \bar{\psi} \end{aligned} \tag{3.13}$$

be the induced isometry between the two corresponding spinor bundles. Recall that if φ, ψ are two sections of $\mathbb{S}$, and Z any vector field on $\tilde{M}$, we have

$$(\varphi,\psi) = (\bar{\varphi},\bar{\psi})_{\bar{g}} \quad \text{and} \quad \bar{Z}\,\bar{\cdot}\,\bar{\psi} = \overline{Z\cdot\psi}\,,$$

where $\bar{Z} = e^{-u}Z$. We will also denote by $\bar{g} = (e^{2u}g)_{|M}$ the restriction of $\bar{g}$ to M.

Note that this isomorphism commutes with the isomorphism "$\star$" given by (2.7). By conformal covariance of the Dirac operator, for $\psi \in \Gamma(\mathbb{S})$, we have,

$$\bar{D}(e^{-\frac{(m-1)}{2}u}\bar{\psi}) = e^{-\frac{(m+1)}{2}u}\overline{D\psi}\,, \tag{3.14}$$

where $\bar{D}$ stands for the Dirac operator w.r.t. to $\bar{g}$. On the other hand, the corresponding mean curvature vector field is given by

$$\tilde{H} = e^{-2u}(H - m\ \mathrm{grad}^N u)\,. \tag{3.15}$$

Now, assume that $\mathrm{grad}^N u = 0$. If $\bar{D}_H$ stands for the submanifold Dirac operator w.r.t. to $\bar{g}$, Eqs. (3.14) and (3.15) imply that $\bar{D}_H$ is a conformally covariant operator, i.e.

$$\bar{D}_H(e^{-\frac{(m-1)}{2}u}\bar{\psi}) = e^{-\frac{(m+1)}{2}u}(\overline{D_H\psi}) \tag{3.16}$$

for any section ψ of $\mathbb{S}$.

From now on, we will only consider regular conformal changes of the metric, i.e. $\bar{g} = e^{2u}g$ with $\mathrm{grad}^N u = 0$, on M.

Theorem 3.6. *Let $M^m \subset \tilde{M}^{m+n}$ be a compact spin submanifold of a Riemannian spin manifold $(\tilde{M}, g)$. Consider a non-trivial spinor field $\psi \in \Gamma(\mathbb{S})$ such that $D_H\psi = \lambda\psi$. For any regular conformal change of the metric $\bar{g} = e^{2u}g$ on $\tilde{M}$, assume that*

$$\bar{\mathrm{R}}\mathrm{e}^{2\mathrm{u}} + \kappa_1 + 4|\mathrm{Q}^\psi|^2 > \|\mathrm{H}\|^2 > 0$$

on M_ψ. Then one has

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\psi}\left(\sqrt{\bar{\mathrm{R}}\mathrm{e}^{2\mathrm{u}} + \kappa_1 + 4|\mathrm{Q}^\psi|^2} - \|H\|\right)^2 . \tag{3.17}$$

Proof. For $\psi \in \Gamma(\mathbb{S})$ an eigenspinor of D_H with eigenvalue λ, let $\bar{\varphi} := e^{-\frac{n-1}{2}u}\bar{\psi}$. Then (3.16) gives $\bar{D}_H\,\bar{\varphi} = \lambda e^{-u}\bar{\varphi}$. Recall that

$$\bar{\nabla}_i\bar{\psi} = \overline{\nabla_i\psi} - \frac{1}{2}\overline{e_i\cdot \mathrm{d}u\cdot\psi} - \frac{1}{2}e_i(u)\bar{\psi}\,,$$

and $\overline{e_i} = e^{-u}e_i$. Now, it is straightforward to get $\bar{Q}^{\bar{\varphi}}_{\bar{i}\bar{j}} = e^{-u}Q^\psi_{ij}$, hence,

$$|\bar{Q}^{\bar{\varphi}}|^2 = e^{-2u}|Q^\psi|^2\,. \tag{3.18}$$

Equation (3.10), which is also true on $(\tilde{M}, \bar{g})$, applied to $\bar{\varphi}$ yields

$$\int_M |\bar{\nabla}^{\bar{Q}}\bar{\varphi}|^2 v_{\bar{g}} = \int_M (1+mq^2)\left[(\lambda e^{-u})^2 - \frac{1}{4}\left(\frac{\bar{R} + \bar{R}^N_{\bar{\varphi}} + 4|\bar{Q}^{\bar{\varphi}}|^2}{(1+mq^2)} - \frac{\|\tilde{H}\|^2_{\bar{g}}}{mq^2}\right)\right]|\bar{\varphi}|^2_{\bar{g}}\,v_{\bar{g}}$$

$$- \frac{1}{4}\int_M (1+mq^2)\left[\frac{2}{mq(1+mq^2)}\left(\|\tilde{H}\|^2_{\bar{g}} - \frac{\langle\tilde{H}\,\bar{\cdot}\,\bar{\varphi}, \overline{\omega_\perp}\,\bar{\cdot}\,\bar{\varphi}\rangle^2_{\bar{g}}}{|\varphi|^4_{\bar{g}}}\right)\right]|\bar{\varphi}|^2_{\bar{g}}\,v_{\bar{g}}\,.$$

Since $\tilde{H} = e^{-u}\bar{H}$, and $\bar{R}^N_{\bar{\varphi}} = e^{-2u}R^N_\psi$, we have

$$\int_M |\bar{\nabla}^{\bar{Q}}\bar{\varphi}|^2 v_{\bar{g}} = \int_M (1+mq^2)e^{-2u}\left[\lambda^2 - \frac{1}{4}\left(\frac{\bar{\mathrm{R}}\mathrm{e}^{2\mathrm{u}} + \mathrm{R}^{\mathrm{N}}_\psi + 4|\mathrm{Q}^\psi|^2}{(1+mq^2)} - \frac{\|H\|^2}{mq^2}\right)\right]|\bar{\varphi}|^2 v_{\bar{g}}$$

$$-\frac{1}{4}\int_M (1+mq^2)e^{-2u}\left[\frac{2}{mq(1+mq^2)}\left(\|H\|^2 - \frac{\langle H\cdot\psi, \omega_\perp\cdot\psi\rangle^2}{|\psi|^4}\right)\right]|\bar{\varphi}|^2 v_{\bar{g}}\,.$$

As in the proof of Theorem 3.5, we finally take

$$q = \sqrt{\frac{\|H\|}{m(\sqrt{\bar{\mathrm{R}}\mathrm{e}^{2\mathrm{u}} + \kappa_1 + 4|\mathrm{Q}^\psi|^2} - \|H\|)}}$$

and use the Cauchy–Schwarz inequality (3.11). □

If the hypothesis in Theorem 3.6 is satisfied by an eigenfunction u_1 associated with the first eigenvalue μ_1 of the Yamabe operator, then one has:

Corollary 3.7. *Under the same conditions as in Theorem* 3.6, *assume that* $m \geq 3$ *and* $\mu_1 + \kappa_1 + 4|Q^\psi|^2 > \|H\|^2 > 0$ *on* M_ψ, *then*

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{\mu_1 + \kappa_1 + 4|Q^\psi|^2} - \|H\|\right)^2 .$$

Corollary 3.8. *Under the same conditions as in Theorem* 3.6, *if* M *is a compact surface of genus zero and* $\frac{8\pi}{\mathrm{Area}(M)} + \kappa_1 + 4|Q^\psi|^2 > \|H\|^2 > 0$ *on* M_ψ, *then*

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{\frac{8\pi}{\mathrm{Area}(M)} + \kappa_1 + 4|Q^\psi|^2} - \|H\|\right)^2 .$$

Now suppose that equality holds in (3.17). Then

$$\bar{\nabla}^{\bar{Q}}\bar{\varphi} = 0\,, \quad \omega_\perp \cdot \psi = \varepsilon \frac{H}{\|H\|} \cdot \psi \quad \text{where } \varepsilon \in \{\pm 1\}\,,$$

$$|\lambda| = \frac{1}{2}\left(\sqrt{\bar{\mathrm{R}}\mathrm{e}^{2\mathrm{u}} + \kappa_1 + 4|\mathrm{Q}^\psi|^2} - \|H\|\right) \quad \text{and} \quad \mathcal{R}^N\psi = \kappa_1\psi\,.$$

Using (3.13) and (3.18), it follows $\varepsilon = \mathrm{sgn}(\lambda)$ and

$$\nabla_i\psi^\star = \frac{1}{2} e_i \underset{M}{\cdot} \mathrm{d}u \underset{M}{\cdot} \psi^\star + \frac{m}{2}\mathrm{d}u(e_i)\psi^\star - \sum_j Q^\psi_{ij}\, e_j \underset{M}{\cdot} \psi^\star \tag{3.19}$$

with $\mathrm{d}u = \frac{2\mathrm{d}(\ln(|\psi|))}{m-1}$. Non-trivial spinor fields satisfying (3.19) will be naturally *called* twisted WEM-spinors (compare with [14]).

4. Final Remark

In this section, we show that the normal bundle of the submanifold can be replaced by an auxiliary arbitrary vector bundle on the submanifold. Thus, all the preceding computations could be done in an intrinsic way to obtain results for a twisted Dirac operator on the manifold.

Let (M^m, g) be a compact Riemannian spin manifold. Let $N \to M$ be a Riemannian vector bundle of rank n over M. Suppose that N is endowed with a metric connection ∇^N and a spin structure. Let ΣM (respectively ΣN) be the spinor bundle of M (respectively N). Set

$$\Sigma := \Sigma M \otimes \Sigma N\,.$$

Recall that Clifford multiplication on $\Gamma(\Sigma)$ by a tangent vector field X is given by:

$$\forall\,\psi \in \Gamma(\Sigma)\,, \quad X\cdot\psi = (\rho_M(X)\otimes \mathrm{Id}_{\Sigma N})(\psi)\,.$$

Define the tensor-product connection ∇ on $\Gamma(\Sigma)$ by

$$\nabla = \nabla^{\Sigma M}\otimes \mathrm{Id}_{\Sigma N} + \mathrm{Id}_{\Sigma M}\otimes \nabla^{\Sigma N}\,,$$

where $\nabla^{\Sigma M}$ and $\nabla^{\Sigma N}$ are the induced connections on $\Gamma(\Sigma M)$ and $\Gamma(\Sigma N)$ respectively. Let $D_M^{\Sigma N}$ be the twisted Dirac operator given by

$$D_M^{\Sigma N} = \sum_{i=1}^m e_i\cdot\nabla_i\,.$$

For any smooth real function f on M, define the modified twisted Dirac operator by

$$D_f = D_M^{\Sigma N} - \frac{f}{2}\,.$$

For $\lambda\in\mathbb{R}$, consider the following modified connections

$$\hat{\nabla}_i^\lambda = \nabla_i + \frac{(1-q)f}{2(1-mq)}\,e_i\cdot + q\lambda e_i\,\cdot$$

$$\hat{\nabla}_i^Q = \nabla_i - \frac{f}{2mq}\,e_i\cdot + (-1)^{n+1}q\lambda e_i\cdot + \sum_j Q_{ij}^\psi\,e_j\,,$$

where Q^ψ is now the intrinsic Energy-Momentum tensor associated with ψ.

Note that these connections can be obtained from those defined in Sec. 3, assuming that

$$H\cdot\psi = f\omega_\perp\cdot\psi\,.$$

In fact, this is the only way to give an intrinsic meaning to the modified connection used before. Then the same computations as in the proofs of Theorems 3.1, 3.5 and 3.6, lead to the following assertions:

Let (M^m, g) be a compact Riemannian spin manifold with $N\to M$ an auxiliary oriented Riemannian spin vector bundle of rank n. Let $\psi\in\Gamma(\Sigma)$ be an eigenspinor for the modified twisted Dirac operator D_f, associated with the eigenvalue λ. Then,

Proposition 4.1. *Assume that $m\geq 2$ and $m(R+\kappa_1) > (m-1)f^2 > 0$ on M_ψ. Then one has*

$$\lambda^2 \geq \frac{1}{4}\inf_{M_\psi}\left(\sqrt{\frac{m}{m-1}(R+\kappa_1)} - |f|\right)^2\,.$$

If equality holds, (M^m,g) admits a twisted Killing spinor.

Following the proof of Theorem 3.6, we can extend the previous theorem by performing a conformal change of the metric on M. For the limiting case, just note that $Q^\psi = \frac{1}{m}\mathrm{tr}(Q^\psi)g$.

Proposition 4.2. *Assume that* $m \geq 2$ *and* $m(\bar{\mathrm{R}}e^{2u} + \kappa_1) > (m-1)f^2 > 0$ *on* M_ψ *for any conformal change of the metric* $\bar{g} = e^{2u}g$ *on* M. *Then one has*

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{\frac{m}{m-1}(\bar{\mathrm{R}}e^{2u} + \kappa_1)} - |f| \right)^2 .$$

If equality holds, $(M^m, \bar{g})$ *admits a twisted WEM-spinor, with* $Q^\psi = \frac{\mu}{m} g$, *where*

$$\mu^2 = \frac{1}{4} \frac{m}{m-1} (\bar{\mathrm{R}}e^{2u} + \kappa_1) .$$

Proposition 4.3. *Assume that* $R + \kappa_1 + 4|Q^\psi|^2 > f^2 > 0$ *on* M_ψ. *Then one has*

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{R + \kappa_1 + 4|Q^\psi|^2} - |f| \right)^2 .$$

If equality holds, (M^m, g) *admits a twisted EM-spinor.*

Proposition 4.4. *Assume that* $\bar{\mathrm{R}}e^{2u} + \kappa_1 + 4|Q^\psi|^2 > f^2 > 0$ *on* M_ψ *for any conformal change of the metric* $\bar{g} = e^{2u}g$ *on* M. *Then one has*

$$\lambda^2 \geq \frac{1}{4} \inf_{M_\psi} \left(\sqrt{\bar{R}e^{2u} + \kappa_1 + 4|Q^\psi|^2} - |f| \right)^2 .$$

If equality holds, (M^m, g) *admits a twisted WEM-spinor.*

Remark 4.5. Assuming the normal curvature tensor is zero and f is constant, then the necessary conditions for the equality cases in Theorems 4.1, 4.2, 4.3 and 4.4 become sufficient conditions. Moreover, when m is odd, the considered Dirac operator may have to be defined with the opposite Clifford multiplication according to the sign of f.

Remark 4.6. We would like to thank Christian Bär for the following suggestion: all inequalities which appear in the hypotheses of our theorems and propositions can be taken in the large. This can be done by choosing an adapted function q_ε depending continuously on a parameter $\varepsilon > 0$ instead of the function q in the proof of the above theorems. We then obtain our inequalities when ε tends towards 0.

Acknowledgment

We would like to thank Oussama Hijazi for his support during the preparation of this paper.

References

1. C. Bär, *Real Killing Spinors and Holonomy*, Comm. Math. Phys. **154** (1993), 525–576.
2. ———, *Extrinsic Bounds for Eigenvalues of the Dirac Operator*, Ann. Glob. Anal. Geom. **16** (1998), 573–596.
3. H. Baum, T. Friedrich, R. Grunewald and I. Kath, *Twistor and Killing Spinors on Riemannian Manifolds*, Teubner Verlag, Stuttgart/Leipzig, 1991.

4. J. P. Bourguignon, O. Hijazi, J.-L. Milhorat and A. Moroianu, *A Spinorial Approach to Riemannian and Conformal Geometry*, Monograph (En préparation).
5. T. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalar-krümmung*, Math. Nach. **97** (1980), 117–146.
6. ———, *Dirac–Operatoren in der Riemannschen Geometrie*, Vieweg, Braunschweig Wiesbaden, 1997.
7. T. Friedrich and E.-C. Kim, *The Einstein-Dirac equation on Riemannian spin manifolds*, J. Geom. Phys. **33**(1–2) (2000), 128–172.
8. O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986), 151–162.
9. O. Hijazi and X. Zhang, *Lower bounds for the Eigenvalues of the Dirac Operator, Part II*, *The Submanifold Dirac Operator*, to appear in Ann. Glob. Anal. Geom.
10. O. Hijazi, X. Zhang and S. Montiel, *Eigenvalues of the Dirac operator on Manifolds with Boundary*, Les Prépublications de l'Institut Elie Cartan, Nancy **33** (2000).
11. ———, *Dirac operator on embedded hypersurfaces*, Math. Res. Lett. **8** (2001), 195–208.
12. H. B. Lawson and M. L. Michelsohn, *Spin Geometry*, Princeton University Press, 1989.
13. J. Milnor, *Remarks Concerning Spin Manifolds*, Princeton University Press, 1965, pp. 55–62.
14. B. Morel, *Eigenvalue Estimates for the Dirac-Schrödinger Operators*, J. Geom. Phys. **38** (2001), 1–18.
15. E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.
16. X. Zhang, *Lower bounds for eigenvalues of hypersurface Dirac operators*, Math. Res. Lett. **5** (1998), 199–210.
17. ———, *A remark: Lower bounds for eigenvalues of hypersurface Dirac operators*, Math. Res. Lett. **6** (1999), 465–466.